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Reconstruction of potential part of 3D vector field by using singular value decomposition

Anna Polyakova
Graduate student, Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia
E-mail: anna.polyakova@ngs.ru

Abstract. In this paper we suggest the method of 3D vector tomography problem solving. The problem consists in determination of potential part of 3D vector field by its known the normal Radon transform. The singular value decomposition of the normal Radon transform operator is obtained. Based on obtained decomposition inversion formula is derived. The decomposition can be the basis for numerical solution of given problem.

1. Introduction
The development both of mathematical approaches and of systems of data measurements and processing induced new mathematical models in tomography such as thermotomography, diffusion tomography, vector and tensor tomography. The models appear due to the necessity of the reconstruction of properties of media with different degree of complication. Thus the tomography of vector fields arises for the description of vector characteristics of currents of fluids, vectors of electromagnetic fields inside the conductor in inhomogeneous media, and many others.[1]

We consider here a method of solving the 3D vector tomography problem in the case of parallel scheme of observation. As the problem of scalar tomography consists in the inversion of the Radon transform for a function, the vector tomography problem is the problem of inversion of normal Radon transform operator applying to potential part of 3D vector fields. In other words, one has to solve operator equations $Af = g$ of the first kind. Here $A$ is a linear, bounded operator. In the operator equation $g$ is a known right hand-side (data of tomographic measurements), and $f$ is an unknown vector field to be determined.

The method of singular value decomposition (SVD) is well known and often used for inversion of compact linear operators. The idea of the approach consists in representing the operator in a form of series of singular numbers and basic elements in the image space. Then the inverse operator is a similar series with the reciprocal of the singular numbers and pre-images of the bases elements. The solution of 3D vector tomography problem, which is proposed in the paper, is based on the possibility of 3D vector field potential part representation by using of potentials, which are constructed as a product of harmonic functions and classical orthogonal polynomials.

2. Definitions and statement of the problem
In this paper we solve the problem of recovery a potential part of vector field, which is given in unit ball $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$ with boundary $\partial B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. A notification $Z = \{(|\xi| = 1, s \in \mathbb{R}) \}$ is used for cylinder.
We use the following differential operators:

1) gradient operator \( d : H^k(B) \to H^{k-1}(S^1(B)) \), which acts on the potential \( \psi \) by formula:

\[
d\psi = \left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z} \right);
\]

2) rotor operator \( \text{rot} : H^k(S^1(B)) \to H^{k-1}(S^1(B)) \), which acts on a vector field \( w \) by next way:

\[
\text{rot} w = \left( \frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z}, \frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x}, \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} \right);
\]

3) divergence operator \( \delta : H^k(S^1(B)) \to H^{k-1}(B) \), which acts on a vector field \( w \) by rule:

\[
\delta w = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}.
\]

A vector field \( u \in H^k(S^1(B)) \) is a potential vector field, if there is \( \phi \in H^{k+1}(B) \) (potential), such as \( u = d\phi \). A vector field \( v \in H^k(S^1(B)) \) is solenoidal, if \( \delta v \in H^{k-1}(B) = 0 \). It is obvious that field \( u = \text{rot}v \) is solenoidal.

It is well known [2] that every vector field \( w \in L^2(S^1(B)) \) in \( R^3 \) can be decomposed uniquely in a sum of potential and solenoidal parts

\[
w = d\phi + \text{rot}v.
\]

where \( \phi \in H^1_0(B) \) and \( v \in H^1(S^1(B)) \).

2.1. The Radon transform

The plane \( P_{\xi,s} \) in \( R^3 \) is given by equation \( \langle \xi, x \rangle - s = 0 \) for \( x = (x, y, z), \xi = (\xi^1, \xi^2, \xi^3), |\xi| = 1 \). There \( |s| \) is distance between the plane and the origin of coordinates, \( \xi \) is a normal vector of the plane.

The Radon transform \( Rf : L^2(R^3) \to L^2(Z, \rho) \) of function \( f(x) \) is given by formula

\[
Rf(s, \xi) = \int_{P_{\xi,s}} f(ue_1 + ve_2 + s\xi) \, du \, dv.
\]

Integral in the right-hand side does not depend on the choice of the basis \( e_1, e_2 \) on the plane of integration.

2.2. The ray transform of a vector field

Let \( T = \{ (u, v, \xi) | u \in [-\sqrt{1-v^2}, \sqrt{1-v^2}], v \in [-1, 1], |\xi| = 1 \} \).

The ray transform \( \mathcal{P} : L^2(S^1(B)) \to L^2(T) \) of a vector field \( w \) is given by formula

\[
(\mathcal{P}w)(u, v, \xi) = \int_{-\infty}^{\infty} \langle w, \xi \rangle \, dt.
\]

Easy to show that the kernel of the operator consist of potential vector fields \( d\phi \in L^2(S^1(B)) \) with potential \( \phi \in H^1_0(B) \). That is if we know the ray transform of a vector field we can reconstruct only its solenoidal part.
2.3. The normal Radon transform

The normal Radon transform $R^\perp : L^2(S^1(B)) \rightarrow L^2(Z, (1 - s^2)^{-1})$ of a vector field $w = w(x, y, z) = (w_1, w_2, w_3)$ is given by formula

$$R^\perp w = \int\int_{P_{\xi,s}} (w_1 \xi_1 + w_2 \xi_2 + w_3 \xi_3) \, du \, dv.$$  

To obtain decomposition we need to use following connection between the normal Radon transform of a vector field and the Radon transform of a potential $f \in H^1_0(B)$:

$$\left( R^\perp (df) \right)(s, \xi) = \frac{\partial}{\partial s} \left( (Rf)(s, \xi) \right).$$  

Lemma 1. The kernel of the normal Radon transform consists of solenoidal vector fields, that is, the following equation holds $R^\perp (\text{rot}w) = 0$ with $w \big|_{\partial B} = 0$.

In other words if we know the normal Radon transform of a vector field, we can reconstruct only its potential part.

2.4. Statement of the problem

Let we have some potential vector field $d\phi \in L^2(S^1(B))$, $\phi \in H^1_0(B)$, which is given in a unit ball $B$. One has to recover this field by its known the normal Radon transform.

3. A singular value decomposition of the normal Radon transform operator

We choose the following system of polynomials as the potentials

$$\Phi_{k,n}(x, y, z) = (1 - x^2 - y^2 - z^2) H_k(x, y, z) P_n^{(k+2.5,k+1.5)}(x^2 + y^2 + z^2), \quad k, n = 0, 1, 2, \ldots,$$

where $H_k(x)$ are harmonic functions and $P_n^{(p,q)}(x)$ are Yakobi polynomials for the interval $[0, 1]$, $k, n = 0, 1, \ldots$. In spherical coordinates $(r, \theta)$ the functions have the form

$$\Phi_{k,n}(r, \theta, \phi) = (1 - r^2) r^k P_n^{(k+2.5,k+1.5)}(r^2) Y_k(\omega).$$

An application of the operator $d$ leads to a set of potential vector fields

$$T_{k,n}(x, y, z) = \text{d}\Phi_{k,n}(x, y, z).$$

Theorem 1. System of potentials (in spherical coordinates)

$$F_{k,n}(r, \theta, \phi) = \frac{\Gamma(n + k + 1.5)}{(n + 1)! \Gamma(k + 1.5)|Y_k(\omega)|} \sqrt{\frac{2n + k + 2.5}{2}} (1 - r^2) r^k P_n^{(k+2.5,k+1.5)}(r^2) Y_k(\omega)$$

forms a system of potential vector fields

$$(F_{k,n})(x, y, z) = \text{d}F_{k,n}(x, y, z),$$

which is orthonormal in space $L^2(S^1(B))$.

The following proposition is very important to get the images of the potential basis vector fields[3].

Proposition 1. Let $\nu > 0.5$, $k, n \geq 0$,

$$\Psi(\omega, s) = (1 - s^2)^{\nu - 0.5} C_{2n+k}(s) Y_k(\omega),$$
with $C^{(\nu)}_{2n+k}(s)$ – Gegenbauer polynomials. Then

$$\Phi = R^{-1}\Psi = c(n, k, \nu) \left(1 - r^2\right)^{\nu-1.5} r^k P_{n}^{(k+\nu,k+1.5)}(r^2) Y_k(\omega),$$

with $c(n, k, \nu) = \frac{(-1)^n 2^{2-2\nu} \Gamma(2n + k + 2\nu) \Gamma(k + n + 1.5)}{\sqrt{\Gamma(\nu) \Gamma(n + \nu - 0.5) \Gamma(k + 1.5)}}$ and $P_n^{(p,q)}(r^2)$— Yakobi polynomials of degree $n$ with indices $p, q$.

With usage of the Proposition 1 and the formula (1), we obtain

**Theorem 2** A system of function

$$G_{k,n} = \frac{(-1)^n \sqrt{2n + k + 2.5}}{\sqrt{(2n + k + 3)(2n + k + 2)}\|Y_k(\omega)\|} (1 - s^2) C^{(1.5)}_{2n+k+1}(s) Y_k(\omega)$$

forms an orthonormal system in space $L_2(Z, (1 - s^2)^{-1})$.

In other words we have the following relation:

$$(R^\perp F_{k,n})(s, \theta, \phi) = \sigma_{k,n} G_{k,n}(s, \theta, \phi), \quad k, n = 0, 1, 2, \ldots,$$

with $\sigma_{k,n} = \frac{2\sqrt{2}}{\sqrt{(2n + k + 2)(2n + k + 3)}}$ – singular values of the operator $R^\perp$. Therefore, a singular value decomposition of the normal Radon transform operator $R^\perp$ has the form

$$R^\perp v = \sum_{k,n=0}^{\infty} \sigma_{k,n} (v, F_{k,n})_{L_2(S^1(B))} G_{k,n},$$

but the inverse operator can be calculated by the formula

$$v = \left(R^\perp\right)^{-1} g = \sum_{k,n=0}^{\infty} \sigma_{k,n}^{-1} (g, G_{k,n})_{L_2(Z,(1-s^2)^{-1})} F_{k,n}.$$  

4. Conclusion
The algorithm can be used for solving of 3D vector tomography problem, namely for reconstruction of a potential part of a 3D vector field if its the normal Radon transforms are known. For this it needs to use truncated singular value decomposition, which is regularization of the problem.

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