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Chaotic geodesic motion around a black hole and disc

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Abstract. We study the chaotic features of time-like free motion around a Schwarzschild black hole, induced by the presence of a static, axially and reflectionally symmetric thin disc or ring. We describe the field by an exact (Weyl-type) solution of Einstein's equations and use Poincaré sections and time-series analysis in order to recognise overall tendencies in the evolution of phase portrait as well as changes that can be traced down to single orbits and their segments.

1. Introduction

Black holes assumed in galactic nuclei certainly dominate their surroundings gravitationally. However, even in non-active nuclei like that in our Galaxy, there is enough mass (in stars and gas) to influence *its own* (and each others') motion in a non-negligible way. One of tiny but robust effects is that the geodesic motion, which is integrable in the field of an *isolated* Schwarzschild or Kerr black hole, becomes chaotic when some additional matter is present. Actually, even if the additional matter does not break the space-time symmetries of the centre, the geodesic motion in the resulting field is no longer completely integrable, so it may become chaotic. This is important for the long-term evolution of the system and may have direct observational consequences, namely it can be recognised in electromagnetic and gravitational radiation emitted by the orbiting particles.

Here we consider exact static and axially symmetric space-times generated by a Schwarzschild black hole surrounded by a thin, reflectionally symmetric disc or ring, in order to show how the dynamics of free timelike motion changes from regular to chaotic (and possibly back) when parameters of the system are changed. The information revealed on Poincaré sections is supplemented by and compared with that provided by time series of phase variables. In particular, we illustrate whether and how the overall tendencies in the evolution of geodesic dynamics correspond to changes taking place on the level of individual orbits.

2. Weyl space-times

Static and axially symmetric vacuum regions of space-time can be described, in Weyl coordinates (t, ρ, z, ϕ) , by the Weyl metric

$$ds^2 = -e^{2\nu(\rho,z)} dt^2 + \rho^2 e^{-2\nu(\rho,z)} d\phi^2 + e^{2\lambda(\rho,z)-2\nu(\rho,z)} (d\rho^2 + dz^2). \quad (1)$$

The Einstein equations yield the Laplace equation for ν , hence this function (representing gravitational potential) superposes linearly. The other unknown function λ can be obtained

by line integration of

$$\lambda = \int_{\text{axis}}^{\rho, z} \rho \left[(\nu_{,\rho}^2 - \nu_{,z}^2) d\rho + 2\nu_{,\rho} \nu_{,z} dz \right]. \quad (2)$$

We have specifically considered the superposition of an (initially) Schwarzschild black hole with three different types of additional source — the inverted Morgan-Morgan counter-rotating discs [1], the discs with power-law density profile [2] and the Bach-Weyl rings [3]. All these sources have finite mass and the disc families were shown to contain physically realistic cases within non-negligible parameter ranges. The only serious limitation of the Weyl solutions is the lack of rotation. Indeed, the systems we would like to simulate (astrophysical black holes with accretion discs or rings) are likely to rotate rapidly rather than being static. The *exact* inclusion of rotation would also be interesting because it has been indicated on approximate solutions in the literature that dragging effects can alter (probably attenuate rather than amplify) the chaotic features significantly. Unfortunately, the superposition of sources is much more difficult in a non-static (stationary) case and no solution of such a type exists that could be submitted to a similar extensive numerical study.

The Weyl space-times are stationary (even static) and axially symmetric, so the energy at infinity per unit mass $E = -g_{tt}u^t$ and the angular momentum at infinity per unit mass $L = g_{\phi\phi}u^\phi$ conserve along the geodesics. In addition, one has normalization of four-velocity, but there is no other isolating integral in general, so the (four-dimensional) motion is not completely integrable and may become chaotic.

The phenomenon of chaos has been detected and studied in many systems, notably in physics, mathematics, biology and economics. It is of special interest in systems of general relativity where the configuration space is itself described by a non-linear theory. Immediately a question thus arises whether and how general relativity contributes to chaos and whether this contribution stems mainly from special relativistic dynamics or rather from dragging effects and spatial curvature. These questions have also practical aspects, because the general relativistic description of a particular system of sources is much more difficult than in Newton's theory and it is often necessary to resort to approximations like pseudo-Newtonian potentials or weak-field limit. It has already been shown in the literature that geodesic dynamics clearly distinguishes between these approximations and full general relativistic system, which indicates that the approximations typically fail to simulate the long-term behaviour correctly.

3. Results

There are many methods how to diagnose and classify chaos. However, in general relativity the usage of some of them is not easy or does not seem to have good sense, because they are usually applied in a particular coordinate system while in relativity coordinates (time, in particular) do not have an invariant meaning. Such queries mainly occur when trying to quantify “sensitive dependence on initial conditions”, namely to compare how fast nearby orbits diverge in various directions. In particular, this requires to compare certain tangent vectors (namely velocities of the neighbouring particles) at different points of the manifold which is not a unique concept as one can use different ways of how to transport the quantities to the same point. It is only to be clarified how to compute, in general relativity, correctly and efficiently the various exponents proposed as a measure of the orbital divergence. It has been proved [4], fortunately, that the sign of the mostly used *Lyapunov* exponents is independent of coordinates, although their value is otherwise not invariant. Recently several new suggestions of how to practically compute the Lyapunov exponents have appeared [5]. We plan to try them in the future.

On the other hand, some of the methods can be adopted to general relativity quite straightforwardly. In particular, the Poincaré surfaces of section are very efficient in revealing the properties of phase space for systems where one is able to find the whole trajectories. Quite naturally, we began the study of geodesics in the fields of black holes surrounded by

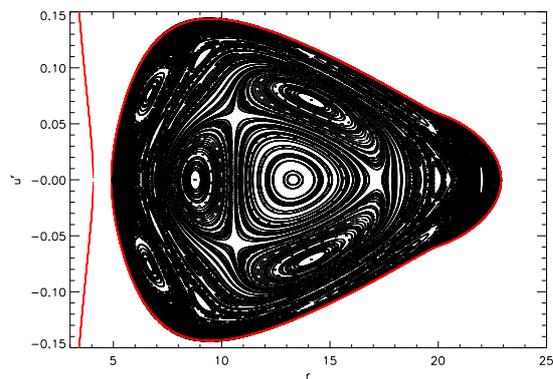


Figure 1. Poincaré's surface of section (equatorial plane) for geodesic motion ($E = 0.954$, $L = 3.75M$) around BH with first iMM disc ($r = 20M$, $\mathcal{M} = 0.5M$).

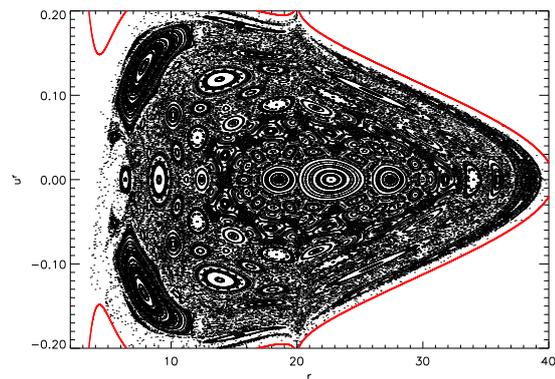


Figure 2. Poincaré's surface of section (equatorial plane) for geodesic motion ($E = 0.977$, $L = 3.75M$) around BH with Bach-Weyl ring ($r = 20M$, $\mathcal{M} = 0.08M$).

discs or rings by plotting the Poincaré sections [6]; two examples for different external sources are given in figure 1 and figure 2. However, it is problematic to compare the Poincaré-section method directly with astrophysical observational data, because the latter mostly do not contain information about all three directions. The data are usually represented by time series of dynamical quantities (e.g. projection of position onto the celestial sphere). In order to decipher them, one tries to recognise their components showing different degrees of regularity, in particular, to decompose the “signal” that is probable to originate from a deterministic system from what appears to be just noise. There exist various routines that can provide the degree of regularity/stochasticity of the time series. We specifically tested the technique proposed in [7] on the data computed for our hole-disc system.

The method lies in analysis of the trajectory determined by a given time series $a(t)$ in a 3D space whose axes are $a(t)$, $a(t - \tau)$ and $a(t - 2\tau)$, where $a(t - \tau)$ and $a(t - 2\tau)$ are “clones” of the original series once and twice delayed by a given time shift τ . The space thus obtained is divided into $m \times m \times m$ boxes and the passages of the trajectory through the boxes is recorded in terms of a normalised directional vector pointing from the point where the trajectory enters a particular box to the point where it leaves the box. The vectors provided by passages through each particular (j -th) box are then summed into a “total j -th box vector” V_j and this vector's norm is computed and divided by the number of passages through the j -th box n_j (it is assumed that the time series is long enough to yield many transits through the box). In autonomous deterministic systems the dynamics is given by equations of motion which depend on position in phase space, so when the boxes are small enough, the tangent vectors recorded for a given box point in similar directions, so their normalised sum $|V_j/n_j|$ is close to one. On the contrary, for random time series, the norm of this average tangent vector decreases with n_j as the average displacement per step for a random walk in a 3D space, i.e., for (large) n_j , as

$$\bar{R}_{n_j}^3 = \frac{4}{(6\pi)^{1/2}} n_j^{-1/2}. \quad (3)$$

For a suitably chosen τ , a deterministic signal yields $|V_j/n_j|$ close to one for arbitrary n_j , whereas for a random signal the value falls off as $n_j^{-1/2}$.

Finally, one can compute a weighted average of $|V_j/n_j|$ over all traversed boxes for a given τ ,

$$\bar{\Lambda}(\tau) = \left\langle \frac{(V_j/n_j)^2 - (R_{n_j}^3)^2}{1 - (R_{n_j}^3)^2} \right\rangle. \quad (4)$$

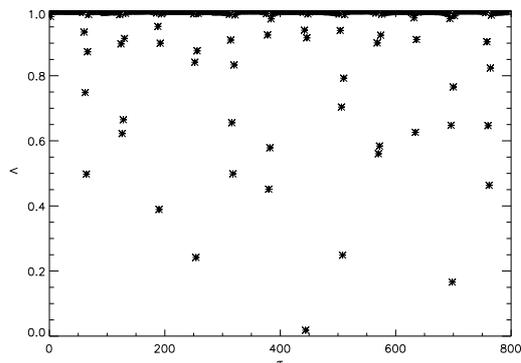


Figure 3. Weighted average of directional vectors.

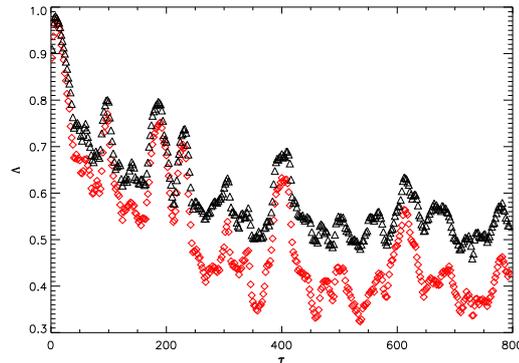


Figure 4. Weighted average of directional vectors ($m = 20$ – red, $m = 32$ – black).

The behaviour of this quantity in dependence on τ is again very different for regular and random trajectories, and we were curious to check what it does for chaotic ones. For regular trajectories the value proved to remain close to one for increasing τ except for certain specific values related to the period of motion. For chaotic trajectories the value turns out to decrease with τ , with some peaks occurring on the curve. The course of $\bar{\Lambda}$ for regular trajectory is plotted in figure 3 while for the chaotic one it is given in figure 4. Both these trajectories have the same global parameters (first inverted Morgan-Morgan disc with $r = 20M$, $\mathcal{M} = 0.5M$ and constants of the motion $E = 0.953$, $L = 3.75M$) but different initial conditions.

The value of $\bar{\Lambda}(t)$ depends on the sampling frequency of the time series (on how dense the data points are) and also on the box size, but the latter dependence (on m) is weak within the range where the elementary boxes are small enough to capture the folds of the trajectory and at the same time large enough to catch at least several points of the trajectory. This is shown in figure 4, where the red data points are computed for $m = 20$ and the black ones are computed for $m = 32$ and their behaviour is very similar.

4. Conclusions

We have shown on Poincaré surfaces of section that in the system of a static black hole surrounded by a thin static and axially symmetric discs or ring the geodesic flow could be chaotic. As usual in the weakly perturbed systems, the chaotic layers spread over the phase space from the unstable periodic points and separatrices between regular islands. We checked the degree of individual-orbit chaoticity by analysing the corresponding time series — we computed their power spectra and also applied the method of computing the weighted average of directional transit vectors suggested by [7].

Acknowledgments

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