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Boltzmann equation with a nonlocal collision term and the resultant dissipative fluid dynamics

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Abstract. Starting with the relativistic Boltzmann equation where the collision term was generalized to include gradients of the phase-space distribution function, we recently presented a new derivation of the equations for the relativistic dissipative fluid dynamics. We compared them with the corresponding equations obtained in the standard Israel-Stewart and related approaches. Our method generates all the second-order terms that are allowed by symmetry, some of which have been missed by the traditional approaches, and the coefficients of other terms are altered. The first-order or Navier-Stokes equation too receives a small correction. Here we outline this work for the general audience.

1. Introduction

The kinetic or transport theory of gases is a microscopic description in the sense that detailed knowledge of the motion of the constituents is required. Fluid dynamics (also sloppily called hydrodynamics) is an effective (macroscopic) theory that describes the slow, long-wavelength motion of a fluid close to local thermal equilibrium. No knowledge of the motion of the constituents is required to describe observable phenomena. Quantitatively, if l denotes the mean free path, τ the mean free time, k the wave number, and ω the frequency, then $kl \ll 1,\ \omega\tau \ll 1$ is the hydro regime, $kl \simeq 1,\ \omega\tau \simeq 1$ the kinetic regime, and $kl \gg 1,\ \omega\tau \gg 1$ the free-particle regime.

Hydrodynamic equations are a set of coupled partial differential equations for number density n, energy density ϵ, pressure P, hydrodynamic four-velocity u^μ, and dissipative fluxes such as bulk viscosity Π, heat current n^μ, and shear stress tensor $\pi^{\mu\nu}$. In addition, the equation of state (EoS) needs to be supplied. Hydrodynamics is a powerful technique: Given the initial conditions and the EoS, it predicts the evolution of the matter. Its limitation is that it is applicable at or near (local) thermal equilibrium only.

Hydrodynamics finds applications in cosmology, astrophysics, high-energy nuclear physics, etc. In relativistic heavy-ion collisions, it is used to calculate the multiplicity and transverse momentum spectra of hadrons, anisotropic flows and femtoscopic radii. Energy density or temperature profiles resulting from the hydrodynamic evolution are needed in the calculations of jet quenching, J/ψ melting, thermal photon and dilepton productions, etc. Thus hydro plays a central role in modeling relativistic heavy-ion collisions.

Hydrodynamics is formulated as an order-by-order expansion in gradients of u^μ, the ideal hydrodynamics being of the zeroth order. The zeroth-, first-, and second-order equations are named after Euler, Navier-Stokes, and Burnett, respectively, in the non-relativistic case (Fig. 1).

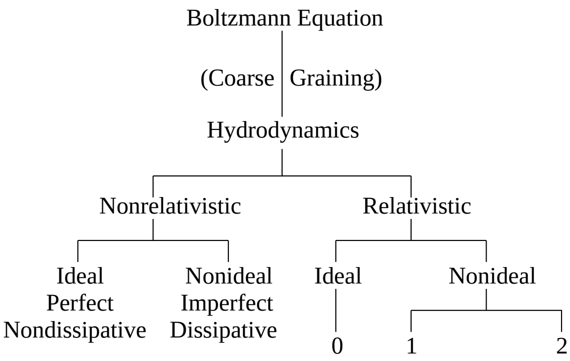

Figure 1. Coarse-graining of kinetic theory.

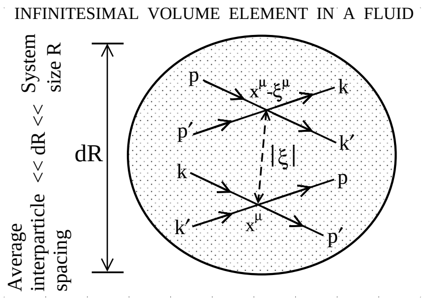

Figure 2. Collisions $kk' \to pp'$ and $pp' \to kk'$ occurring at points x^μ and $x^\mu - \xi^\mu$ within an infinitesimal fluid element of size dR, containing a large number of particles represented by dots.

The relativistic Navier-Stokes equations are parabolic in nature and exhibit acausal behaviour, which was rectified in the (second-order) Israel-Stewart (IS) theory [1]. The formulation of the relativistic dissipative second-order hydrodynamics ("2" in Fig. 1) is currently under intense investigation [2, 3, 4, 5, 6, 7, 8, 9, 10].

Hydrodynamics has traditionally been derived either from entropy considerations (i.e., the generalized second law of thermodynamics) or by taking the second moment of the Boltzmann equation; for a review, see [11]. The former approach captures a subset of the terms allowed by symmetry [12] in the evolution equations for the dissipative quantities. The latter approach captures some more terms but not all. The question 'why do the traditional approaches not generate *all* the allowed terms', was the main motivation of our work [13].

2. Present work

In [13] we presented a new derivation of the dissipative hydrodynamic equations within kinetic theory but using a nonlocal collision term in the Boltzmann equation. We obtained all the second-order terms that are allowed by symmetry [12] and showed that the coefficients of the existing terms in the widely used traditional IS equations were altered. These modifications do have a rather strong influence on the evolution of the viscous medium as we demonstrated in the case of one-dimensional scaling expansion.

It is important to recall at the outset that an infinitesimal volume element in the fluid is always supposed to be large compared with the mean interparticle spacing and hence contains "a very great" number of particles [14]; see also Fig. 2. In kinetic theory, the single-particle phase-space distribution function $f(x,p)$ is assumed to vary slowly over space-time, i.e., it changes negligibly over the range of interparticle interaction [15], which in fact is much smaller than the mean interparticle spacing. It is important to keep this hierarchy of length scales in mind.

It is also interesting to recall Israel and Stewart's classic paper [1] where they list the properties of the collision term $C[f]$. We quote: "We require only the following general properties: (i) $C[f]$ is a purely local function or functional of f, independent of $\partial_\mu f$. (ii) The form of $C[f]$ is consistent with conservation of 4-momentum and number of particles at collisions. (iii) $C[f]$ yields a non-negative expression for the entropy production .... These requirements are of course met by Boltzmann's ansatz for 2-particles collisions, and, indeed, one may hope that they hold somewhat more generally, although the locality assumption (i) is a powerful restriction." Note the words *hope* and *assumption*. The locality assumption is questionable and we relaxed it on the length scale of dR [13]. Despite the long history of the Boltzmann equation, a collision term containing $\partial_\mu f$, to our knowledge, has never been used to derive hydrodynamic equations. Such a collision term brings about a change not only in hydrodynamics but also in the kinetic theory.

Our starting point is the relativistic Boltzmann equation with the modified collision term:

$$p^\mu \partial_\mu f = C_m[f] = C[f] + \partial_\mu (A^\mu f) + \partial_\mu \partial_\nu (B^{\mu\nu} f) + \cdots, \tag{1}$$

where A^μ and $B^{\mu\nu}$ depend on the type of the collisions $(2 \leftrightarrow 2,\ 2 \leftrightarrow 3, \ldots)$.

For instance, for $2 \leftrightarrow 2$ elastic collisions,

$$C[f] = \frac{1}{2} \int dp' dk\ dk'\ W_{pp' \to kk'}\ (f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} - f_p f_{p'} \tilde{f}_k \tilde{f}_{k'}), \tag{2}$$

where $W_{pp'\to kk'}$ is the transition rate, $f_p \equiv f(x,p)$ and $\tilde{f}_p \equiv 1 - rf(x,p)$ with $r = 1, -1, 0$ for Fermi, Bose, and Boltzmann gas, and $dp = g d\mathbf{p}/[(2\pi)^3 \sqrt{\mathbf{p}^2 + m^2}]$, g and m being the degeneracy factor and particle rest mass. The first and second terms in Eq. (2) refer to the gain and loss processes $kk' \to pp'$ and $pp' \to kk'$, respectively, occurring *anywhere* in the infinitesimal fluid element located at the space-time point x^μ. These processes have traditionally been assumed to occur at the same point x^μ with an underlying assumption that $f(x,p)$ is constant not only over the range of interparticle interaction but also over the entire fluid element of size dR. Boltzmann equation together with this crucial assumption has been used to derive the standard second-order dissipative hydrodynamic equations [1, 9, 11]. We, however, emphasize that the variation of $f(x,p)$ over the span of the fluid element may not be negligible, and hence the space-time points at which the above two kinds of processes occur should be separated by an interval $|\xi^\mu| \le dR$ within the volume d^4R (Fig. 2). The large number of particles within d^4R collide among themselves with various separations ξ^μ. Of course, the points $(x^\mu - \xi^\mu)$ must lie within the past light-cone of the point x^μ (i.e., $\xi^2 > 0$ and $\xi^0 > 0$) to ensure that the evolution of $f(x,p)$ in Eq. (1) does not violate causality. With this realistic viewpoint, the second term in Eq. (2) involves $f(x-\xi,p) f(x-\xi,p') \tilde{f}(x-\xi,k) \tilde{f}(x-\xi,k')$, which on Taylor expansion at x^μ up to second order in ξ^μ, results in the modified Boltzmann equation (1) with

$$A^\mu = \frac{1}{2} \int dp' dk\ dk'\ \xi^\mu W_{pp'\to kk'} f_{p'} \tilde{f}_k \tilde{f}_{k'}, \text{ and } B^{\mu\nu} = -\frac{1}{4} \int dp' dk\ dk'\ \xi^\mu \xi^\nu W_{pp'\to kk'} f_{p'} \tilde{f}_k \tilde{f}_{k'}. \tag{3}$$

In general, for all collision types $(2 \leftrightarrow 2,\ 2 \leftrightarrow 3, \ldots)$, the momentum dependence of the coefficients A^μ and $B^{\mu\nu}$ can be made explicit by expressing them in terms of the available tensors p^μ and the metric $g^{\mu\nu}$ as $A^\mu = a(x) p^\mu$ and $B^{\mu\nu} = b_1(x) g^{\mu\nu} + b_2(x) p^\mu p^\nu$. Equation (1) with this A^μ and $B^{\mu\nu}$ forms the basis of our derivation of the second-order dissipative hydrodynamics. Arguments in the previous paragraph were meant only to provide a physical motivation for the mathematical form of $C_m[f]$ in Eq. (1).

Conservation of current, $\partial_\mu N^\mu = 0$ and energy-momentum tensor, $\partial_\mu T^{\mu\nu} = 0$, yield the fundamental evolution equations for n, ϵ and u^μ (defined in the Landau frame)

$$\begin{aligned} Dn + n\partial_\mu u^\mu + \partial_\mu n^\mu &= 0, \\ D\epsilon + (\epsilon + P + \Pi)\partial_\mu u^\mu - \pi^{\mu\nu} \nabla_{(\mu} u_{\nu)} &= 0, \\ (\epsilon + P + \Pi) Du^\alpha - \nabla^\alpha (P + \Pi) + \Delta^\alpha_\nu \partial_\mu \pi^{\mu\nu} &= 0. \end{aligned} \tag{4}$$

Equations (4) together with the EoS constitute six equations in fifteen unknowns. How to derive the extra nine equations that would give us a closed set of equations? Boltzmann equation provides a way: The requirement of the conservation of energy-momentum and current implies vanishing zeroth and first moments of the collision term $C_m[f]$ in Eq. (1), i.e., $\int dp\ C_m[f] = 0 = \int dp\ p^\mu C_m[f]$ at each order in ξ^μ. In order to obtain the evolution equations for the dissipative quantities, we follow the IS approach [1] and consider the second moment of the modified Boltzmann equation (1)

$$\int dp\ p^\alpha p^\beta p^\gamma \partial_\gamma f = \int dp\ p^\alpha p^\beta [C[f] + p^\gamma \partial_\gamma (af) + \partial^2 (b_1 f_0) + (p\cdot\partial)^2 (b_2 f_0)], \tag{5}$$

and then take recourse to Grad's 14-moment approximation [16] for the single-particle distribution in orthogonal basis [9]. This gives the desired equations:

$$\begin{aligned}\Pi &= \tilde{a}\Pi_{\rm NS} - \beta_{\dot\Pi}\tau_\Pi\dot\Pi + \tau_{\Pi n} n\cdot\dot u - l_{\Pi n}\partial\cdot n - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi n} n\cdot\nabla\alpha + \lambda_{\Pi\pi}\pi_{\mu\nu}\sigma^{\mu\nu}\\ &\quad +\Lambda_{\Pi\dot u}\dot u\cdot\dot u + \Lambda_{\Pi\omega}\omega_{\mu\nu}\omega^{\nu\mu} + (8 \text{ terms}), \qquad (6)\\ n^\mu &= \tilde{a} n^\mu_{\rm NS} - \beta_{\dot n}\tau_n \dot n^{\langle\mu\rangle} + \lambda_{nn} n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu\theta + l_{n\Pi}\nabla^\mu\Pi - l_{n\pi}\Delta^{\mu\nu}\partial_\gamma\pi^\gamma_\nu - \tau_{n\Pi}\Pi\dot u^\mu\\ &\quad -\tau_{n\pi}\pi^{\mu\nu}\dot u_\nu + \lambda_{n\pi} n_\nu\pi^{\mu\nu} + \lambda_{n\Pi}\Pi n^\mu + \Lambda_{n\dot u}\omega^{\mu\nu}\dot u_\nu + \Lambda_{n\omega}\Delta^\mu_\nu\partial_\gamma\omega^{\gamma\nu} + (9 \text{ terms}), \qquad (7)\\ \pi^{\mu\nu} &= \tilde{a}\pi^{\mu\nu}_{\rm NS} - \beta_{\dot\pi}\tau_\pi\dot\pi^{\langle\mu\nu\rangle} + \tau_{\pi n} n^{\langle\mu}\dot u^{\nu\rangle} + l_{\pi n}\nabla^{\langle\mu} n^{\nu\rangle} + \lambda_{\pi\pi}\pi_\rho^{\langle\mu}\omega^{\nu\rangle\rho} - \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha - \tau_{\pi\pi}\pi_\rho^{\langle\mu}\sigma^{\nu\rangle\rho}\\ &\quad -\delta_{\pi\pi}\pi^{\mu\nu}\theta + \Lambda_{\pi\dot u}\dot u^{\langle\mu}\dot u^{\nu\rangle} + \Lambda_{\pi\omega}\omega_\rho^{\langle\mu}\omega^{\nu\rangle\rho} + \chi_1\dot b_2\pi^{\mu\nu} + \chi_2\dot u^{\langle\mu}\nabla^{\nu\rangle} b_2 + \chi_3\nabla^{\langle\mu}\nabla^{\nu\rangle} b_2, \qquad (8)\end{aligned}$$

where $\tilde{a} = (1-a)$ and $\dot X = DX$. The "8 terms" ("9 terms") involve second-order, linear scalar (vector) combinations of derivatives of b_1, b_2. All the terms in the above equations are inequivalent, i.e., none can be expressed as a combination of others via equations of motion [12]. All the coefficients in Eqs. (6)-(8) can be written as functions of hydrodynamic variables [13].

In [13], we demonstrated the numerical significance of the new dissipative equations derived here, by considering the evolution of a massless Boltzmann gas, with the equation of state $\epsilon = 3P$, at vanishing net baryon number density, in the Bjorken model [17].

3. Summary

To summarize, we have presented a new derivation of the relativistic dissipative hydrodynamic equations by introducing a nonlocal generalization of the collision term in the Boltzmann equation. The first-order (Navier-Stokes) and second-order (Israel-Stewart) equations are modified: new terms occur and coefficients of others are altered. While it is well known that the derivation based on the generalized second law of thermodynamics misses some terms in the second-order equations, we have shown that the standard derivation based on kinetic theory also misses other terms. The method presented here is able to generate all possible terms to a given order that are allowed by symmetry.

[1] Israel W and Stewart J M 1979 *Annals Phys.* **118** 341
[2] Baier R, Romatschke P and Wiedemann U A 2006 *Phys. Rev.* C **73** 064903
[3] Baier R, Romatschke P, Son D T, Starinets A O and Stephanov M A 2008 *J. High Energy Phys.* JHEP04(2008)100
[4] Bhattacharyya S, Hubeny V E, Minwalla S and Rangamani M 2008 *J. High Energy Phys.* JHEP02(2008)045
[5] Natsuume M and Okamura T 2008 *Phys. Rev.* D **77**, 066014 [Erratum 2008 *Phy. Rev.* D **78** 089902]
[6] Bhalerao R S and Gupta S 2008 *Phys. Rev.* C **77** 014902
[7] El A, Muronga A, Xu Z and Greiner C 2009 *Phys. Rev.* C **79** 044914
[8] El A, Xu Z and Greiner C 2010 *Phys. Rev.* C **81** 041901
[9] Denicol G S, Koide T and Rischke D H 2010 *Phys. Rev. Lett.* **105** 162501
[10] Denicol G S, Niemi H, Molnar E and Rischke D H 2012 *Phys. Rev.* D **85** 114047
[11] Romatschke P 2010 *Int. J. Mod. Phys.* E **19** 1
[12] Bhattacharyya S 2012 *J. High Energy Phys.* JHEP07(2012)104
[13] Jaiswal A, Bhalerao R S and Pal S 2012 *Preprint* 1204.3779 [nucl-th] (revised)
[14] Landau L D and Lifshitz E M 1987 *Fluid Mechanics* (Oxford: Butterworth-Heinemann) p 1
[15] de Groot S R, van Leeuwen W A and van Weert Ch G 1980 *Relativistic Kinetic Theory — Principles and Applications* (Amsterdam: North-Holland) p 18
[16] Grad H 1949 *Comm. Pure Appl. Math.* **2** 331
[17] Bjorken J D 1983 *Phys. Rev.* D **27** 140