Phase Diagram of the Blume-Emery-Griffiths-Vannimenus Model on the Cayley Tree

This content has been downloaded from IOPscience. Please scroll down to see the full text.
2013 J. Phys.: Conf. Ser. 410 012034
(http://iopscience.iop.org/1742-6596/410/1/012034)

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 54.191.40.80
This content was downloaded on 25/08/2017 at 01:21

Please note that terms and conditions apply.

You may also be interested in:

Two Potts Models on Cayley Tree of Arbitrary Order
Nasir Ganikhodjaev, Ashraf Mohamed Nawi and Mohd Hirzie Mohd Rodzhan

Strange Attractors in the Vannimenus Model on an Arbitrary Order Cayley Tree
Nasir Ganikhodjaev, Hasan Akin, Seyit Temir et al.

On extreme Gibbs measures of the Vannimenus model
Nasir Ganikhodjaev, Hasan Akin, Selman Uuz et al.

On Paramagnetic Phases of the Potts Model on a Bethe Lattice in the Presence of Competing Interactions
Nasir Ganikhodjaev and Anasbek Rahmatullaev

Modulated structures of an Ising model with competing nearest-neighbour interactions
M J de Oliveira and S R Salinas

The magnetization process of an Ising-type frustrated S = 1 spin chain
M Kaburagi, M Kang, T Tonegawa et al.

Surface incommensurate structure in an anisotropic model with competing interactions on semi-infinite triangular lattice
Pavol Pajerský and Anton Surda

An iterative scheme for the 2D ANNNI model
M A S Saqi and D S McKenzie

Critical behaviour of a two-dimensional bonded lattice model
A P Young and D A Lavis
Phase Diagram of the Blume-Emery-Griffiths-Vannimenus Model on the Cayley Tree

Nasir Ganikhodjaev
Department of Computational and Theoretical Sciences, Faculty of Science, IIUM, 25200 Kuantan, Malaysia
E-mail: nasirgani@hotmail.com

Abstract. We study the phase diagram of the Blume-Emery-Griffiths-Vannimenus model on a Cayley tree with competing nearest-neighbour couplings and next-nearest-neighbour couplings and show that a detailed study of its properties can be carried out with essentially exact results, using rather simple numerical methods. In addition to the expected paramagnetic and ferromagnetic phases, we find modulated phase for Blume-Emery-Griffiths model and hence for Blume-Emery-Griffiths-Vannimenus model also.

1. Introduction

The existence of competing interactions lies at the heart of a variety of original phenomena in magnetic systems, ranging from the spin-glass transitions found in many disordered materials to the modulated phases with an infinite number of commensurate regions, that are observed in certain models with periodic interactions [1]. Vannimenus [1] and Mariz et al [2] examine the local properties of the Ising model on a Cayley tree with nearest-neighbour interactions and competing next-nearest-neighbour interactions. By a process of iteration, Vannimenus found a new modulated phase, in addition to the expected paramagnetic, ferromagnetic and antiferromagnetic phases, which consisted of commensurate (periodic) and incommensurate (aperiodic) regions corresponding to the so-called "devil’s staircase", found also in the ANNNI [3] and other competing interaction models [4]. As expanded Ising model, the Blume-Emery-Griffiths (BEG) model, which is characterized by bilinear and biquadratic exchange interactions and crystal-field interaction, has played important role in the development of the theory of tricritical phenomena. The Hamiltonian of the BEG model on the Cayley tree is given by

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y) - K \sum_{<x,y>} \sigma^2(x)\sigma^2(y) + D \sum_x \sigma^2(x)$$

where $\sigma(x) \in \{-1,0,1\}$ is the spin at site $x$, first and second summation run over all the nearest-neighbour pairs, and last summation runs over all the sites. Here $J, K, D$ describe the bilinear exchange, biquadratic interactions, and crystal-field interaction. This Hamiltonian was originally proposed to explain the phase separation and superfluidity in $^3He-^4He$ mixtures [5]. The present paper is devoted to the study of a Blume-Emery-Griffiths-Vannimenus model on a Cayley tree with competing nearest-neighbour couplings and next-nearest-neighbour couplings and to show that a detailed study of its properties can be carried out with essentially exact results, using rather simple numerical methods.
The model considered consists of spins \( \{\sigma(x) \in \{-1, 0, 1\}\} \) on a a semi-infinite Cayley tree \( \Gamma^2 = (V, L) \) of second order, i.e., an infinite graph without cycles with 3 edges issuing from each vertex except for \( x^0 \), so-called a root of the tree, which has only 2 edges, where \( V \) is the set of vertices and \( L \) is the set of edges. Two kinds of bonds are present: nearest-neighbours interactions and prolonged next-nearest-neighbour interactions, these being restricted to spin belonging to the same branch of the tree.

**The Model:** The Hamiltonian of the Blume-Emery-Griffiths-Vannimenus (BEGV) model on the Cayley tree is defined by

\[
H(\sigma) = -J_p \sum_{<x,y>} \sigma(x)\sigma(y) - J_1 \sum_{<x,y>} \sigma(x)\sigma(y) - K \sum_{x} \sigma^2(x)\sigma^2(y) + D \sum_{x} \sigma^2(x) \tag{2}
\]

where \( \sigma(x) \in \{-1, 0, 1\} \) is the spin at site \( x \in V \), the first summations runs over all prolonged next-nearest-neighbours, the second and third summation runs over all nearest-neighbours pairs, and last summation runs over all the sites. Here \( J_p, J_1, K, D \) describe the bilinear exchange, biquadratic interactions and crystal-field interaction respectively. The Hamiltonian (2) with \( J_p = 0 \) defines the BEG model [5] and with \( K = D = 0 \) defines Vannimenus model [1].

**3. Recurrence equations**

In order to produce the recurrence equations, we consider the relation of the partition function of an \( n \)- generation tree \( V_n \) to the partition function of its subsystems containing \( (n-1) \) generations \( V_{n-1} \).

Let \( <x^0, x> = l \in L \) be an edge of semi-infinite Cayley tree \( \Gamma^2 \) of second order. The infinite subtree \( \Gamma^2_+(l) = (V^l, L^l) \) is called a single-trunk Cayley tree, if from vertex \( x^0 \) a single edge \( l \) emanates and from any other vertex \( x \in V^l \), \( x \neq x^0 \) exactly 3 edges emanate. Let \( W_1 = \{x_1, x_2\} \) and \( <x^0, x^1> = l_1, <x^0, x^2> = l_2 \) be two edges emanating from \( x^0 \). It is evident that semi-infinite Cayley tree \( \Gamma^2_+ \) splits into two components - two single-trunk Cayley trees \( \Gamma^2_+(l_i), i = 1, 2 \).

Assume \( Z^{(n)}(i,j) \) be a partition function on \( V^l \) with the configuration \( (i, j) \) on an edge \( l = <x^0, x> \), where \( i,j = -1, 0, 1 \); and \( Z^{(n)}(i_1, i_0, i_2) \) be a partition function on \( V_n \) where the spin in the root \( x^0 \) is \( i_0 \) and the two spins in the proceeding vertices \( x^1, x^2 \) are \( i_1 \) and \( i_2 \), respectively.

There are 27 different partition functions \( Z^{(n)}(i_1, i_0, i_2) \) and the partition function \( Z^{(n)} \) in volume \( V_n \) can be written as follows

\[
Z^{(n)} = \sum_{i_1, i_0, i_2 = -1}^{1} Z^{(n)}(i_1, i_0, i_2). \tag{3}
\]

and

\[
Z^{(n)}(\sigma(x^1), \sigma(x^0), \sigma(x^2)) = Z^{(n)}(\sigma(x^0), \sigma(x^1)) \cdot Z^{(n)}(\sigma(x^0), \sigma(x^2)). \tag{4}
\]

Let

\[
a = \exp\left(\frac{J_p}{k_BT}\right), b = \exp\left(\frac{J_1}{k_BT}\right), c = \exp\left(\frac{K}{k_BT}\right), d = \exp\left(-\frac{D}{2k_BT}\right).
\]

From (4) one can select only nine new independent variables

\[
\begin{align*}
  u_1^{(n)} &= \sqrt{Z^{(n)}(-1, -1, -1)}, & u_2^{(n)} &= \sqrt{Z^{(n)}(-1, 0, -1)}, & u_3^{(n)} &= \sqrt{Z^{(n)}(-1, 1, -1)}, \\
  u_4^{(n)} &= \sqrt{Z^{(n)}(0, -1, 0)}, & u_5^{(n)} &= \sqrt{Z^{(n)}(0, 0, 0)}, & u_6^{(n)} &= \sqrt{Z^{(n)}(0, 1, 0)}, \\
  u_7^{(n)} &= \sqrt{Z^{(n)}(1, -1, 1)}, & u_8^{(n)} &= \sqrt{Z^{(n)}(1, 0, 1)}, & u_9^{(n)} &= \sqrt{Z^{(n)}(1, 1, 1)}.
\end{align*}
\]
We note that, in the paramagnetic phase (high symmetry phase), \( u_1 = u_9, u_2 = u_8, u_3 = u_7 \) and \( u_4 = u_6 \). For discussing the phase diagram, the following choice of reduced variables is convenient:

\[
x_1 = \frac{u_2 + u_8}{u_1 + u_9}, \quad x_2 = \frac{u_3 + u_7}{u_1 + u_9}, \quad x_3 = \frac{u_4 + u_6}{u_1 + u_9}, \quad x_4 = \frac{2u_5}{u_1 + u_9},
\]

\[
y_1 = \frac{u_4 - u_9}{u_1 + u_9}, \quad y_2 = \frac{u_2 - u_8}{u_1 + u_9}, \quad y_3 = \frac{u_3 - u_7}{u_1 + u_9}, \quad y_4 = \frac{u_4 - u_6}{u_1 + u_9}.
\]

It is straightforward to establish the following recursive relations:

\[
x_1' = \frac{1 + x_2 + x_3}{6cD} + y_1, \quad y_1' = \frac{2(a + a^{-1} x_2 + x_3) (y_1 - a^{-1} y_3 + y_4)}{6D},
\]

\[
x_2' = \frac{a^{-1} + a x_2 + x_3}{6D} + |y_1 - a^{-1} y_3 + y_4|, \quad y_2' = \frac{2(1 + x_2 + x_3) (y_1 - y_3 + y_4)}{6D},
\]

\[
x_3' = \frac{(a - a^{-1}) x_1 + x_2 + x_3 + (a - a^{-1}) y_2}{6D}, \quad y_3' = \frac{2(a^{-1} + a x_2 + x_3) (a - y_3 + y_4)}{6D},
\]

\[
x_4' = \frac{2[x_1 + x_3]}{6D}, \quad y_4' = \frac{2(a - a^{-1}) (a + a^{-1} x_1 + x_3) y_2}{6D},
\]

where

\[D = (a + a^{-1} x_2 + x_3)^2 + (ay_1 - a^{-1} y_3 + y_4)^2,\]

and primed variables correspond to the \( Z^{(n+1)}(i_1, i_0, i_2) \). The system of eight equations (5) are essentially complicated than the similar basic equations of the Ising model [1], [2].

The total partition function is given in terms of \( (u_i^{(n)}) \) by

\[Z^{(n)} = (u_1^{(n)} + u_4^{(n)} + u_7^{(n)}) + (u_2^{(n)} + u_5^{(n)} + u_8^{(n)}) + (u_3^{(n)} + u_6^{(n)} + u_9^{(n)})^2.\]

The average magnetization \( m^{(n)} \) for the \( n \)th generation is given by

\[m^{(n)} = -\frac{4(1 + x_2 + x_3)(y_1 - y_3 + y_4)}{(2x_1 + x_4)^2 + 2(1 + x_2 + x_3)^2 + (y_1 - y_3 + y_4)^2}.
\]

4. Phase Diagram

In this chapter we consider the broad features of the phase diagram. This can be achieved numerically in a straightforward fashion. Starting from random initial conditions (with \( y_1, y_2, y_3, y_4 \neq 0 \)), one iterates the recurrence relations (5) and observes their behaviour after a large number of iterations. In the simplest situation a fixed point \( (x_1^*, x_2^*, x_3^*, x_4^*, y_1^*, y_2^*, y_3^*, y_4^*) \) is reached. It corresponds to a paramagnetic phase (P) if \( y_1^* = y_2^* = y_3^* = y_4^* = 0 \), or to a ferromagnetic phase (F) if \( y_1^*, y_2^*, y_3^*, y_4^* \neq 0 \). Secondary, the system may be periodic with period \( p \), where case \( p = 2 \) corresponds to antiferromagnetic phase and case \( p = 4 \) corresponds to so-called antiphase, that denoted \( < 2 > \) for compactness. Finally, the system may remain aperiodic. The distinction between a truly aperiodic case and one with a very long period is difficult to make numerically. Below we consider periodic phases with period \( p \) where \( p \leq 12 \). All periodic phases with period \( p > 12 \) and aperiodic phase we consider as modulated phase (M). We plot phase diagrams in \( (k_B T/J_1, D/J_1) \) plane for some fixed values of \( p = -J_p/J_1 \) and \( k = K/J_1 \) with \( J_1 > 0 \) and \( D > 0 \).

The phase diagrams of BEG model, i.e., \( J_p = 0 \), are shown in Fig.1. Here \( p = 0 \) and \( k = \pm 0.5 \). One can see that the phase diagram contains modulated phase, in addition to the expected paramagnetic and ferromagnetic ones. As shown by Vannimenus [1] for the Ising model one can reach modulated phase only in presence of competing next-nearest-neighbours interactions, meanwhile for the BEG model with nearest-neighbour interactions we have modulated phase. If \( K > 0 \) then the phase diagram contains paramagnetic, ferromagnetic and modulated phases and all three phases meet at the multicritical point \( (T/J_1 = 1.6, D/J_1 = 1) \), and if \( K < 0 \) then the phase diagram contains ferromagnetic and modulated phases only.
The phase diagrams of BEGV model with $p = 0.6$ and $k = \pm 0.5$ are shown in Fig. 2. If $K > 0$ then the phase diagram contains antiphase and modulated phases only, and if $K < 0$ the phase diagram contains antiphase, modulated phase and regions with period 5, 6, and 9.

5. Conclusion
We show that the local magnetization for BEG model with nearest-neighbour interactions on a Cayley tree is mainly chaotic with oscillatory glasslike behavior (see [1], [3], [4]) and for BEGV model the local magnetization is fully chaotic. For BEG model on periodic lattice one can expect similar result. The structure of modulated phase and stability of the phases will be considered in another paper.

5.1. Acknowledgments
This work is supported by the FRGS Grant Phase 2/2010, FRGS 10-022-0141.

6. References