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To cite this article: K Moaddy *et al* 2013 *J. Phys.: Conf. Ser.* **423** 012024

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Parametric Control on Fractional-Order Response for Lü Chaotic System

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Abstract. This paper discusses the influence of the fractional order parameter on conventional chaotic systems. These fractional-order parameters increase the system degree of freedom allowing it to enter new domains and thus it can be used as a control for such dynamical systems. This paper investigates the behaviour of the equally-fractional-order Lü chaotic system when changing the fractional-order parameter and determines the fractional-order ranges for chaotic behaviour. Five different parameter values and six fractional-order cases are discussed through this paper. Unlike the conventional parameters, as the fractional-order increases the system response begins with stability, passing by chaotic behaviour then reaches periodic response. As the system parameter α increases, a shift in the fractional order is required to maintain chaotic response. Therefore, the range of chaotic response can be expanded or minimized by controlling the fractional-order parameter. The non-standard finite difference method is used to solve the fractional-order Lü chaotic system numerically to validate these responses.

1. Introduction

Recently, many natural systems which dynamics are complex can be modelled as chaotic dynamical systems which are extremely sensitive to the initial conditions as weather. The fundamentals of the chaotic system and some of its characteristics in integer and fractional-order domain were discussed in [1]. In addition, the impact of the chaotic systems become seriously once many chaotic responses have been implemented practically and especially on the transistor-level [2,3] which is suitable in the VLSI technologies. Many chaotic systems have been introduced last few decades as Lorenz, Rössler, and Lu chaotic systems with many applications such as in control, communication, and encryption.

Integer calculus is considered a very narrow subset from fractional-calculus, which made it a must for dynamical systems to be generally re-discussed based on fractional-order differentiation due to the extra degrees of freedom, flexibility, and better characterization it offers. During the last few decades, fractional calculus has become a powerful tool to describe the dynamics of complex systems which appear frequently in several branches of science and engineering. This generalization showed many new fundamentals which exist in the fractional-order domain and disappear in the integer-order case. One of the main goals for many applications is to discuss the effect of the fractional-order parameter on the system response where the results can be used in better description and control of these systems

in many applications. The characterization of real dynamical systems has proved to be superior to the fundamentals of traditional calculus; this opened the door for the use of fractional calculus in numerous applications in the field of viscoelasticity, robotics, feedback amplifiers, electrical circuits, control theory, electro analytical chemistry, fractional multi-poles, chemistry and biological sciences [4-14]. For example the stability analysis of fractional-order systems has been discussed in [6], the fractional order PID controller became ubiquitous in industry as shown in [9 - 11], The behaviour of the fractional-order chaotic systems has been discussed in [13], and the modelling of some complex dynamics in biological tissues using fractional calculus in [14, 15].

One of the main differences between the ordinary differential equation systems (integer order) and the corresponding fractional-order-differential equation systems is that, the definition of the fractional order system depends on the whole history of the responses instead of the finite memory in the integer case. Chaotic systems have a profound effect on their numerical solutions and are highly sensitive to time step sizes. Thus it is beneficial to find a reliable analytical tool to test its long-term accuracy and efficiency. Also hyper-chaotic systems have more complex dynamical behaviours because they are defined as chaotic systems with two positive Lyapunov exponents. In addition to the previous, since its discovery, chaotic synchronization has been applied in many different fields, including biological and physical systems, structural engineering, and ecological models [16].

A great deal of effort has been directed recently in an attempt to find robust and stable numerical and analytical methods for solving fractional equations. Such as the fractional difference method [17], the Adomian decomposition method [18, 19], the variational iteration method [20] and the Adams-Bashforth-Moulton method [21-22].

In this paper, the non-standard finite difference method (NSFD) [23-25] is implemented to give numerical solutions for various types of differential equations of integer and fractional orders [26-29]. NSFD has developed as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, biological, and chaotic systems. The technique has many advantages over the classical techniques, and provides an efficient numerical solution. The main aim of this paper is to study the proper fractional-order range in which the Lu system exhibits chaotic behaviour. Several cases are investigated for different orders and changing only a single system parameter. Stable, periodic and chaotic responses are shown for each system parameter but with different fractional order ranges.

2. Preliminaries and Notations

In this section we give some basic definitions and properties of the fractional calculus theory and non-standard discretization which will be used further in this paper.

2.1. Grünwald-Letnikov approximation.

We will begin with the single fractional differential equation, [26]

$$D^\alpha x(t) = f(t, x(t)), T \geq t \geq 0, x(t_0) = x_0, \quad (1)$$

where $\alpha > 0$ and D^α denotes the fractional derivative, defined by

$$D^\alpha x(t) = J^{n-\alpha} D^n x(t), \quad (2)$$

where $n - 1 < \alpha \leq n, n \in \mathbb{N}$ and J^n in the n^{th} -order Riemann – Liouville integral operator defined as

$$J^n x(t) = \frac{1}{\Gamma(n)} \int_0^t (t - \tau)^{n-1} x(\tau) d\tau. \quad (3)$$

To apply Mickens scheme, we have chosen the Grünwald-Letnikov method of approximation for the one-dimensional fractional derivative as follows:

$$\lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^N (-1)^j \binom{\alpha}{j} x(t - jh), \quad (4)$$

where $N = t/h$ and t denotes the integer part of t and h is the step size. Therefore, equation (4) is discretized as follow

$$\sum_{j=0}^{n+1} c_j^\alpha x(t - jh) = f(t_n, x(t_n)), n = 1, 2, 3, \dots, \quad (5)$$

where $tn = nh$ and c_j^α are the Grunwald-Letnikov coefficients defined as

$$c_j^\alpha = \left(1 - \frac{1 + \alpha}{j}\right) c_{j-1}^\alpha, \text{ and } c_0^\alpha = h^{-\alpha}, j = 1, 2, 3, \dots. \quad (6)$$

2.2. Non-standard discretization

We seek to obtain the NSFD solution for a system of fractional differential equations of the form

$$D^{\alpha_1}(x) = f_1(x, y, z), \quad (7)$$

$$D^{\alpha_2}(y) = f_2(x, y, z), \quad (8)$$

$$D^{\alpha_3}(z) = f_3(x, y, z), \quad (9)$$

where $0 < \alpha \leq 1$, and initial conditions x_0, y_0, z_0 .

Applying the NSFD scheme by replacing the step size h by a function $\varphi(h)$ and using the Grünwald-Letnikov discretization method, yields the following relations

$$x_{n+1} = \frac{\left(-\sum_{j=1}^{n+1} c_j^{\alpha_1} x_{n+1-j} + f_1(x_{n+1}, y_n, z_n)\right)}{c_0^{\alpha_1}}, \quad (10)$$

$$y_{n+1} = \frac{\left(-\sum_{j=1}^{n+1} c_j^{\alpha_2} y_{n+1-j} + f_2(x_{n+1}, y_{n+1}, z_n)\right)}{c_0^{\alpha_2}}, \quad (11)$$

$$z_{n+1} = \frac{\left(-\sum_{j=1}^{n+1} c_j^{\alpha_3} z_{n+1-j} + f_3(x_{n+1}, y_{n+1}, z_{n+1})\right)}{c_0^{\alpha_3}}, \quad (12)$$

where $c_0^{\alpha_i} = \varphi_i(h)^{-1}$ are functions of the step size $h = \Delta t$, with the following properties:

$$\varphi_i(h) = h + O(h^2), \text{ where } h \rightarrow 0. \quad (13)$$

Examples of functions $\varphi_i(h)$ that satisfy (13) are: $h, \sin h, \sinh h, e^h - 1$, and in most applications, the general choice of $\varphi_i(h)$ is

$$\frac{1 - e^{-R_i h}}{R_i}, \quad (14)$$

the function R_i can be chosen as the following form

$$R_i = \max \left(\left| \frac{\partial f_i}{\partial x_i} \right|_{x=x_i} \right). \quad (15)$$

The nonlinear terms in system (10)-(12) can be in general being replaced by nonlocal discrete representations. For example,

$$y^2 \approx y_k y_{k+1}, \quad (16)$$

$$xy \approx 2xy - xy \rightarrow 2x_{n+1}y_n - x_{n+1}y_{n+1}, \quad (17)$$

where

$$h = \frac{T}{N}, t_n = nh, n = 0, 1, \dots, N \in \mathbb{Z}^+. \quad (18)$$

3. NSFD Simulation Results

The Fractional order Lü system is the lowest-order chaotic system among all the chaotic systems [30]. The minimum effective dimension reported is 0.30. The system is given by

$$D^\alpha x = a(y - x), \quad (19)$$

$$D^\alpha y = by - xy, \quad (20)$$

$$D^\alpha z = xy - cz, \quad (21)$$

where a, b , and c are the system parameters, (x, y, z) are the state variables, and α is the fractional order. Now, we apply the NSFD to obtain the numerical solution for the fractional-order Lü system. Using the Grünwald-Letnikov discretization method and applying NSFD scheme by replacing the step size h by a function $\varphi_i(h)$ and using the form in (16) and (17) for the nonlinear term xy the system (19)–(21) yields

$$\sum_{j=0}^{n+1} c_j^\alpha x(t-jh) = a(y(t_n) - x(t_n)), \quad (22)$$

$$\sum_{j=0}^{n+1} c_j^\alpha y(t-jh) = by(t_n), -(2x(t_{n+1})y(t_n) - x(t_{n+1})y(t_{n+1})), \quad (23)$$

$$\sum_{j=0}^{n+1} c_j^\alpha z(t-jh) = 2x(t_{n+1})y(t_n), \quad (24)$$

where $x(t_0) = x_0$, $y(t_0) = y_0$, and $z(t_0) = z_0$. Doing some algebraic manipulation to systems (22)–(24), yields the following relations

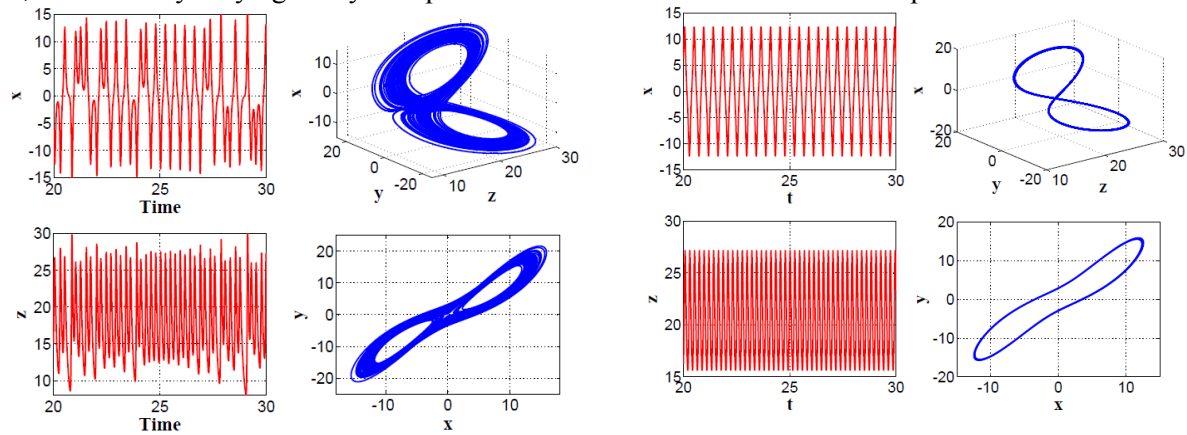
$$x(t_{n+1}) = \varphi^h \left(- \sum_{j=1}^{n+1} c_j^\alpha x(t-jh) a(y(t_n) - x(t_n)) \right), \quad (25)$$

$$y(t_{n+1}) = \frac{- \sum_{j=1}^{n+1} c_j^\alpha x(t-jh) + (b - 2x(t_{n+1}))y(t_n)}{c_0^\alpha - x(t_{n+1})}, \quad (26)$$

$$z(t_{n+1}) = \varphi^h \left(- \sum_{j=1}^{n+1} c_j^\alpha z(t-jh) + 2x(t_{n+1})y(t_n) - x(t_{n+1})y(t_{n+1}) - cz(t_n) \right), \quad (27)$$

where $c_0^\alpha = \varphi^{-h}$ and we chose $\varphi(h) = \sin h$ as a suitable function [26]. Conventionally when $\alpha = 1$, the system has two equilibrium points at $(0,0,0)$ and $(b, b, b^2/c)$ which depend on the parameters b and c only. The system exhibits chaotic behavior when the parameters set $(a, b, c) = (36.0, 28.0, 3.0)$. In the following simulations the effect of the parameter a ; which doesn't affect the equilibrium points; in addition to different values of the fractional-order parameter α will be studied for more than thirty cases to determine the region where chaotic responses appear.

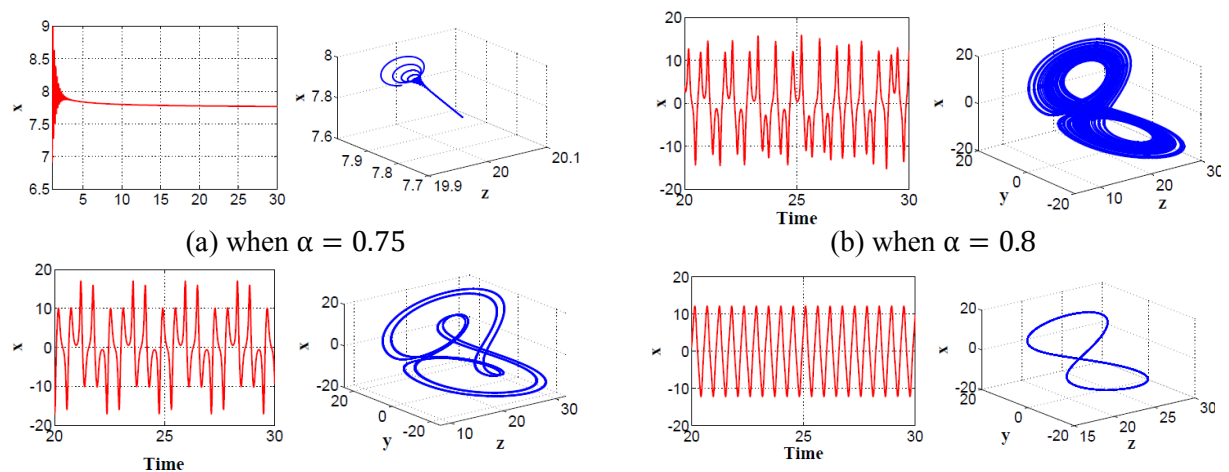
All the following simulations are performed using NSFD method using (25) up to (28) with $b = 28$, $c = 3$ and by varying the system parameter a and also the fractional-order parameter α .



(a) when $\alpha = 0.75$

(b) when $\alpha = 0.8$

Figure 1. System response when $a = 19.5$



(a) when $\alpha = 0.75$

(b) when $\alpha = 0.8$

(c) when $\alpha = 0.85$

(d) when $\alpha = 0.9$

Figure 2. System response when $a = 22$

Figure 1 shows the system responses when $a = 19.5$ for two different fractional-orders. When α less than 0.75 the system displays stable response. However, as α increases to 0.75, the system behaves chaotically as shown in Fig.1(a). But for $\alpha = 0.8$ and higher orders, the system response is periodic with period 1. Therefore, the system response will be chaotic around $\alpha = 0.75$ with a small range.

As the system parameter a increases to 22 and when $\alpha < 1$, the system response and x time waveform pass by stable, chaotic, high-period, and single-period responses when the fractional-order α equals to 0.75, 0.8, 0.85, and 0.9 cases as shown in Fig.2. Therefore, the range of the fractional-order α for chaotic response increases as the parameter a increases. In addition, when $\alpha = 0.85$, the system is periodic with period 5 as shown in Fig. 2(c).

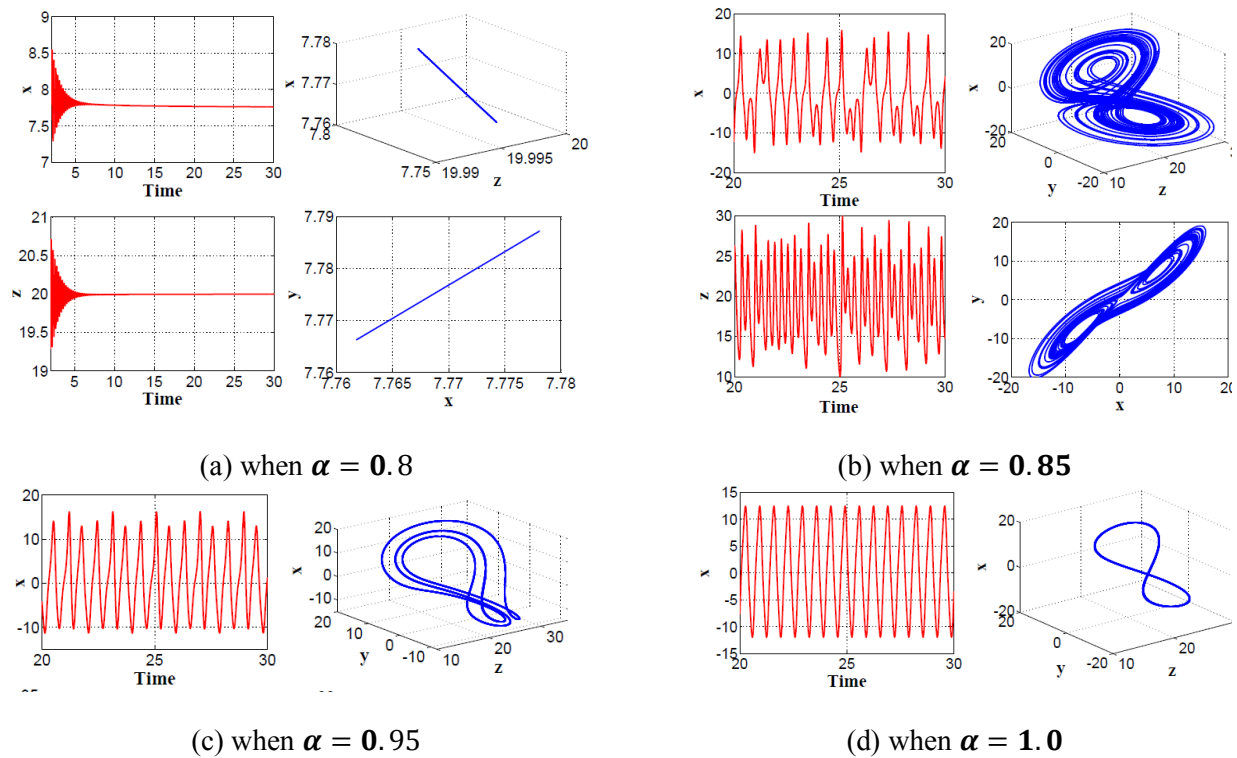


Figure 3. System response when $a = 25$

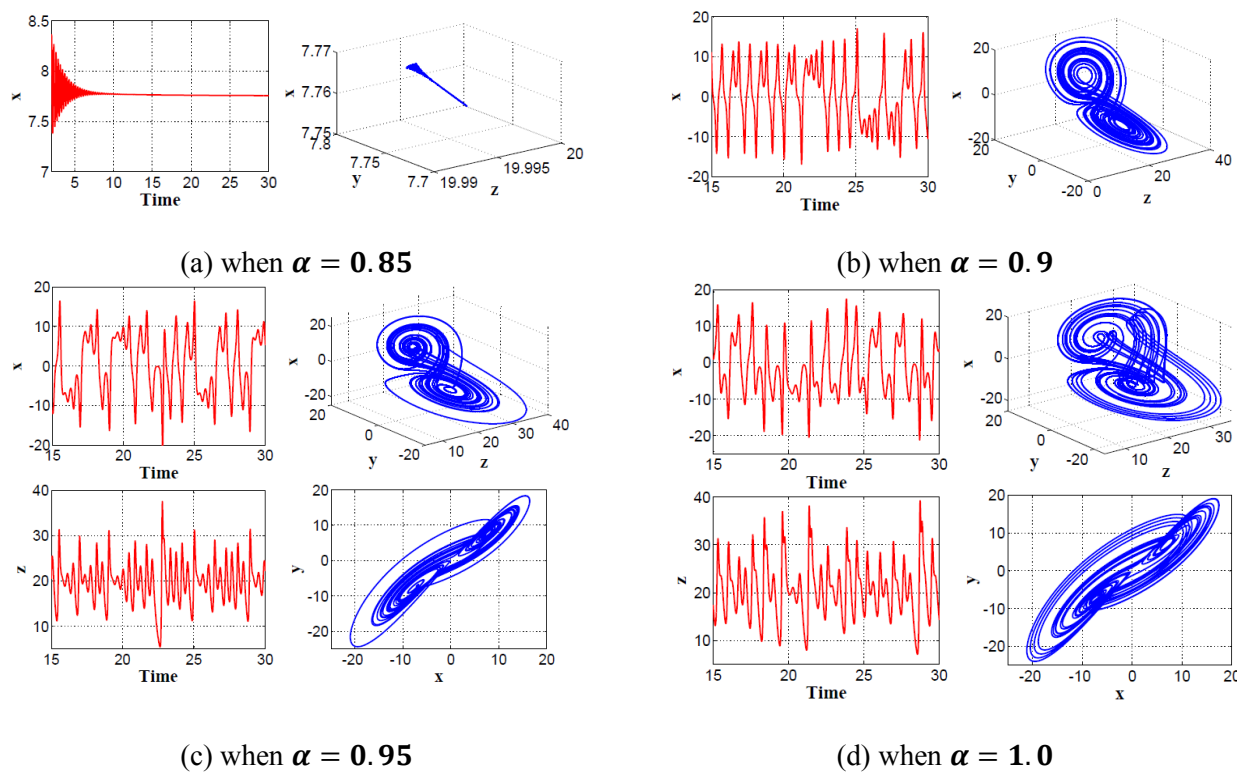


Figure 4. System response when $a = 30$

The responses when the system parameter equals to 25 with different values of the fractional-order α are shown in Fig.3 where the range of chaotic responses increases and different periodic attractors

are obtained. Moreover, the range of the stable responses is shifted up and appears when $\alpha = 0.8$ with the same equilibrium value as the previous case when $a = 19.5$. The system shows chaotic responses and high periods when the fractional-order α belongs to the interval from 0.85 up to 0.95.

Similarly, as the system parameter a increases the chaotic response is shifted up as shown in Fig. 4 and figure 5 with the same equilibrium point. While when $a = 30$, the system becomes stable where α is less than or equal to 0.85 and the chaotic response starts to appear in the range $[0.9, 1.0]$ as shown in Fig. 4. In this case the periodic response is expected to exist when the fractional-order $\alpha > 1$. When $a = 36$, the system will be stable up to $\alpha = 0.9$ and the chaotic response appears when $\alpha = 1.0$ which is the conventional case.

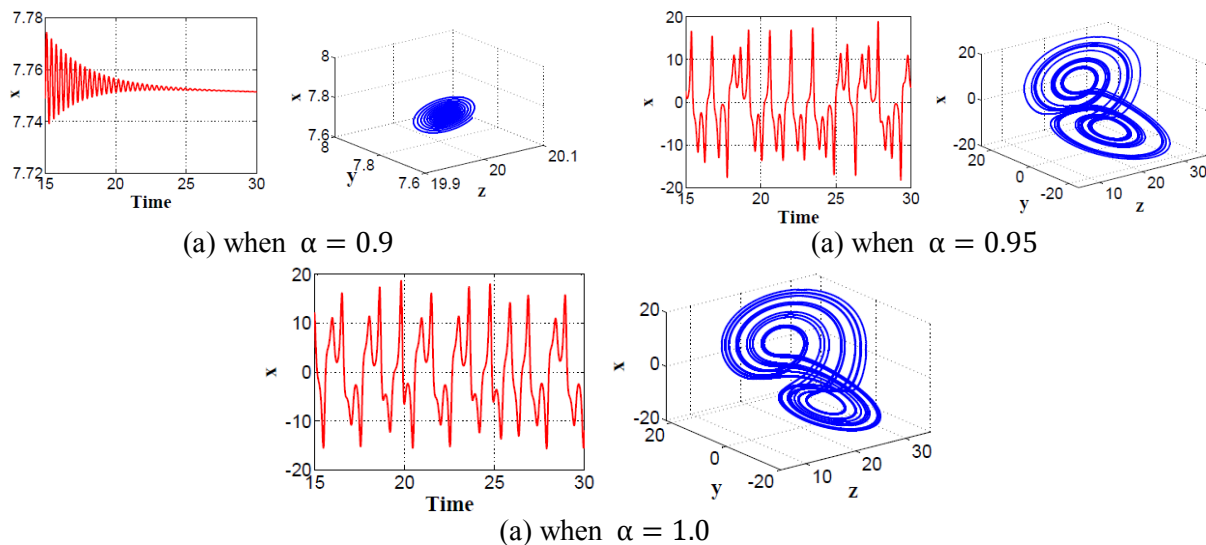


Figure 5. System response when $a = 36$

From the previous figures, we can conclude the results in Table 1, where the chaotic responses appear for a wide range of the system parameter a but in different ranges of the fractional-order parameter α . Therefore, as a increases the range of α for chaotic response increases and is shifted down. Moreover, it is expected that the Lü system can behave chaotically for larger values of $a > 36$ but with fractional-order $\alpha > 1$. In addition, as the range of α increases, more cases of high-periodic responses will appear.

Table 1. System performance versus the parameters

a	19.5	22	25	30	36
$\alpha < 7.5$	Stable	Stable	Stable	Stable	Stable
$\alpha = 7.5$	Chaotic	Stable	Stable	Stable	Stable
$\alpha = 0.8$	Period1	Chaotic	Stable	Stable	Stable
$\alpha = 0.85$	Period1	Period5	Chaotic	Stable	Stable
$\alpha = 0.9$	Period1	Period1	Chaotic	Chaotic	Stable
$\alpha = 0.95$	Period1	Period1	Period3	Chaotic	Chaotic
$\alpha = 1.0$	Period1	Period1	Period1	Chaotic	Chaotic

Figure 6 shows the response in the case when $\alpha = 0.8$ and with time varying parameter $a = 22 + 3\sin(0.025\pi t)$. It is clear that at the beginning when a close to 22, the system response behave chaotically and as a increases close to 25 the system response changed into stable then back to chaotic in the middle cycle where a back close to 22. Moreover, when the system parameter a reaches its

minimum value 19, the system become periodic as shown in the last interval of Fig.6. Therefore, this figure shows the continuous behaviour of the system when a changed smoothly with time.

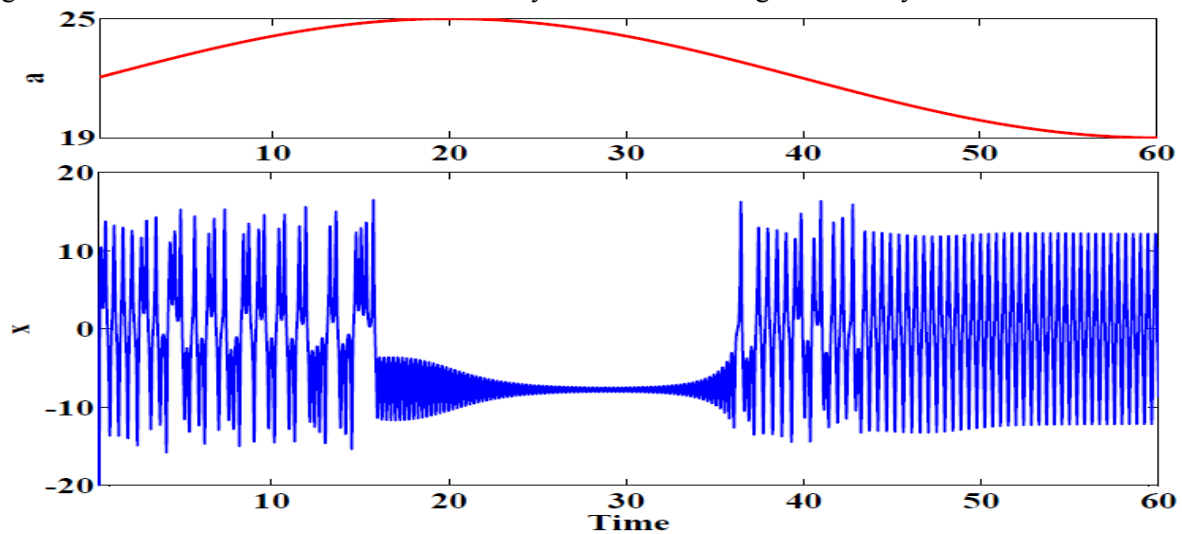


Figure 6. The x response when $\alpha = 0.8$ with time variant parameter $a = 22 + 3\sin(0.025\pi t)$.

4. Conclusion

This paper studied the fractional-order Lü system with different system parameters and for equal fractional-orders. As the parameter a increases, the system behaviour changes and a higher fractional-order is needed to get similar response. Thus, the conventional integer system with stable responses can be chaotic for lower fractional-orders under the same circuit parameters. Moreover, for each fixed set of parameters, there is a range of fractional-orders at which the response is stable, chaotic, and periodic. More than thirty cases have been investigated and simulated using the non-standard finite difference scheme to validate the smooth and gradual change in the system response as the system parameter a or the fractional-order parameter α is changed smoothly with time.

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