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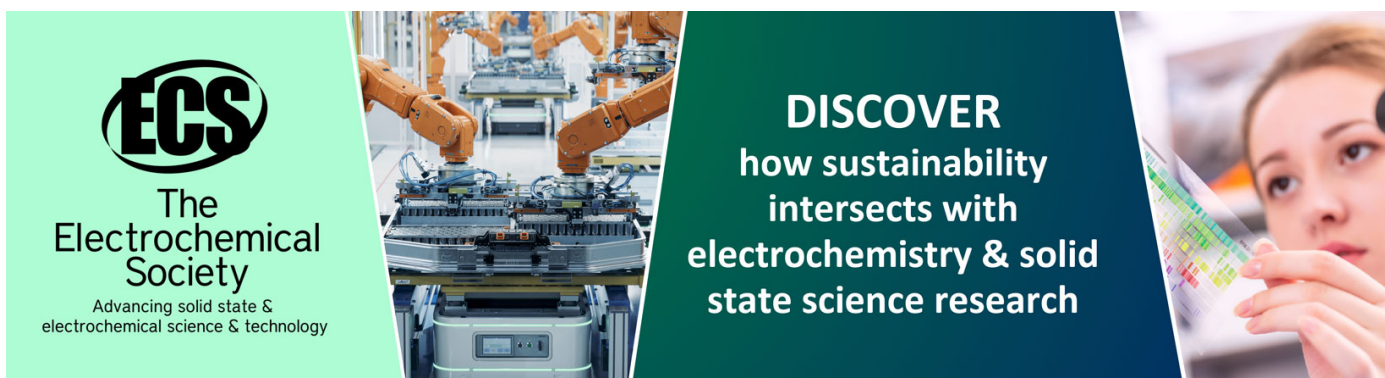
Review of stochastic mechanics

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Review of stochastic mechanics

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Abstract. Stochastic mechanics is an interpretation of nonrelativistic quantum mechanics in which the trajectories of the configuration, described as a Markov stochastic process, are regarded as physically real. The natural stochastic generalization of classical variational principles leads to a derivation of the Schrödinger equation. A brief review of the successes and failures of the theory is given, with references.

Stochastic mechanics has been the work of many people. It is described in the books [1] and [2], the second of which contains many references. It is an attempt to derive and explain nonrelativistic quantum mechanics as an emergent theory in which particle trajectories are physically real and governed by stochastic laws of motion.

1. Classical mechanics

Consider n particles in d space dimensions, with configuration space $\mathbb{R}^{dn}$. Let m_k be the mass of the k th particle and let m_{ij} be the diagonal matrix with entries m_k on the diagonal when $(k-1)d < i = j < kd$. Then m_{ij} is a Riemann metric, and even though it is flat it is convenient to use tensor notation with the summation convention.

Then $T = \frac{1}{2}m_{ij}\dot{x}^i\dot{x}^j$ (where $\dot{x}$ is the velocity) is the kinetic energy. Let V be the potential energy, so $L = T - V$ is the Lagrangian, and

$$F_i = -\frac{\partial v}{\partial x^i} + \frac{d}{dt} \frac{\partial V}{\partial v^i}$$

is the force (where $v^i = \dot{x}^i$).

The *state* of the system is (x, v) and a *dynamical variable* is a function of the state. We assume that $V = \varphi - A_i \dot{x}^i$, where φ is the *scalar potential* and A is the *covector potential*. This form of the potential energy is to ensure that the force will be a dynamical variable. From a variational equation one derives the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla^i S - A^i)(\nabla_i S - A_i) + \varphi = 0$$

with the Newton equation $F_i = m_{ij}\ddot{x}^j$.

2. Kinematics of stochastic mechanics

A stochastic process is a *Markov process* in case the past and future are conditionally independent given the present. That is, knowledge of the past gives no more information about the future than does knowledge of the present, and vice versa. The notion is time symmetric.

In stochastic mechanics, the trajectory $x(t)$ is a Markov process governed by a stochastic differential equation of the form

$$dx(t) = dw(t) + b(x(t), t)dt$$

Here w is a Wiener process (Brownian motion process) with

$$E_t dw^i(t) dw^j(t) = \hbar m^{ij} dt$$

where E_t is the conditional expectation with respect to the present at time t . Then $dw(t)$ is singular, of order $dt^{1/2}$. Such a process has continuous but nowhere differentiable trajectories.

The assumption that physical particles behave in this way is just on the borderline of being falsifiable. We can measure the position at two times t_1 and t_2 with an error given by the uncertainty principle. With a constant bigger than $\hbar$ in the diffusion tensor $\hbar m^{ij}$ we could determine that particles do not move in the way described by stochastic mechanics.

The *mean forward derivative* $DF(t)$ of a stochastic process F is defined by

$$DF(t) = \lim_{dt \rightarrow 0+} E_t \frac{F(t+dt) - F(t)}{dt}$$

and the *mean backward derivative* is

$$D_*F(t) = \lim_{dt \rightarrow 0+} E_t \frac{F(t) - F(t-dt)}{dt}$$

Then $b(x(t), t)$ is $Dx(t)$, the *mean forward velocity* of the process. It is one substitute for the derivative, which does not exist. By time reversal symmetry, there is also a *mean backward velocity* $b_*(x(t), t)$, and we also form the *current velocity* $v = \frac{1}{2}(b + b_*)$ and the *osmotic velocity* $u = \frac{1}{2}(b - b_*)$.

Let $\rho(x, t)$ be the probability density of the configuration at time t . Then we have the *osmotic equation*

$$u^i = \frac{1}{2} \frac{\nabla^i \rho}{\rho}$$

and the *current equation*

$$\frac{\partial \rho}{\partial t} = -\nabla_i (v^i \rho)$$

3. Dynamics of stochastic motion

Classical dynamics comes from a variational principle applied to action integrals. How can we formulate the action in the absence of derivatives? The contribution $\int \varphi(x(t), t)dt$ from the scalar potential is an ordinary Riemann integral, and the contribution from the covector potential can be expressed as a Fisk-Stratonovich time-symmetric stochastic integral $\int A_j(x(t), t)dx^j(t)$. The kinetic action is more subtle and was formulated by Francesco Guerra and Laura Morato [3]. There is an account of this in §9 of [2].

Briefly, let dx^i/dt be a difference quotient with $dt > 0$, not a derivative. We need to calculate

$$E_t \frac{1}{2} \frac{dx^i}{dt} \frac{dx_i}{dt}$$

to $\mathcal{O}(1)$. Let

$$W^k = \int_t^{t+dt} [w^k(r) - w^k(t)]dr$$

We find

$$dx^i dx_i = b^i b_i dt^2 + 2b^i dw_i dt + 2\nabla_k b^i W^k dw_i + o(dt^2)$$

First miracle: the term $2b^i dw_i dt$ is singular, of order $dt^{3/2}$, but $E_t 2b^i dw_i dt = 0$ since $2b^i$ lives in the present and dw_i is in the future with mean 0.

Now we use the fact that w has orthogonal increments and calculate further, finding

$$E_t \frac{1}{2} \frac{dx^i}{dt} \frac{dx_i}{dt} = \frac{1}{2} b^i b_i + \frac{1}{2} \nabla_i b^i + \frac{nd}{2dt} + o(1)$$

Second miracle: the singular term $nd/2dt$ is a constant, independent of the trajectory, so it drops out when a variational principle is applied to the action.

Let $R = \frac{\hbar}{2} \log \rho$, so $\nabla^i \rho$ is the osmotic velocity u^i . Apply the variational principle to the expected action. We find the stochastic Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2} (\nabla^i S - A_i) (\nabla_i S - A_i) + \varphi - \frac{1}{2} \nabla^i R \nabla_i R - \frac{\hbar}{2} \nabla^i \nabla_i R = 0$$

which without the terms containing R is the classical Hamilton-Jacobi equation. Write the current equation in terms of R and S and find

$$\frac{\partial R}{\partial t} + \nabla_i R (\nabla^i S - A^i) + \frac{\hbar}{2} \nabla^i \nabla_i S - \frac{\hbar}{2} \nabla_i A^i = 0$$

We have a pair of coupled nonlinear partial differential equations.

Third miracle: Let $\psi = e^{\frac{1}{\hbar}(R+iS)}$. Then these equations are equivalent to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{1}{2} \left(\frac{\hbar}{i} \nabla^j - A^j \right) \left(\frac{\hbar}{i} \nabla_j - A_j \right) + \varphi \right] \psi$$

We also have the *stochastic Newton equation* $F_i = m_{ij} a^j$ where a is the *mean acceleration* given by $\frac{1}{2} (DD_* x + D_* Dx)$.

We can apply the same procedure to any Riemannian manifold. We find the Schrödinger equation with an additional term $(\hbar^2/12)\bar{R}$ where $\bar{R}$ is the scalar curvature. This is the Bryce DeWitt term [4].

4. Successes of stochastic mechanics

Here is a list of the main successes of stochastic mechanics.

- A classical derivation of the Schrödinger equation, by Guerra and Morato [3].
- The probability density ρ of the Markov process agrees with $|\psi|^2$ at all times.
- A stochastic explanation of the relation between momentum and the Fourier transform of the wave function, by David Shucker [5].
- A proof of the existence of the Markov process under the physically natural assumption of finite action, by Eric Carlen [6]. This is perhaps the most technically demanding work in the entire subject.
- A stochastic explanation of why identical particles satisfy either Bose-Einstein or Fermi-Dirac statistics if $d \geq 3$, with parastatistics possible if $d = 2$. This is not contained in §20 of [2], but it follows from the discussion there.
- A stochastic explanation of spin and why it is integral or half-integral, work of Thaddeus Dankel [7], Timothy Wallstrom [8], and of Daniela Dohrn and Francesco Guerra jointly [9].
- If the force is time-independent, the expected stochastic energy $E_t(\frac{1}{2}u^i u_i + \frac{1}{2}v^i v_i + \varphi)$ is conserved; see §14 of [2].
- A stochastic picture of the two-slit experiment, explaining how particles have trajectories going through just one slit or the other, but nevertheless produce a probability density as for interfering waves; see §17 of [2].

5. Failures of stochastic mechanics

In quantum mechanics, if there are two dynamically uncoupled systems, an alteration of the second system in no way affects the first, even if the two systems are entangled. This is not so in stochastic mechanics. With two dynamically uncoupled particles, a force applied to one can immediately affect the motion of the other, in a way independent of their spatial separation; see §23 of [2]. This makes it unrealistic to regard the trajectories as physically real.

There is a more serious problem. Since $\rho = |\psi|^2$ at all times, stochastic mechanics gives the same prediction as quantum mechanics for a measurement performed at a single time. But it can give wrong predictions for measurements performed at two different times; see Chapter 10 of [10]. Consider two entangled but dynamically uncoupled harmonic oscillators. Let $X_i(t)$ be the Heisenberg position operator of oscillator i at time t . Each is periodic in t , so the correlation of $(X_1(t_1), X_2(t_2))$ does not decay as $t_2 \rightarrow \infty$. Let $x_i(t)$ be the position of oscillator i at time t according to stochastic mechanics. Then $x_i(t)$ has the same probability distribution as $X_i(t)$ for each i and each t , but $(x_1(t_1), x_2(t_2))$ does not have the same probability distribution as $(X_1(t_1), X_2(t_2))$. In fact, the correlation of $(x_1(t_1), x_2(t_2))$ decays to 0 as $t_2 \rightarrow \infty$. The oscillators are uncoupled, so $X_1(t_1)$ and $X_2(t_2)$ commute, and according to quantum mechanics, the probability distribution is that of $(X_1(t_1), X_2(t_2))$. If $(x_1(t_1), x_2(t_2))$ represented the real physical situation, theirs would be the probability distribution. Thus stochastic mechanics and quantum mechanics give different predictions for the result. Why do I not suggest that the experiment be done? Because if a record of the observation of the first oscillator at time t_1 is made by some physical means, and similarly for the second oscillator, and the two records are compared at a common later time t_3 , this is an observation at a single time, for which quantum mechanics and stochastic mechanics agree. The nonlocality of stochastic mechanics conspires to bring the records into agreement.

How can a theory be so right and yet so wrong? The most natural explanation is that stochastic mechanics is an approximation to a correct theory of quantum mechanics as emergent. But what is the correct theory?

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