

OPEN ACCESS

An iteration problem

To cite this article: Ural Bekbaev 2013 *J. Phys.: Conf. Ser.* **435** 012007

View the [article online](#) for updates and enhancements.

You may also like

- [LIMITS OF INDETERMINACY IN MEASURE OF \$T\$ -MEANS OF TRIGONOMETRIC SERIES](#)
D Men'sov
- [Extraction Technique of Trap Density at Grain Boundaries in Polycrystalline-Silicon Thin-Film Transistors with Device Simulation](#)
Kiyoshi Harada, Takuto Yoshino, Tohru Yasuhara et al.
- [Factors influencing the ill-posedness of nonlinear problems](#)
B Hofmann and O Scherzer



ECS
The
Electrochemical
Society
Advancing solid state &
electrochemical science & technology

DISCOVER
how sustainability
intersects with
electrochemistry & solid
state science research

An iteration problem

Ural Bekbaev

Department of Science in Engineering, Faculty of Engineering, International Islamic University
Malaysia, 57528, P.O. 10, Kuala Lumpur, Malaysia
e-mail: bekbaev@iiu, bekbaev2011@gmail.com

Abstract. Let F stand for the field of real or complex numbers, $\varphi : F^n \rightarrow F^n$ be any given polynomial map of the form $\varphi(x) = x + \text{"higher order terms"}$. We attach to it the following operator $D : F[x] \rightarrow F[x]$ defined by $D(f) = f - f \circ \varphi$, where $F[x] = F[x_1, x_2, \dots, x_n]$ -the F -algebra of polynomials in variables x_1, x_2, \dots, x_n , $f \in F[x]$ and \circ stands for the composition(superposition) operation. It is shown that trajectory of any $f \in F[x]$ tends to zero, with respect to a metric, and stabilization of all trajectories is equivalent to the stabilization of trajectories of x_1, x_2, \dots, x_n .

Let S^{n-1} be $n - 1$ dimensional simplex in the real vector space and $D : S^{n-1} \rightarrow S^{n-1}$ be a(for example, polynomial or continuous) map. The behavior investigation of the trajectories $(D^m(x))_{m \in N}$, where $x \in S^{n-1}$, D^m means m times application of D , is one of the important problems in Discrete Dynamical Systems theory. In particular characterization of polynomial maps D , for which trajectories $(D^m(x))_{m \in N}$ converge at any $x \in S^{n-1}$, is one of the unsolved problems. One can consider stronger case: When does $(D^m(x))_{m \in N}$ stabilize, that is there exists $m_0 = m_0(x) \in N$ such that $D^m(x) = D^{m_0}(x)$ at any $m > m_0$, for any $x \in S^{n-1}$? This problem is not solved even for polynomial maps of degree ≤ 2 . In this paper we consider a linear transformation D of the space of multivariate polynomials which appears as a regular pattern in the formal inversion formula for multivariate power series [1,2]. It will be shown that for this type transformations trajectories stabilization problem at all points can be reduced to the stabilization of the trajectories of a finite system of points. Moreover it is noted that for these transformations the stabilization problem is closely related to the Jacobian conjecture. It is interesting to note that the same regular pattern is the main object of investigation in [3], where it is considered by functional analysis's point of view with applications to problems of Dynamical System and Functional Analysis.

Let F stand for the field of real or complex numbers, $\varphi : F^n \rightarrow F^n$ be any given polynomial map. We attach to it the following transformation $D_\varphi = D : F[x] \rightarrow F[x]$ defined by $D(f) = f - f \circ \varphi$, where $F[x] = F[x_1, x_2, \dots, x_n]$ -the F -algebra of polynomials in variables x_1, x_2, \dots, x_n , $f \in F[x]$ and \circ stands for the composition(superposition) operation. The following Proposition shows that this transformation reminds a differential operator.

Proposition 1. *The following identities are true*

1. D is a F -linear map.
2. $D(fg) = D(f)(g \circ \varphi) + fD(g)$ whenever $f, g \in F[x]$.
3. $D(f) \circ \varphi = D(f \circ \varphi)$

Proof. Proof is an easy one, for example, here is a proof the second identity.

$$D(fg) = fg - (fg) \circ \varphi = (f - f \circ \varphi)(g \circ \varphi) + f(g - g \circ \varphi) = D(f)(g \circ \varphi) + fD(g)$$



In future it will be assumed that the polynomial map $\varphi(x)$ is of the form $\varphi(x) = x +$ "higher order terms". In this case one can check easily that $\text{ord}(D(f)) > \text{ord}(f)$ whenever $\text{ord}(f) < \infty$, where $\text{ord}(f)$ stands for the least degree of nonzero terms of f , $\text{ord}(0) = \infty$. Therefore D_φ has only one fixed point, namely zero polynomial. If one considers $F[x]$ with respect to the metric

$$d(f, g) = 2^{-\text{ord}(f-g)}$$

then every trajectory tends to this fixed point. But here we are interested in the following problem: When does every trajectory terminate(stabilize)? In other words when for every $f \in F[x]$ one can find $m = m(f)$ such that $D^m(f) = 0$, where $D_\varphi^m = D^m$ stands for $D_\varphi^{\circ m} = D^{\circ m}$.

Let $K_{\varphi, m} = K_m = \text{Ker}(D^m)$.

Proposition 2. *The following equalities are true.*

$$D^{m+1}(f) = D^m(f) - D^m(f) \circ \varphi$$

$$D^m(f) = \sum_{k=0}^m (-1)^k \binom{m}{k} f \circ \varphi^{\circ k}$$

$$D^m(fg) = \sum_{k=0}^m \binom{m}{k} D^k(f) D^{m-k}(g \circ \varphi^{\circ k})$$

$$K_m K_l \subset K_{m+l}$$

Proof. A proof of this Proposition can be done by induction on m . Here is a proof of the last but one and the last properties. At $m = 0$ and $m = 1$ the equality is true. Due to Proposition 1 and induction

$$D^{m+1}(fg) = D^1(D^m(fg)) = D^1\left(\sum_{k=0}^m \binom{m}{k} D^k(f) D^{m-k}(g \circ \varphi^{\circ k})\right) =$$

$$\sum_{k=0}^m \binom{m}{k} (D^{k+1}(f) D^{m-k}(g \circ \varphi^{\circ k}) \circ \varphi + D^k(f) D^{m-k+1}(g \circ \varphi^{\circ k})) = \sum_{k=0}^{m+1} \binom{m+1}{k} D^k(f) D^{m+1-k}(g \circ \varphi^{\circ k})$$

To prove the last property let $f \in K_m$, $g \in K_l$ and consider

$$D^{m+l}(fg) = \sum_{k=0}^{m+l} \binom{m+l}{k} D^k(f) D^{m+l-k}(g \circ \varphi^{\circ k})$$

As far as at least one of $D^k(f)$, $D^{m+l-k}(g \circ \varphi^{\circ k})$ is zero whenever $0 \leq k \leq m+l$ the above sum has to be zero that is $fg \in K_{m+l}$.

If $K_\varphi = K$ stands for $\bigcup_{m=0}^{\infty} K_{\varphi, m}$ then due to the Propositions it is clear that K is a subalgebra of $F[x]$, $K \circ \varphi \subset K$, $D(K) \subset K$ and $D^{-1}(K) \subset K$.

Here is the main result of this paper.

Theorem. *All trajectories stabilize if and only if trajectories of all x_1, x_2, \dots, x_n stabilize. In other words $K = F[x]$ if and only if $\{x_1, x_2, \dots, x_n\} \subset K$.*

Proof. If all trajectories stabilize then, in particular, trajectories of x_1, x_2, \dots, x_n stabilize. If trajectories of x_1, x_2, \dots, x_n stabilize then due to finiteness of this system one can find natural m such that all x_1, x_2, \dots, x_n are in K_m . For any polynomial f of degree l due to Proposition 2 one has $f \in K_{ml}$ that is trajectory of f stabilizes. This the end of the proof.

Let id stand for the identity map $\text{id} : F[x] \rightarrow F[x]$, $\text{id}(x) = x$. Termination of trajectories of all x_1, x_2, \dots, x_n means that there exists m such that

$$D^m(\text{id}) = \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k} = 0$$

, where $D^m(id)$ stands for $(D^m(x_1), D^m(x_2), \dots, D^m(x_n))$. But due to [1,2] it implies that the inverse map φ^{-1} is also a polynomial map. Therefore one has the following Corollary.

Corollary. *If all trajectories stabilize then φ^{-1} is also a polynomial map.*

If one knows that φ^{-1} is also a polynomial map can he be sure that all trajectories stabilize? It is also an open problem.

Though termination of trajectories of all elements $F[x]$ is reduced to the termination of trajectories of the finite number elements x_1, x_2, \dots, x_n checking termination of trajectories of these elements is not an easy task. But termination of trajectories of all x_1, x_2, \dots, x_n can be written in the following equivalent form: There exists m such

$$(\partial\varphi)(x)(mE + \sum_{k=2}^m (-1)^{k-1} \binom{m}{k} (\partial\varphi)|_{\varphi(x)} (\partial\varphi)|_{\varphi^{\circ 2}(x)} \dots (\partial\varphi)|_{\varphi^{\circ(k-1)}(x)}) = E \quad (1)$$

, where $\partial_k = \frac{\partial}{\partial x_k}$, $(\partial\varphi(x))_j^i = \partial_i \varphi_j(x)$, $(\partial\varphi)|_{\varphi^{\circ(k-1)}(x)} = (\partial\varphi)(\varphi^{\circ(k-1)}(x))$, E is n order identity matrix. By taking determinant of both sides (1) one can see that for termination of every trajectory the condition $\det((\partial\varphi(x))_j^i) = 1$ is a necessary condition. Checking of this necessary condition can be considered as an easy thing. Therefore one can ask the following natural question.

Question. *Does $\det((\partial\varphi(x))_j^i) = 1$ imply that every trajectory stabilizes?*

In common case the answer to this question seems to be negative. But we have no corresponding example.

Here we would like to offer an approach to construct such an example. It has been noted that equality (1) guarantees that the inverse map φ^{-1} is also a polynomial map. Therefore the above theorem is closely related to the well known unsolved Jacobian Conjecture [4]. It is known ([5]) that to prove the Jacobian Conjecture it is enough to prove it for polynomials of the form

$$\varphi(x) = x + ((xA_1)^{*3}, (xA_2)^{*3}, \dots, (xA_n)^{*3}) = x + (xA)^{*3}$$

, where x^{*3} stands for $(x_1^3, x_2^3, \dots, x_n^3)$, $A_i = (a_i^1, a_i^2, \dots, a_i^n)$ is a column vector consisting of numbers, $A = (A_1, A_2, \dots, A_n)$ is the corresponding matrix. Therefore one can look for a possible counterexample to our question among such polynomial maps. For such polynomial $\varphi(x)$ the equality $\det((\partial\varphi(x))_j^i) = 1$ is equivalent to the system of equalities

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (xA_{i_1})^2 (xA_{i_2})^2 \dots (xA_{i_k})^2 \det((a_{i_t}^{i_s})_{s,t=1,2,\dots,k}) = 0$$

, where $k = 1, 2, \dots, n$. In particular $\det A = 0$. Let us consider only $rk(A) = n - 1$ case. In this case one can assume that $A_n = \lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_{n-1} A_{n-1}$ and therefore $y_n = \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_{n-1} y_{n-1}$, where $y_i = xA_i$. For given n one can first consider ideal I generated by polynomials

$$y_n - \sum_{i=1}^{n-1} \lambda_i y_i, a_n^k - \sum_{i=1}^{n-1} \lambda_i a_i^k, \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (y_{i_1})^2 (y_{i_2})^2 \dots (y_{i_k})^2 \det((a_{i_t}^{i_s})_{s,t=1,2,\dots,k})$$

, where $k = 1, 2, \dots, n$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n-1})$, in $F[y, \lambda, A]$ then, due to $D^1(x) = \Phi_1(y) = -y^{*3}$ and $\Phi_{m+1}(y) = \Phi_m(y) - \Phi_m(y + y^{*3}A)$, check if all components of $\Phi_m(y)$ are in I . If "Yes" consider next n , if, for example, for all $m = 1, 2, \dots, 100$ not all components of $\Phi_m(y)$ are in I then he should consider the corresponding $\varphi(x)$ as a possible counterexample to our question. Of course after that the obtained $\varphi(x)$ should be checked whether it is a counterexample or not.

References

- [1] Ural Bekbaev, *An inversion formula for multivariate power series*, arXiv:1203.3834v1 [math. AG] 17 Mar. 2012.
- [2] Wenhua Zhao, *Exponential formulas for the Jacobians and Jacobian matrices of analytic maps*, *Journal of Pure and Applied Algebra*, 166(2002) 321-336.
- [3] Yu.I.Lyubich, *The cohomological equations in nonsmooth categories*, arXiv:1211.0229v1 [math. FA] 1 Nov.2012.
- [4] H.Bass, E.Connell, D.Wtight, *The Jacobian conjecture, reduction of degree and formal expansion of the inverse*, *Bull.Amer.Soc.*, 7(1982) 287-330.
- [5] Druzkowski L.M., *An effective approach to Keller's Jacobian Conjecture*, *Math. Ann.*, 264(1983),303-313.