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Classification of $\xi^{(s)}$-Quadratic Stochastic Operators on 2D simplex

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Abstract. A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of differing species in biology. Some QSO has been studied by Lotka and Volterra. The general problem in the nonlinear operator theory is to study the behavior of operators. This problem was not fully finished even for the quadratic stochastic operators. To study this problem it was investigated several classes of such QSO. In this paper we study $\xi^{(s)}$-QSO class of operators. We study such kind of operators on 2D simplex. We first classify these $\xi^{(s)}$-QSO into 20 classes. Further, we investigate the dynamics of one class of such operators.

1. Introduction

A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of differing species in biology (see [7, 6]). Note that, in general, such operators are defined on $n-1$-dimensional simplex of $\mathbb{R}^n$. Some QSO has been studied by Lotka and Volterra [11]. Many natural phenomena are being modeled by Lotka-Volterra (in short LV) quadratic systems. The investigation of dynamical properties and modeling in various fields running from economy to population dynamics have been using the LV system to be the source of analysis. The fascinating applications of quadratic stochastic operators to population genetics were given in the book [7]. It describes a distribution of species in the next generation if the distribution of these species of the current generation is given. In [1], it was given along self-contained exposition of the recent achievements and open problems in the theory of quadratic stochastic operators. A general problem in the nonlinear operator theory is to study the behavior of operators. This problem was not fully finished even for the quadratic stochastic operators (for the simplest nonlinear operators). The asymptotical behavior of the QSO (even) on small dimensional simplex is complicated (see [5]). In order to solve this problem, many researchers always introduced a certain class of quadratic operators and studied their behaviors: Volterra QSO [2, 10, 12], ℓ–Volterra QSO [9], Non-Volterra QSO, Strictly non-Volterra QSO [13], F-QSO, Separable QSO, Quadratic doubly stochastic operators and so on. For more information, one may refer to [1]. However, all these class of operators together would not cover all QSO. Therefore, there are many classes of QSO which were not studied yet.

In this paper we are going to study $\xi^{(s)}$-QSO class of operators which has been introduced in [8]. This class of operators depend on a partition of the index set. In case of two dimensional simplex, the index set has five possible partitions. In [8] it has been investigated $\xi^{(s)}$-QSO related to the point partition. In the present paper, we are going to describe and classify such operators generated by other three partitions. Further, we also investigate the dynamics of one class of such operators.

2. Preliminaries

Recall that a quadratic stochastic operator (QSO) is a mapping of the simplex

$$S^{n-1} = \{x = (x_1, x_2, ..., x_m)\} \in \mathbb{R}^m : \; x_i \geq 0, \;\; \sum_{i=1}^{m} x_i = 1 \tag{2.1}$$

into itself, of the form

$$V : x_k = \sum_{i,j=1}^{m} P_{i,j,k} x_i x_j, \tag{2.2}$$

where $P_{i,j,k}$ are coefficient of heredity, which satisfy the following conditions

$$P_{i,j,k} \geq 0, \quad P_{i,j,k} = P_{j,i,k}, \quad \sum_{i,j=1}^{m} P_{i,j,k} = 1. \tag{2.3}$$

Thus, each quadratic stochastic operator V can be uniquely defined by a cubic matrix $P = \left(P_{i,j,k}\right)_{i,j,k=1}^{m}$ with conditions (2.3). For a given $x^{(0)} \in S^{m-1}$ the trajectory $\{x^{(n)}\}$, $n = 0, 1, 2, ...$ of $x^{(0)}$ under the action of QSO V is defined by $x^{(n+1)} = V(x^{(n)})$, where $n = 0, 1, 2, ...$ By $W(x^{(0)})$ we denote the set of the limiting points of the trajectory. Since $x^{(n)} \subset S^{m-1}$ and S^{m-1} is compact, it follows that $W(x^{(0)}) \neq \emptyset$. Obviously, if $W(x^{(0)})$ consists of a single point, then the trajectory converges, and $W(x^{(0)})$ is fixed point of QSO.

In order to introduce a new class QSO, we need some auxiliary notations.

Note that each element $x \in S^{m-1}$ is a probability distribution on $I = \{1, ..., m\}$. Therefore, take two $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ from S^{m-1}. We say that x *is equivalent to* y if $x_k = 0 \Leftrightarrow y_k = 0$.

Let us denote $supp(x) = \{i : x_i \neq 0\}$. We say that x *is singular to* y ($x \perp y$) if $supp(x) \cap supp(x) = \phi$. Note that if $x, y \in S^{m-1}$ then $x \perp y$ if and only if $(x, y) = 0$, here $(\cdot, \cdot)$ stands usual inner product in $\mathbb{R}^m$.

By $\mathbf{P}$ we denote the set of ordered pairs of I, i.e, $\mathbf{P} = \{(i, j) : \;\; i, j \in I, \; i < j\}$. Let $\xi = \{A_i\}_{i=1}^{N}$ be a partition of $\mathbf{P}$, i.e. $A_i \bigcap A_j = \emptyset, \bigcup_{i=1}^{N} A_i = \mathbf{P}$.

Definition 2.1. A quadratic stochastic operator V given by (2.2),(2.3) is called $\xi^{(s)}$-*QSO* if the following conditions are satisfied:

(i) for each A_k and every $(i, j), (u, v) \in A_k$ one has

$$(P_{i,j,1}, P_{i,j,2}, ..., P_{i,j,m}) \sim (P_{u,v,1}, P_{u,v,2}, ..., P_{u,v,m});$$

(ii) for every $(i, j) \in A_k$, $(u, v) \in A_l$, $(k \neq l)$ one has

$$(P_{i,j,1}, P_{i,j,2}, ..., P_{i,j,m}) \perp (P_{u,v,1}, P_{u,v,2}, ..., P_{u,v,m});$$

(iii) for every $i, j \in I$ one has

$$(P_{i,j,1}, P_{i,j,2}, ..., P_{i,j,m}) \perp (P_{u,v,1}, P_{u,v,2}, ..., P_{u,v,m}).$$

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3. Classification of $\xi^{(s)}$-QSO on 2Dsimplex

In this section we are going to study $\xi^{(s)}$-QSO in two dimensional simplex, i.e. $m = 3$. In this case, we have the following possible partitions

$$\begin{aligned}
&\xi_1 = \{\{(1,2)\},\{(1,3)\},\{(2,3)\}\},\ |\xi_1| = 3,\\
&\xi_2 = \{\{(1,2)\},\{(1,3),(2,3)\}\},\ |\xi_2| = 2,\\
&\xi_3 = \{\{(1,3)\},\{(1,2),(2,3)\}\},\ |\xi_3| = 2,\\
&\xi_4 = \{\{(2,3)\},\{(1,2),(1,3)\}\},\ |\xi_4| = 2,\\
&\xi_5 = \{\{(1,2),(1,3),(2,3)\},\ |\xi_5| = 1.
\end{aligned}$$

In [8] it has been investigated $\xi^{(s)}$-QSO related to the partition ξ_1. In the present paper we are going to described all $\xi^{(s)}$-QSO related to the partitions ξ_2, ξ_3 and ξ_4. One can calculate that there are 108 parmetrical $\xi^{(s)}$-QSOs related to the said partitions.

Let us recall that two QSO V_1, V_2 are *conjugate*, if one can find a permutation π of $\{1,2,3\}$ such that one has $\pi V_1 \pi^{-1} = V_2$. Note that the $\xi^{(s)}$-QSO related partitions ξ_3 and ξ_4 are conjugate to $\xi^{(s)}$-QSO generated by the partition ξ_2. More exactly, these operators are conjugate by means of the permutation $\pi_1 = \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$. Therefore, below we are listing $\xi^{(s)}$-QSO related to the partition ξ_2. Let us introduce some notations

$$\begin{aligned}
&h(x) = x(1-x), \quad b = 1-a,\\
&g(x,y,z;\lambda;\alpha,\beta,\gamma) = x^2 + 2\lambda x^\alpha y^\beta z^\gamma, \quad \alpha,\beta,\gamma = \{0,1\}.
\end{aligned}$$

Using these notation, we are ready to list the $\xi^{(s)}$-QSOs which are the following ones:

$$V_1 : \begin{cases} x' = x^2 + 2ah(x) \\ y' = y^2 + 2bh(x) \\ z' = g(z,x,y;1;1,0,1) \end{cases} \quad V_2 : \begin{cases} x' = y^2 + 2ah(x) \\ y' = x^2 + 2bh(x) \\ z' = g(z,x,y;1;1,0,1) \end{cases} \quad V_3 : \begin{cases} x' = z^2 + 2ah(x) \\ y' = y^2 + 2bh(x) \\ z' = g(x,y,z;1;0,1,1) \end{cases}$$

$$V_4 : \begin{cases} x' = x^2 + 2ah(x) \\ y' = z^2 + 2bh(x) \\ z' = g(y,x,z;1;1,0,1) \end{cases} \quad V_5 : \begin{cases} x' = y^2 + 2ah(x) \\ y' = z^2 + 2bh(x) \\ z' = g(x,y,z;1;0,1,1) \end{cases} \quad V_6 : \begin{cases} x' = z^2 + 2ah(x) \\ y' = x^2 + 2bh(x) \\ z' = g(y,x,z;1;1,0,1) \end{cases}$$

$$V_7 : \begin{cases} x' = g(x,y,z;1;0,1,1) \\ y' = y^2 + 2ah(x) \\ z' = z^2 + 2bh(x) \end{cases} \quad V_8 : \begin{cases} x' = g(y,x,z;1;1,0,1) \\ y' = x^2 + 2ah(x) \\ z' = z^2 + 2bh(x) \end{cases} \quad V_9 : \begin{cases} x' = g(z,x,y;1;1,0,1) \\ y' = y^2 + 2ah(x) \\ z' = x^2 + 2bh(x) \end{cases}$$

$$V_{10} : \begin{cases} x' = g(x,y,z;1;0,1,1) \\ y' = z^2 + 2ah(x) \\ z' = y^2 + 2bh(x) \end{cases} \quad V_{11} : \begin{cases} x' = g(y,x,z;1;1,0,1) \\ y' = z^2 + 2ah(x) \\ z' = x^2 + 2bh(x) \end{cases} \quad V_{12} : \begin{cases} x' = g(z,x,y;1;1,0,1) \\ y' = x^2 + 2ah(x) \\ z' = y^2 + 2bh(x) \end{cases}$$

$$V_{13} : \begin{cases} x' = x^2 + 2ah(x) \\ y' = g(y,x,z;1;1,0,1) \\ z' = z^2 + 2bh(x) \end{cases} \quad V_{14} : \begin{cases} x' = y^2 + 2ah(x) \\ y' = g(x,y,z;1;0,1,1) \\ z' = z^2 + 2bh(x) \end{cases} \quad V_{15} : \begin{cases} x' = z^2 + 2ah(x) \\ y' = g(y,x,z;1;1,0,1) \\ z' = x^2 + 2bh(x) \end{cases}$$

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$$V_{16}: \begin{cases} x' = x^2 + 2ah(x) \\ y' = g(z,x,y;1;1,0,1) \\ z' = y^2 + 2bh(x) \end{cases} \quad V_{17}: \begin{cases} x' = y^2 + 2ah(x) \\ y' = g(z,x,y;1;1,0,1) \\ z' = x^2 + 2bh(x) \end{cases} \quad V_{18}: \begin{cases} x' = z^2 + 2ah(x) \\ y' = g(x,y,z;1;0,1,1) \\ z' = y^2 + 2bh(x) \end{cases}$$

$$V_{19}: \begin{cases} x' = g(x,y,z;a;0,1,1) \\ y' = g(y,x,z;b;1,0,1) \\ z' = z^2 + 2h(x) \end{cases} \quad V_{20}: \begin{cases} x' = g(y,x,z;a;1,0,1) \\ y' = g(x,y,z;b;0,1,1) \\ z' = z^2 + 2h(x) \end{cases} \quad V_{21}: \begin{cases} x' = g(z,x,y;a;1,0,1) \\ y' = g(y,x,z;b;1,0,1) \\ z' = x^2 + 2h(x) \end{cases}$$

$$V_{22}: \begin{cases} x' = g(x,y,z;a;0,1,1) \\ y' = g(z,x,y;b;1,0,1) \\ z' = y^2 + 2h(x) \end{cases} \quad V_{23}: \begin{cases} x' = g(y,x,z;a;1,0,1) \\ y' = g(z,x,y;b;1,0,1) \\ z' = x^2 + 2h(x) \end{cases} \quad V_{24}: \begin{cases} x' = g(z,x,y;a;1,0,1) \\ y' = g(x,y,z;b;0,1,1) \\ z' = y^2 + 2h(x) \end{cases}$$

$$V_{25}: \begin{cases} x' = x^2 + 2h(x) \\ y' = g(y,x,z;a;1,0,1) \\ z' = g(z,x,y;b;1,0,1) \end{cases} \quad V_{26}: \begin{cases} x' = y^2 + 2h(x) \\ y' = g(x,y,z;a;0,1,1) \\ z' = g(z,x,y;b;1,0,1) \end{cases} \quad V_{27}: \begin{cases} x' = z^2 + 2h(x) \\ y' = g(y,x,z;a;1,0,1) \\ z' = g(x,y,z;b;0,1,1) \end{cases}$$

$$V_{28}: \begin{cases} x' = x^2 + 2h(x) \\ y' = g(z,x,y;a;1,0,1) \\ z' = g(y,x,z;b;1,0,1) \end{cases} \quad V_{29}: \begin{cases} x' = y^2 + 2h(x) \\ y' = g(z,x,y;a;1,0,1) \\ z' = g(x,y,z;b;0,1,1) \end{cases} \quad V_{30}: \begin{cases} x' = z^2 + 2h(x) \\ y' = g(x,y,z;a;0,1,1) \\ z' = g(y,x,z;b;1,0,1) \end{cases}$$

$$V_{31}: \begin{cases} x' = g(x,y,z;a;0,1,1) \\ y' = y^2 + 2h(x) \\ z' = g(z,x,y;b;1,0,1) \end{cases} \quad V_{32}: \begin{cases} x' = g(y,x,z;a;1,0,1) \\ y' = x^2 + 2h(x) \\ z' = g(z,x,y;b;1,0,1) \end{cases} \quad V_{33}: \begin{cases} x' = g(z,x,y;a;1,0,1) \\ y' = y^2 + 2h(x) \\ z' = g(x,y,z;b;0,1,1) \end{cases}$$

$$V_{34}: \begin{cases} x' = g(x,y,z;a;0,1,1) \\ y' = z^2 + 2h(x) \\ z' = g(y,x,z;b;1,0,1) \end{cases} \quad V_{35}: \begin{cases} x' = g(y,x,z;a;1,0,1) \\ y' = z^2 + 2h(x) \\ z' = g(x,y,z;b;0,1,1) \end{cases} \quad V_{36}: \begin{cases} x' = g(z,x,y;a;1,0,1) \\ y' = x^2 + 2h(x) \\ z' = g(y,x,z;b;1,0,1) \end{cases}$$

Theorem 3.1 *All $\xi^{(s)}$-QSO defined with respect to the partition ξ_2 on 2D simplex are classified into 20 non conjugate classes and they are listed below*

$$\begin{array}{llll} K_1 = \{V_1, V_{13}\}, & K_2 = \{V_2, V_{15}\}, & K_3 = \{V_3, V_{14}\}, & K_4 = \{V_4, V_{16}\} \\ K_5 = \{V_5, V_{18}\}, & K_6 = \{V_6, V_{17}\}, & K_7 = \{V_7\}, & K_8 = \{V_8, V_9\} \\ K_9 = \{V_{10}\}, & K_{10} = \{V_{11}, \quad V_{12}\}, & K_{11} = \{V_{19}, V_{31}\}, & K_{12} = \{V_{20}, V_{33}\} \\ K_{13} = \{V_{21}, V_{32}\}, & K_{14} = \{V_{22}, V_{34}\}, & K_{15} = \{V_{23}, V_{36}\}, & K_{16} = \{V_{24}, V_{35}\} \\ K_{17} = \{V_{25}\}, & K_{18} = \{V_{26}, V_{27}\}, & K_{19} = \{V_{28}\}, & K_{20} = \{V_{29}, V_{30}\} \end{array}$$

Proof. We are going to classify the above given operators with respect to permutation $\pi_3 = \begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}$. One can see that

$$\begin{aligned} \pi_3 \circ V_1 \circ \pi_3^{-1} &= \pi_3(x^2 + 2ax(1-x), z^2 + 2(1-a)x(1-x), y(1-x+z)) \\ &= (x^2 + 2ax(1-x), y(1-x+z), z^2 + 2(1-a)x(1-x)) \\ &= V_{13} \end{aligned}$$

From $\pi_3 = \pi_3^{-1}$, we infer that V_1, V_{13} are in the same class, which is denoted by K_1. The other operators can be classified by the same way, so one gets the above given list. This completes the proof.

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4. Dynamics of $\xi^{(s)}$-QSO from the class K_{17}

In this section, for the sake of simplicity, we are going to study dynamics of $\xi^{(s)}$ -QSO taken from the class K_{17}. Namely, we consider V_{25} which is given by

$$V_{25} : \begin{cases} x' = x^2 + 2x(1-x) \\ y' = z^2 + 2ayx \\ z' = y^2 + 2(1-a)yz \end{cases} \tag{4.1}$$

where $0 < a < 1$. In the sequel, by e_1, e_2, e_3 we denote the vertices of the simplex S^2. i.e. $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.

In what follows, for the sake of simplicity, by $\mathbf{V}_a$ we denote V_{25}.

Theorem 4.1 *Let $\mathbf{V}_a : S^2 \to S^2$ be a $\xi^{(s)}$-QSO given by* (4.1) *Then the following statements hold true:*

(i) if $a \neq \frac{1}{2}$ then

1. $Fix(\mathbf{V}_a) = \left\{ e_1, \left(0, \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}, \frac{-1-2a+\sqrt{4+(2a-1)^2}}{2(1-2a)} \right) \right\}$

2. $Per((\mathbf{V}_a) = \{e_2, e_3\}$

(ii) if $a = \frac{1}{2}$ then

1. $Fix(\mathbf{V}_a) = \left\{ \left(0, \frac{1}{2}, \frac{1}{2} \right) \right\}.$

2. $Per(\mathbf{V}_a) = \{x = 0\}$

(iii) Let $x^{(0)} = (x, y, z) \in S^2$ be an initial point with $x = 0$, then we have three cases:

1. if $0 < a < \frac{1}{2}$, then $W(x^{(0)}) = \{e_3\}$

2. if $\frac{1}{2} < a < 1$, then $W(x^{(0)}) = \{e_2\}$

3. if $a = \frac{1}{2} <$, then $W(x^{(0)}) = \{1 - x^{(0)}\}$

(iv) Let $x^{(0)} = (x, y, z) \in S^2$ be an initial point with $x \neq 0$, then $W(x^{(0)}) = \{e_1\}$

Proof. (i) Assume that $a \neq 1/2$. To find fixed points of $\mathbf{V}_a$, we need to solve the equation, $Vx = x$, namely

$$\begin{aligned} &x^2 + 2x(1-x) = x \\ &z^2 + 2ayz = y \\ &y^2 + 2(1-a)yz = z. \end{aligned}$$

Solutions of the first equation $x^2 + 2x(1-x) = x$ are $x = 0, x = 1$. One can easily check that if $x = 1$, we have $z = y = 0$, and if $x = 0$ then $z + y = 1$. So, we have to solve the

$$(1-y)^2 + 2ay(1-y) = y. \tag{4.2}$$

One can find that solutions of (4.2) are $y = \frac{3-2a\pm\sqrt{4+(2a-1)^2}}{2(1-2a)}$ when $a \neq \frac{1}{2}$. We can verify that the only solution $y = \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}$ is positive. Therefore, we have $z = \frac{-1+2a+\sqrt{4+(2a-1)^2}}{2(1-2a)}$. Hence,

$$Fix(\mathbf{V}_a) = \left\{e_1, \left(0, \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}, \frac{-1-2a+\sqrt{4+(2a-1)^2}}{2(1-2a)}\right)\right\},\ a \neq \frac{1}{2}.$$

In order to find 2-periodic points of $\mathbf{V}_a$, we have to solve the following equation: $\mathbf{V}_a^2(x) = x$. Let us consider first component of the last equation:

$$(x^2 + 2x(1-x))^2 + 2(x^2 + 2x(1-x))(1-(x^2+2x(1-x)) = x$$

One can find solutions of this equation are $\left\{0, 1, \frac{3\pm\sqrt{3}i}{2}\right\}$. So, a possibility of appearance of the periodic point is $x = 0$ or $x = 1$. Let us find the corresponding value of y when $x = 0$. To do that, we have to solve the following one

$$(y^2 + 2(1-a)y(1-y))^2 + 2a(1-y)^2 + 2ay(1-y)(y^2 + 2(1-a)y(1-y)) = y \tag{4.3}$$

This equation has solutions $\left\{0, 1, \frac{3-2a\pm\sqrt{4+(2a-1)^2}}{2(1-2a)}\right\}$. So, the periodic points are $(0,0,1), (0,1,0)$. Therefore, we have $Per(\mathbf{V}_a) = \{e_2, e_3\}$, when $a \neq \frac{1}{2}$.

(ii) Now, let us consider the case when $a = \frac{1}{2}$. Then (4.2) has a solution $y = \frac{1}{2}$. This means that $z = \frac{1}{2}$. Therefore, one has

$$Fix(\mathbf{V}_a) = \left\{\left(0, \frac{1}{2}, \frac{1}{2}\right)\right\},\ a = \frac{1}{2}.$$

It is clear that the equation (4.3) has infinitely many solutions. Therefore, all points in the line $\{x = 0\}$ are periodic.

(iii) First of all let us consider the case when $a \neq \frac{1}{2}$. One can easily check that the line $\{x = 0\}$ is invariant w.r.t. $\mathbf{V_a}$. We want to study the behavior of the operator over this line. So, assume that $x = 0$. Then $\mathbf{V_a}$ takes the following form:

$$V_a : \begin{cases} x' = 0 \\ y' = z^2 + 2ayz \\ z' = y^2 + 2(1-a)yz \end{cases}$$

Let us consider the function $f_a(y) = (1-y)^2 + 2ay(1-y)$. One can check easily that $f_a(y)$ is decreasing on $[0,1]$, when $a \neq \frac{1}{2}$. We immediately find that $f_a(f_a(y)) > y$, when $y > \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}$, and $f_a(f_a(y)) < y$, when $y < \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}$. Therefore, we have

(a) for any $n \in N$ one has $f_a^{(2n+2)}(y) \geq f_a^{(2n)}(y)$, when $y > \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}$. So, the sequence $f_a^{(2n)}(y)$ is increasing and bounded, moreover $\{f_a^{(2n)}(y)\}$ converges to y^*. One can see that y^* is a fixed point of $f_a(f_a(y))$. The only possibility is $y^* = 1$. Hence, from $f_a^{(2n)}(y) \to 1$, one concludes that $z^{(2n)} \to 0$. Thus, we get $W(x^{(0)}) = (0,1,0)$ if $a \neq \frac{1}{2}$ and $y > \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}$.

(b) Similarly, one has $f_a^{(2n+2)}(y) \leq f_a^{(2n)}(y)$, for any $n \in N$, when $y < \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}$. So, the sequence $f_a^{(2n)}(y)$ is decreasing and bounded, moreover $\{f_a^{(2n)}(y)\}$ converges to y^*. One can see that y^* is a fixed point of $f_a(f_a(y))$. The only possibility is $y^* = 0$. Therefore, one finds $z^{(2n)} \to 1$. So, we get $W(x^{(0)}) = (0, 0, 1)$, if $a \neq \frac{1}{2}$ and $y < \frac{3-2a-\sqrt{4+(2a-1)^2}}{2(1-2a)}$.

Now assume that $a = \frac{1}{2}$, then $\mathbf{V_a}$ has the following form.

$$V_{\frac{1}{2}} : \begin{cases} x' = 0 \\ y' = z^2 + yz \\ z' = y^2 + yz \end{cases}$$

Obviously, one can see that $y' = 1 - y$. This means that all points in the line $x = 0$ are periodic points. So, $W(x^{(0)}) = \{1 - x^{(0)}\}$.

(iv) Assume that $x \neq 0$. Now, let us consider the function $f(x) = x^2 + 2x(1 - x)$. One can show that $f^{(n+1)}(x) \geq f^{(n)}(x)$, for any $n \in N$. We find that if $x \geq \frac{1}{2}$, then $x' \geq \frac{1}{2}$. That means the region when $x \geq \frac{1}{2}$ is invariant. Let us study the behavior of $\mathbf{V_a}$ in the $int(S^2)$. In order to do that we shall divide $int(S^2)$ into two subregions. Namely,

$$S_1 = \left\{(x, y, z) : x \geq \frac{1}{2}\right\}, \quad S_2 = \left\{(x, y, z) : x < \frac{1}{2}\right\}.$$

From the above observation, one concludes S_1 is invariant w.r.t. $\mathbf{V_a}$. Now, if $x < \frac{1}{2}$, then there exists n such that $x^{(n)} \geq \frac{1}{2}$. Since $x^{(n+1)} = (x^{(n)})^2 + 2x^{(n)}(1 - x^{(n)}) \geq x^{(n)}, x \in [0, 1]$, and $x^{(n)}$ converge to 1. This means for us it is enough to study the dynamics of $\mathbf{V_a}$ on S_1. Now we are going to show that $W(x^{(0)}) = e_1$, when $x^{(0)} \in S_1$.

Now, we want to show that the Lyapunov function $\varphi(x, y) = x - y$ is increasing when $0 < a < \frac{1}{2}$, $x \geq \frac{1}{2}$. It is sufficient to prove that $x' - y' \geq x - y$, i.e.

$$2x - x^2 - (1 - x - y)^2 - 2ay(1 - x - y) - x + y \geq 0,$$

when $x \geq \frac{1}{2}$. So, we need to prove that the minimum value of the expression in the LHS is greater than or equal zero.

One can calculate that only critical point of this function is $\left(\frac{2a^2+2a}{2a^2+4a-2}, f\left(\frac{2a^2+2a}{2a^2+4a-2}\right)\right)$. We easily can check this point doesn't belong to the simplex, so the minimum value will be in the boundary of the region $\Omega = \left\{(x, y) : x \geq \frac{1}{2}\right\}$. We can find the minimum value is 0, and it is clear in figure 1.

So, the Lyapunov function $\varphi(x, y) = x - y$ is increasing, this means that $x^n - y^n \to \alpha$. Since the sequence $\{x^{(n)}\}$ converges, therefore, one finds that $y^{(n)}$ is also converging. So, $(x^n, y^n, z^n) \to (x^*, y^*, z^*)$ one can see that (x^*, y^*, z^*) is a fixed point, and the only possibility is $(1, 0, 0)$. So, $W(x^{(0)} = e_1$. By the same way, we can also show that the Lyapunov function $\varphi(x, z) = x - z$ is increasing, when $\frac{1}{2} \leq a \leq 1$ and $x \geq \frac{1}{2}$. Consequently, we obtain $W(x^{(0)}) = e_1$.

This completes the proof.

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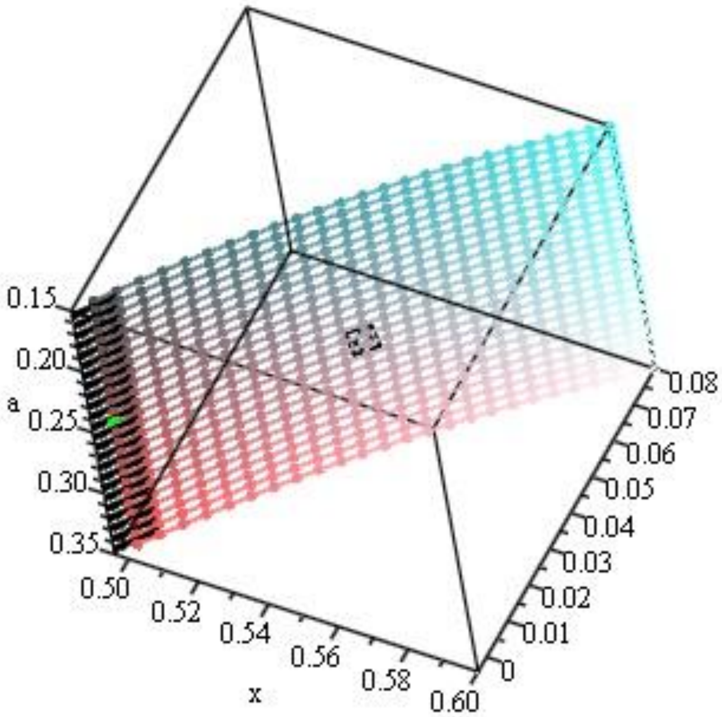

Figure 1.

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