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Topics on n-ary algebras

J.A. de Azcárraga

Dept. of Theoretical Physics and IFIC (CSIC-UVEG), University of Valencia, 46100-Burjassot (Valencia), Spain

E-mail: `j.a.de.azcarraga@ific.uv.es`

J. M. Izquierdo

Dept. of Theoretical Physics, University of Valladolid, 47011-Valladolid, Spain

E-mail: `izquierd@fta.uva.es`

Abstract. We describe the basic properties of two n-ary algebras, the Generalized Lie Algebras (GLAs) and, particularly, the Filippov ($\equiv n$-Lie) algebras (FAs), and comment on their n-ary Poisson counterparts, the Generalized Poisson (GP) and Nambu-Poisson (N-P) structures. We describe the Filippov algebra cohomology relevant for the central extensions and infinitesimal deformations of FAs. It is seen that semisimple FAs do not admit central extensions and, moreover, that they are rigid. This extends the familiar Whitehead's lemma to all $n \geq 2$ FAs, $n = 2$ being the standard Lie algebra case. When the n-bracket of the FAs is no longer required to be fully skewsymmetric one is lead to the n-Leibniz (or Loday's) algebra structure. Using that FAs are a particular case of n-Leibniz algebras, those with an anticommutative n-bracket, we study the class of n-Leibniz deformations of simple FAs that retain the skewsymmetry for the first $n-1$ entires of the n-Leibniz bracket.

1. Introduction

The Jacobi identity (JI) for Lie algebras $\mathfrak{g}$, $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$, may be looked at in two ways. First, it may be seen as a consequence of the associativity of the composition of the generators in the Lie bracket. Secondly, it may be viewed as the statement that the adjoint map is a derivation of the Lie algebra, $ad_X\,[Y,Z] = [ad_X\,Y, Z] + [Y, ad_X\,Z]$.

A natural generalization is to consider n-ary algebras $\mathfrak{H}$. In its more general form, the problem goes back to the multioperator linear algebras of Kurosh (see [1, 2]). In our Lie algebra context, we have to look for the possible characteristic identities that a fully antisymmetric n-ary bracket,

$$(X_1, \ldots, X_n) \in \mathfrak{H} \times \cdots \times \mathfrak{H} \mapsto [X_1, \ldots, X_n] \in \mathfrak{H} \;, \tag{1.1}$$

may satisfy (the n-Leibniz case is discussed in Sec. 7). When $n > 2$ the above two aspects of the JI are no longer equivalent and two n-ary generalizations of the Lie algebra structure immediately suggest themselves. These depend on which aspect of the $n = 2$ JI is retained to define their corresponding *characteristic identity*. This leads to[1]:

[1] It is also possible to consider intermediate possibilities between the two below: see [3, 4].

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(a) ***Higher order Lie algebras or generalized Lie algebras (GLAs)*** $\mathcal{G}$.
They were proposed independently in [5–7] and [8–11]. Their bracket is defined by the full antisymmetrization

$$[X_{i_1}, \ldots, X_{i_n}] := \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} X_{i_{\sigma(1)}} \ldots X_{i_{\sigma(n)}} \ . \tag{1.2}$$

For n even, this definition implies the *generalized Jacobi identity* (GJI)

$$\sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \Big[[X_{i_{\sigma(1)}}, \ldots, X_{i_{\sigma(n)}}], X_{i_{\sigma(n+1)}} ..., X_{i_{\sigma(2n-1)}} \Big] = 0 \ , \quad i = 1, \ldots, \dim \mathcal{G} \ , \tag{1.3}$$

which follows from the associtivity of the products in (1.2) (for n odd, the *r.h.s* is $n!(n-1)!\,[X_{i_1}, \ldots, X_{i_{2n-1}}]$ rather than zero giving rise to a *mixed GJI*[2]). Chosen a basis of $\mathcal{G}$, the bracket may be written as $[X_{i_1}, \ldots, X_{i_{2p}}] = \Omega_{i_1 \ldots i_{2p}}{}^j X_j$, where the $\Omega_{i_1 \ldots i_{2p}}{}^j$ are the *GLA structure constants*. Thus, for n even, a GLA is defined by an n-linear antisymmetric bracket (1.2) closed in $\mathcal{G}$ that satisfies the GJI (1.3).

(b) ***n-Lie or Filippov algebras (FAs)*** $\mathfrak{G}$.
The characteristic identity that generalizes the $n = 2$ JI is the *Filippov identity* (FI) [13]

$$[X_1, \ldots, X_{n-1}, [Y_1, \ldots Y_n]] = \sum_{a=1}^{n} [Y_1, \ldots Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots Y_n] \ . \tag{1.4}$$

Let $\mathscr{X} = (X_1, \ldots, X_{n-1})$ be antisymmetric in its $(n-1)$ entries, $\mathscr{X} \in \wedge^{n-1}\mathfrak{G}$. The $\mathscr{X}$s will be called [14] *fundamental objects*, and act on $\mathfrak{G}$ by

$$\mathscr{X} \cdot Z \equiv ad_{\mathscr{X}} Z := [X_1, \ldots, X_{n-1}, Z] \quad \forall Z \in \mathfrak{G} \ . \tag{1.5}$$

Thus, the FI just expresses that (note the dot) $\mathscr{X} \cdot \equiv ad_{\mathscr{X}}$ is a derivation of the FA bracket,

$$ad_{\mathscr{X}} [Y_1, \ldots, Y_n] = \sum_{a=1}^{n} [Y_1, \ldots, ad_{\mathscr{X}} Y_a, \ldots, Y_n] \ . \tag{1.6}$$

Chosen a basis, $\mathfrak{G}$ may be defined by the *FA structure constants*,

$$[X_{a_1} \ldots X_{a_n}] = f_{a_1 \ldots a_n}{}^d X_d \ , \quad a, d = 1, \ldots \dim \mathfrak{G} \ , \tag{1.7}$$

in terms of which the FI is written as

$$f_{b_1 \ldots b_n}{}^l \, f_{a_1 \ldots a_{n-1} l}{}^s = \sum_{k=1}^{n} f_{a_1 \ldots a_{n-1} b_k}{}^l \, f_{b_1 \ldots b_{k-1} l b_{k+1} \ldots b_n}{}^s \ . \tag{1.8}$$

Note. There is a considerable confusion in the literature concerning the names of the above two n-ary algebras and those of the characteristic identities they satisfy; we refer to Sec. 1 in [15] for a justification of the terminology we advocate.

[2] When more than two nested brackets are used, other identities follow from associativity; see [12].

2. Some definitions and properties of FAs

The definitions of ideals, solvable ideals and semisimple Lie algebras can be extended to the $n > 2$ case [13, 16–18] following the pattern of the Lie algebra one (for a review of FAs and their applications with further references, see [15]). For instance, a subalgebra $I \subset \mathfrak{G}$ is an ideal of $\mathfrak{G}$ if $[X_1, \ldots, X_{n-1}, Z] \subset I \quad \forall X \in \mathfrak{G}\,, \forall Z \in I$. An ideal I is (n-)solvable if the series

$$I^{(0)} := I \ , \ I^{(1)} := [I^{(0)}, \ldots, I^{(0)}] \ , \ldots , \ I^{(s)} := [I^{(s-1)}, \ldots, I^{(s-1)}] \ , \ldots \tag{2.9}$$

terminates. A FA $\mathfrak{G}$ is then semisimple if it does not have solvable ideals, and simple if $[\mathfrak{G}, \ldots, \mathfrak{G}] \neq \{0\}$ and does not contain non-trivial ideals. There is also a Cartan-like criterion for semisimplicity [18]: a FA is semisimple iff

$$k(\mathscr{X}, \mathscr{Y}) = k(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}) := Tr(ad_{\mathscr{X}} ad_{\mathscr{Y}}) \tag{2.10}$$

is non-degenerate in the sense that

$$k(Z, \mathfrak{G}, \overset{n-2}{\ldots}, \mathfrak{G}, \mathfrak{G}, \overset{n-1}{\ldots}, \mathfrak{G}) = 0 \ \Rightarrow \ Z = 0 \ . \tag{2.11}$$

A semisimple FA is the sum of simple ideals $\mathfrak{G} = \mathfrak{G}_{(1)} \oplus \cdots \oplus \mathfrak{G}_{(k)}$.

The derivations of a FA $\mathfrak{G}$ generate a Lie algebra. To see it, introduce first the composition of fundamental objects [19],

$$\mathscr{X} \cdot \mathscr{Y} := \sum_{a=1}^{n-1} (Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots, Y_{n-1}) \ , \tag{2.12}$$

which reflects that $\mathscr{X}$ acts as a derivation. It is then seen that the FI implies that

$$\mathscr{X} \cdot (\mathscr{Y} \cdot \mathscr{Z}) - \mathscr{Y} \cdot (\mathscr{X} \cdot \mathscr{Z}) = (\mathscr{X} \cdot \mathscr{Y}) \cdot \mathscr{Z} \qquad , \quad \forall \mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \wedge^{n-1}\mathfrak{G} \ , \tag{2.13}$$

$$ad_{\mathscr{X}} ad_{\mathscr{Y}} Z - ad_{\mathscr{Y}} ad_{\mathscr{X}} Z = ad_{\mathscr{X} \cdot \mathscr{Y}} Z \qquad , \quad \forall \mathscr{X}, \mathscr{Y} \in \wedge^{n-1}\mathfrak{G}, \forall Z \in \mathfrak{G} \ , \tag{2.14}$$

which means that $ad_{\mathscr{X}} \in \mathrm{End}\,\mathfrak{G}$ satisfies $ad_{\mathscr{X} \cdot \mathscr{Y}} = -ad_{\mathscr{Y} \cdot \mathscr{X}}$. These two identities show that the inner derivations $ad_{\mathscr{X}}$ associated with the fundamental objects $\mathscr{X}$ generate (the ad map is not necessarily injective) an ordinary Lie algebra, the Lie algebra $\mathrm{Lie}\,\mathfrak{G}$ associated with the FA $\mathfrak{G}$.

An important type of FAs is the class of metric Filippov algebras. These are relevant in physical applications (where a scalar product is needed), as in the Bagger-Lambert-Gustavsson model [20–22] in M-theory. These FAs are endowed with a metric $< \ , \ >$, $\langle Y \, , \, Z \rangle = g_{ab} Y^a \, Z^b$, $\forall \ Y, Z \in \mathfrak{G}$ which is invariant *i.e.*,

$$\begin{aligned} \mathscr{X} \cdot \langle Y \, , \, Z \rangle \ &= \ \langle \mathscr{X} \cdot Y \, , \, Z \rangle + \langle Y \, , \, \mathscr{X} \cdot Z \rangle \\ &= \ \langle [X_1, \ldots, X_{n-1}, Y] \, , \, Z \rangle + \langle Y \, , \, [X_1, \ldots, X_{n-1}, Z] \rangle = 0 \ . \end{aligned} \tag{2.15}$$

As a result, the structure constants with all indices down $f_{a_1 \ldots a_{n-1} bc}$ are completely antisymmetric since the invariance of g above implies $f_{a_1 \ldots a_{n-1} b}{}^l \, g_{lc} + f_{a_1 \ldots a_{n-1} c}{}^l \, g_{bl} = 0$. The $f_{a_1 \ldots a_{n+1}}$ define a skewsymmetric invariant tensor f under the action of $\mathscr{X}$, since the FI implies

$$\sum_{i=1}^{n+1} f_{a_1 \ldots a_{n-1} b_i}{}^l \, f_{b_1 \ldots b_{i-1} l b_{i+1} \ldots b_{n+1}} = 0 \qquad \text{or} \qquad L_{\mathcal{X}} . f = 0 \ . \tag{2.16}$$

3. Examples of n-ary algebras

3.1. Examples of GLAs

Let n be *even*, $n = 2p$. We look for structure constants $\Omega_{i_1\ldots i_{2p}}{}^j$ satisfying the GJI (1.3) *i.e.*, such that

$$\Omega_{[j_1\ldots j_{2p}}{}^l\Omega_{j_{2p+1}\ldots j_{4p-1}]l}{}^s = 0 \quad , \quad i = 1, \ldots, \dim \mathcal{G}. \tag{3.17}$$

It turns out [7,6] that given a simple compact Lie algebra, the coordinates of the (odd) cocyles for the corresponding Lie algebra cohomology satisfy the GJI identity (3.17). Thus, these provide the structure constants of an infinity of GLAs with brackets with $n = 2(m_i-1)$ entries, where $i = 1, \ldots, \ell$, ℓ is the rank of the algebra and the m_i are the ranks of the ℓ Casimir-Racah primitive symmetric invariants associated with the corresponding $(2m_i - 1)$-cocycles; see further [15,23].

3.2. Examples of FAs

A very important class of finite Filippov algebras is provided by the real simple n-Lie algebras defined on $(n+1)$-dimensional vector spaces [13]. Chosen a basis $\{e_a\}$ $(a = 1, \ldots, n+1)$, their n-brackets are given by

$$[e_1 \ldots \hat{e}_a \ldots e_{n+1}] = (-1)^{i+1}\varepsilon_a e_a \quad \text{or} \quad [e_{a_1} \ldots\ e_{a_n}] = (-1)^n \sum_{a=1}^{n+1} \varepsilon_a \epsilon_{a_1\ldots a_n}{}^a e_a \ , \tag{3.18}$$

where, using Filippov's notation, the $\varepsilon_a = \pm 1$ are sign factors. In particular, the Euclidean $(\varepsilon_a = +1)$ simple FAs A_{n+1} are constructed on Euclidean $(n+1)$-dimensional vector spaces. Thus, in contrast with the $n = 2$ (Lie) algebra case, simple n-Lie algebras have a very rigid structure for $n \geq 3$: they reduce to the Euclidean (A_{n+1}) and Lorentzian $(A_{s,t}\ ,\ s+t = n+1)$ generalizations of the $n = 2$ $so(3)$ and $so(1,2)$ Lie algebras, $[e_i, e_j] = \sum_k \varepsilon_k \epsilon_{ijk} e_k$, $i, j, k = 1, 2, 3$.

There are also infinite-dimensional FAs that generalize the ordinary Poisson algebra by means of the bracket of n functions $f_i = f_i(x_1, x_2, \ldots, x_n)$ defined by

$$[f_1, f_2, \ldots, f_n] := \epsilon^{i_1\ldots i_n}_{1\ldots n}\ \partial_{i_1} f^1 \ldots \partial_{i_n} f^n = \left| \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x^1, x^2, \ldots, x^n)} \right| \quad . \tag{3.19}$$

This bracket was considered by Nambu [24] (who discussed it specially for the $n = 3$ case) and by Filippov [13]. The above Jacobian n-bracket satisfies the FI, which can be checked *e.g.* by using the 'Schouten identities' trick; we denote the resulting FA by $\mathfrak{N}$. These FAs are also metric FAs. For the simple infinite-dimensional FAs see further [25] and references therein.

For $n = 2$, GLAs, FAs and Lie algebras coincide.

4. n-ary Poisson structures

Both GLAs and FAs have their n-ary Poisson structure counterparts. These satisfy the associated GJI and FI characteristic identities, to which Leibniz's rule is added.

(a) ***Generalized Poisson structures (GPS)***

The generalized Poisson structures [5,6] (GPS) are naturally introduced for $n = 2s$ even (see [26] for n odd and [27] for the Z_2-graded case). They are defined by brackets $\{f_1, \ldots, f_n\}$ where the f_i, $i = 1\ldots, n = 2s$, are functions on a manifold. They are fully antisymmetric

$$\{f_1, \ldots, f_i, \ldots, f_j, \ldots, f_n\} = -\{f_1, \ldots, f_j, \ldots, f_i, \ldots, f_n\} \ , \tag{4.20}$$

satisfy Leibniz's rule ,

$$\{f_1, \ldots, f_{n-1}, gh\} = g\{f_1, \ldots, f_{n-1}, h\} + \{f_1, \ldots, f_{n-1}, g\}h \ , \tag{4.21}$$

and the characteristic identity of the GLAs, the GJI (1.3), which now reads

$$\sum_{\sigma\in S_{4s-1}} (-1)^{\pi(\sigma)}\{f_{\sigma(1)},\ldots,f_{\sigma(2s-1)},\{f_{\sigma(2s)},\ldots,f_{\sigma(4s-1)}\}\} = 0 \ . \tag{4.22}$$

As with ordinary Poisson structures, there are linear GPS given *e.g.* by the coordinates of the primitive, odd cocyles of the compact simple $\mathfrak{g}$. *Linear GPS* are defined by *linear GPS tensors i.e.*, by multivectors of the form

$$\Lambda = \frac{1}{(2m-2)!}\Omega_{i_1\ldots i_{2m-2}.}{}^{\sigma} x_\sigma \partial^{i_1}\wedge\cdots\wedge\partial^{i_{2m-2}} \tag{4.23}$$

which have zero Schouten-Nijenhuis bracket with themselves [6, 7]. Indeed, as it may be checked, $[\Lambda,\Lambda]_{SN}=0$ expresses the GJI (eq. (3.17)); this is satisfied when the $\Omega_{i_1\ldots i_{2m-2}}{}^{\sigma}$ are the $(2m-1)$-cocycle coordinates [6, 7]. In fact, all the $(2m_i-2)$-GLAs associated with the simple Lie algebras cohomology $(2m_i-1)$-cocycles define linear GPS.

(b) *Nambu-Poisson structures (N-P)*
These are defined by relations (4.20) and (4.21), but now the characteristic identity is the FI,

$$\begin{aligned}&\{f_1,\ldots,f_{n-1},\{g_1,\ldots,g_n\}\} = \{\{f_1,\ldots,f_{n-1},g_1\},g_2,\ldots,g_n\} + \\ &\{g_1,\{f_1,\ldots,f_{n-1},g_2\},g_3,\ldots,g_n\} + \cdots + \{g_1\ldots,g_{n-1},\{f_1,\ldots,f_{n-1},g_n\}\} \ .\end{aligned} \tag{4.24}$$

N-P structures were studied in general in [28].

For $n=2$, the two n-ary Poisson structures above reproduce the standard Poisson one.

The Filippov identity for the jacobian of n functions was first written by Filippov [13], and later by Sahoo and Valsakumar [29] and by Takhtajan [28] (who called it fundamental identity) in the context of Nambu mechanics [24]. Physically, the FI is a consistency condition for the time evolution [29, 28], which is given in terms of $(n-1)$ 'hamiltonian' functions that determine an $ad_{\mathscr{X}}$ derivation of the Nambu FA $\mathfrak{N}$. Every even N-P structure is also a GPS, but the converse does not hold (see [15]).

As it is the case of the finite-dimensional FAs, the $n>2$ N-P Poisson structures are extremely rigid; the N-P tensors defining them have the property of being decomposable *i.e.*, they may be given locally by $\partial_{x_1}\wedge\partial_{x_2}\wedge\cdots\wedge\partial_{x_n}$ [30, 19] so that the 'canonical form' of the N-P bracket has the form (3.19) (see [15] for more references on this point).

The question of the quantization of N-P mechanics has been the subject of a vast amount of literature. It is fair to say that (for arbitrary $n>2$) it remains a problem in general, aggravated by the fact that there are not so many physical examples of N-P mechanical systems waiting to be quantized when $n\neq 2$. We just refer here to [26,31,32] and to [15] for further discussion and references.

5. Central extensions and deformations of FAs

It is well known that the Whitehead lemma for semisimple Lie algebras states the vanishing of the second cohomology groups, $H_0^2(\mathfrak{g})=0$, $H_\rho^2(\mathfrak{g},\mathfrak{g})=0$, where ρ is a representation of $\mathfrak{g}$ (in particular, ad or trivial, $\rho=0$). Hence, semisimple Lie algebras do not admit non-trivial central extensions and are moreover rigid (non-deformable) since their central extensions and infinitesimal deformations are governed, respectively, by $H_0^2(\mathfrak{g})$ and $H_{ad}^2(\mathfrak{g},\mathfrak{g})$. Let us now turn to the $n>2$ FA case [14].

5.1. Central extensions of a FA

Given a Filippov algebra $\mathfrak{G}$ with n-bracket (1.7), a central extension $\widetilde{\mathfrak{G}}$ of $\mathfrak{G}$ is a FA of the form

$$\begin{aligned}&[\widetilde{X}_{a_1},\ldots,\widetilde{X}_{a_n}] := f_{a_1\ldots a_n}{}^{d}\,\widetilde{X}_d + \alpha^1(X_1,\ldots,X_n)\Xi \ , \\ &[\widetilde{X}_1,\ldots,\widetilde{X}_{n-1},\Xi] = 0 \quad , \quad \widetilde{X}\in\widetilde{\mathfrak{G}} \ , \ \alpha^1\in\wedge^{n-1}\mathfrak{G}^*\wedge\mathfrak{G}^* \ ,\end{aligned} \tag{5.25}$$

where $\mathfrak{G}^*$ is the dual of the $\mathfrak{G}$ vector space. If we now introduce p-cochains as maps

$$\alpha^p \in \wedge^{n-1}\mathfrak{G}^* \otimes \cdots \otimes \wedge^{n-1}\mathfrak{G}^* \wedge \mathfrak{G}^* \ , \ \alpha^p : (\mathscr{X}_1, \ldots, \mathscr{X}_p, Z) \mapsto \alpha^p(\mathscr{X}_1, \ldots, \mathscr{X}_p, Z) \ , \tag{5.26}$$

the above $\alpha^1(X_1, \ldots, X_n) = \alpha^1(\mathscr{X}, X_n)$ is a one-cochain. Note that the order of the p-cochains α^p for the cohomology of FAs $\mathfrak{G}$ ($n \geq 3$) is naturally defined as the number p of fundamental objects among the arguments of the cochain (for a Lie algebra $\mathfrak{g}$, $\mathscr{X} = X$ and p counts the number of algebra elements so that the α above would be a two- rather than a one-cocyle on $\mathfrak{g}$).

Since the centrally extended $\widetilde{\mathfrak{G}}$ is a FA, the FI for the n-bracket in $\widetilde{\mathfrak{G}}$ implies that the one-cochain $\alpha^1(\mathscr{X}, Z)$ in (5.25) (with $X_n = Z$) has to satisfy the condition

$$\alpha^1(\mathscr{X}, \mathscr{Y} \cdot Z) - \alpha^1(\mathscr{X} \cdot \mathscr{Y}, Z) - \alpha^1(\mathscr{Y}, \mathscr{X} \cdot Z) \equiv (\delta\alpha^1)(\mathscr{X}, \mathscr{Y}, Z) = 0 \ . \tag{5.27}$$

A central extension is actually trivial if it is possible to find new generators $\widetilde{X}' = \widetilde{X} - \beta(X)\Xi$ (where β is a zero-cochain, $\beta \in \mathfrak{G}^*$) such that

$$[\widetilde{X}'_{a_1}, \ldots, \widetilde{X}'_{a_n}] = f_{a_1 \ldots a_n}{}^d \widetilde{X}'_d = \ f_{a_1 \ldots a_n}{}^d \widetilde{X}_d - \beta([X_{a_1}, \ldots, X_{a_n}])\Xi$$

i.e., $\alpha^1(X_1, \ldots, X_{n-1}, Z) = -\beta([X_1, \ldots, X_{n-1}, Z])$, again with $X_{a_n} = Z$. This may be rewritten in the form

$$\alpha^1(\mathscr{X}, Z) = -\beta([X_1, \ldots, X_{n-1}, Z]) \equiv (\delta\beta)(X_1, \ldots, X_{n-1}, Z) \equiv (\delta\beta)(\mathscr{X}, Z) \ , \tag{5.28}$$

where β is the zero-cochain generating the trivial one-cocycle, $\alpha^1 = \delta\beta$. Therefore, central extensions of FAs are characterized by one-cocycles modulo one-coboundaries.

The above suffices to infer the form of the full FA cohomology complex suitable for central extensions. Let α^p be a generic p-cochain. Then, $(C_0^\bullet(\mathfrak{G}), \delta)$ is defined by (see [26])

$$\begin{aligned}(\delta\alpha)(\mathscr{X}_1, \ldots, \mathscr{X}_{p+1}, Z) \qquad &= \sum_{1 \leq i < j}^{p+1} (-1)^i \alpha(\mathscr{X}_1, \ldots, \hat{\mathscr{X}_i}, \ldots, \mathscr{X}_i \cdot \mathscr{X}_j, \ldots, \mathscr{X}_{p+1}, Z) \\ &+ \sum_{i=1}^{p+1} (-1)^i \alpha(\mathscr{X}_1, \ldots, \hat{\mathscr{X}_i}, \ldots, \mathscr{X}_{p+1}, \mathscr{X}_i \cdot Z) \ ,\end{aligned} \tag{5.29}$$

which, for $n = 2$, reproduces the Lie algebra cohomology complex for the trivial action. Defining p-cocycles and p-coboundaries as usual, the p-th FA cohomology group (for the trivial action) is $H_0^p(\mathfrak{G})$ $= Z_0^p(\mathfrak{G})/B_0^p(\mathfrak{G})$. Therefore, a FA $\mathfrak{G}$ admits non-trivial central extensions when $H_0^1(\mathfrak{G}) \neq 0$.

5.2. Infinitesimal deformations of FAs

A similar approach may be used for deformations. An infinitesimal deformation in Gerstenhaber's sense [33] of a FA is obtained by modifying the n-bracket as

$$[X_1, \ldots, X_n]_t = [X_1, \ldots, X_n] + t\alpha^1(X_1, \ldots, X_n) \ , \tag{5.30}$$

where t is the deformation parameter and $\alpha^1 : \wedge^{n-1}\mathfrak{G} \wedge \mathfrak{G} \to \mathfrak{G}$ is now $\mathfrak{G}$-valued, so that $\mathfrak{G}$ will act on it. Again, the FI for the deformed FA n-bracket $[X_1, \ldots, X_n]_t$ constrains α^1. The FI is

$$[X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_n]_t]_t = \sum_{a=1}^{n} [Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a]_t, Y_{a+1}, \ldots, Y_n]_t \ ; \tag{5.31}$$

with $Y_n = Z$, it may we rewritten as

$$[\mathscr{X}, (\mathscr{Y} \cdot Z)_t]_t = [(\mathscr{X} \cdot \mathscr{Y})_t, Z]_t + [\mathscr{Y}, (\mathscr{X} \cdot Z)_t]_t \ . \tag{5.32}$$

At first order in t, the FI gives the following condition on the one-cochain α^1:

$$\begin{aligned}
&[X_1, \dots, X_{n-1}, \alpha^1(Y_1, \dots, Y_n)] + \alpha^1(X_1, \dots, X_{n-1}, [Y_1, \dots, Y_n]) \\
&= \sum_{a=1}^{n} [Y_1, \dots, Y_{a-1}, \alpha^1(X_1, \dots, X_{n-1}, Y_a), Y_{a+1}, \dots, Y_n] \\
&+ \sum_{a=1}^{n} \alpha^1(Y_1, \dots, Y_{a-1}, [X_1, \dots, X_{n-1}, Y_a], Y_{a+1}, \dots, Y_n) \ .
\end{aligned} \tag{5.33}$$

In terms of fundamental objects and with $Y_n = Z$, this may be read as a one-cocycle conditon for α^1,

$$\begin{aligned}
(\delta\alpha^1)(\mathscr{X}, \mathscr{Y}, Z) = ad_{\mathscr{X}} \alpha^1(\mathscr{Y}, Z) - ad_{\mathscr{Y}} \alpha^1(\mathscr{X}, Z) - (\alpha^1(\mathscr{X}, \quad) \cdot \mathscr{Y}) \cdot Z \\
-\alpha^1(\mathscr{X} \cdot \mathscr{Y}, Z) - \alpha^1(\mathscr{Y}, \mathscr{X} \cdot Z) + \alpha^1(\mathscr{X}, \mathscr{Y} \cdot Z) = 0 \quad ,
\end{aligned} \tag{5.34}$$

where, for instance for n=3,

$$\begin{aligned}
\alpha^1(\mathscr{X}, \quad) \cdot \mathscr{Y} \ &:= \ (\alpha^1(\mathscr{X}, \quad) \cdot Y_1, Y_2) \ + \ (Y_1, \alpha^1(\mathscr{X}, \quad) \cdot Y_2) \\
&= \ (\alpha^1(\mathscr{X}, Y_1), Y_2) \ + \ (Y_1, \alpha^1(\mathscr{X}, Y_2)) \ .
\end{aligned} \tag{5.35}$$

To see whether the $\mathfrak{G}$-valued cocycle α^1 is a one-coboundary, we look for the possible triviality of the infinitesimal deformation. It will be trivial if new generators can been found in terms of a $\beta : \mathfrak{G} \to \mathfrak{G}$, $X'_i = X_i - t\beta(X_i)$, such that

$$[X'_1, \dots, X'_n]_t = [X_1, \dots, X_n]' \equiv [X_1, \dots, X_n] - t\beta([X_1, \dots, X_n]) \ . \tag{5.36}$$

At first order in t this implies

$$\begin{aligned}
&[X'_1, \dots, X'_n]_t = [X_1, \dots, X_n]_t - t \sum_{a=1}^{n} [X_1, \dots, X_{a-1}, \beta(X_a), X_{a+1}, \dots, X_n]_t \\
= \ &[X_1, \dots, X_n] + t\alpha^1(X_1, \dots, X_n) - t \sum_{a=1}^{n} [X_1, \dots, X_{a-1}, \beta(X_a), X_{a+1}, \dots, X_n] \ .
\end{aligned} \tag{5.37}$$

Therefore, a deformation is trivial if

$$(\alpha^1)(X_1, \dots, X_n) := -\beta([X_1, \dots, X_n]) + \sum_{a=1}^{n} [X_1, \dots, X_{a-1}, \beta(X_a), X_{a+1}, \dots, X_n] \equiv (\delta\beta)(\mathscr{X}, X_n) \tag{5.38}$$

i.e., when the one-cocycle α^1 is the one-coboundary $\alpha^1 = \delta\beta$,

$$\alpha^1(\mathscr{X}, Z) = (\delta\beta)(\mathscr{X}, Z) = -\beta(\mathscr{X} \cdot Z) + (\beta(\quad) \cdot \mathscr{X}) \cdot Z + \mathscr{X} \cdot \beta(Z) \ . \tag{5.39}$$

If all one-cocycles are trivial, the FA is *stable* or *rigid*.

The above may be used to write the full complex $(C^\bullet_{ad}(\mathfrak{G}, \mathfrak{G}), \delta)$ adapted to the deformations of FA problem (see [14, 15] for details), introduced by Gautheron [19] in the context of Nambu-Poisson cohomology and also considered by Rotkiweicz [34], but it will not be needed here. We shall just mention that general p-cochains are now $\mathfrak{G}$-valued maps $\alpha^p : \wedge^{(n-1)}\mathfrak{G} \otimes \overset{p}{\cdots} \otimes \wedge^{(n-1)} \mathfrak{G} \wedge \mathfrak{G} \to \mathfrak{G}$.

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6. Whitehead lemma for FAs

It follows from the above discussion that an analogue of the Whitehead lemma for FAs would require having $H_0^1(\mathfrak{G}) = 0$ and $H_{ad}^1(\mathfrak{G},\mathfrak{G}) = 0$ for $\mathfrak{G}$ semisimple. That this is indeed the case was proven in [14], taking advantage of the fact that all simple FAs have the same general structure [17,13] given in eq. (3.18).

Characterizing the real-valued $Z_0^1(\mathfrak{G})$ and the $\mathfrak{G}$-valued $Z_{ad}^1(\mathfrak{G},\mathfrak{G})$ one-cocycles for central extensions and deformations of a FA by their coordinates,

$$\alpha^1_{a_1\dots a_n} = \alpha^1(e_{a_1},\dots,e_{a_n}) \quad , \quad {\alpha^1_{a_1\dots a_n}}^d = \alpha^1(e_{a_1},\dots,e_{a_n})^d \ , \quad a,d = 1,\dots,(n+1) \tag{6.40}$$

and using the explicit form of the n-brackets of the simple FAs, it is possible to show [14] that the above one-cocycles are necessarily one-coboundaries generated, respectively, by zero-cochains of coordinates β_a , β_a^d .

Therefore, $H_0^1(\mathfrak{G}) = 0$, $H_{ad}^1(\mathfrak{G},\mathfrak{G}) = 0$ for simple FAs, which therefore do not admit non-trivial central extensions nor deformations. Using now that a semisimple FA is the sum of simple ideals the following lemma is proved in [14]:

Lemma (*Whitehead lemma for semisimple n-Lie algebras*)
Semisimple Filippov algebras, $n \geq 2$, do not admit non-trivial central extensions and are, moreover, rigid.

7. Relaxing anticommutativity: n-Leibniz algebras and cohomology

Leibniz (Loday's) algebras [35] $\mathscr{L}$ are a non-commutative version of Lie algebras: their bracket need not be anticommutative ($[X,Y] \neq -[Y,X]$) but still satisfies the (left, say) 'Leibniz' identity

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[[X,Z]] \quad ; \tag{7.41}$$

right Leibniz algebras are defined in an analogous form.

Similarly, (left, say) n-Leibniz algebras $\mathfrak{L}$ [36,37] are defined by removing the anticommutatitivity requirement for the n-Leibniz bracket while keeping the (left) FI. Introducing also fundamental objects for n-Leibniz algebras $\mathfrak{L}$, the identity reads

$$\mathscr{X}\cdot(\mathscr{Y}\cdot\mathscr{Z}) = (\mathscr{X}\cdot\mathscr{Y})\cdot\mathscr{Z} + \mathscr{Y}\cdot(\mathscr{X}\cdot\mathscr{Z}) \qquad \forall \mathscr{X},\mathscr{Y},\mathscr{Z} \in \otimes^{n-1}\mathfrak{L}\ . \tag{7.42}$$

Note that now $\mathscr{X} \in \otimes^{n-1}\mathfrak{L}$ since, in contrast with FAs, the anticommutativity of the $(n-1)$ arguments in the fundamental object $\mathscr{X}$ is no longer assumed since the n-bracket in (1.5) is no longer antisymmetric for $\mathfrak{L}$. Nevertheless, the above is still the (left) FI (1.4) previously defining FAs. As a result, the characteristic FI

$$\mathscr{X}\cdot(\mathscr{Y}\cdot\mathscr{Z}) - \mathscr{Y}\cdot(\mathscr{X}\cdot\mathscr{Z}) = (\mathscr{X}\cdot\mathscr{Y})\cdot\mathscr{Z} \qquad \forall\ \mathscr{X},\mathscr{Y},\mathscr{Z} \in \otimes^{n-1}\mathfrak{L} \quad , \tag{7.43}$$

which already guaranteed the nilpotency of the coboundary operator δ for the different FA cohomology complexes (as the JI does in the ordinary Lie algebra cohomology), will also do the job for the various n-Leibniz cohomologies. Therefore, and with the proper definition of p-cochains on $\mathfrak{L}$, the n-Leibniz [38,37] and the FA cohomological complexes have the same structure (see [15] for details): n-Leibniz cohomology underlies n-Lie cohomology. This is why the N-P cohomology may be studied from the point of view of n-Leibniz cohomology, as pointed out and discussed by Daletskii and Takhtajan [36].

Our proof of the Whitehead Lemma above for FAs [14], however, relied on the antisymmetry of the FA n-bracket, and thus it does not hold when the anticommutativity is relaxed. On the other hand, n-Lie algebras $\mathfrak{G}$ may be considered as a particular case of n-Leibniz ones $\mathfrak{L}$: FAs are n-Leibniz algebras with a fully skewsymmetric n-bracket. Thus, we may look for n-Leibniz central extensions and deformations of FAs considering these as n-Leibniz ones and expect, in general, to find a richer structure. This has been observed explicitly for the $n = 2$ case [39] by looking at Leibniz deformations of the Heisenberg Lie algebra; also, for $n = 3$, a specific 3-Leibniz deformation of the simple Euclidean 3-Lie algebra A_4 has been given in [40]. Thus, a natural extension of our work above is to look for n-Leibniz deformations of simple n-Lie algebras to see whether this opens more possibilities.

It is natural to relax the skewsymmetry of the FA n-bracket in such a way that we remain within the class of n-Leibniz algebras that have fully skewsymmetric fundamental objects; this corresponds (see eq. (1.5)) to having n-Leibniz brackets that are antisymmetric in their first $n-1$ arguments. For $n = 3$, this type of real Leibniz algebras have in fact appeared in the study of multiple M2-branes [41]. Other examples of weakening the skewsymmetry have been considered in the same M-theory context, as the complex 'hermitean (right) three-algebras' introduced by Bagger and Lambert [42] that are behind the Aharony, Bergman, Jafferis, and Maldacena theory [43]; see further [44].

Our results on the class of real n-Leibniz deformations and central extensions of simple FAs which retain the skewsymmetry of the FA fundamental objects may be summarized by the following two theorems, both proven in [45]:

Theorem 1 (*A class of n-Leibniz deformations of simple FAs*)
The n-Leibniz algebra deformations of the $(n+1)$-dimensional simple FA's that preserve the skewsymmetry of the $(n-1)$ first elements in the n-Leibniz bracket (or that of the fundamental objects) are all trivial for $n > 3$. For $n = 3$, there is a non-trivial one-cocycle with coordinates

$$\alpha^1_{a_1 a_2 c d} \propto \epsilon_{a_1 a_2}{}^{eg} \varepsilon_c \epsilon_{egcd} = 2\varepsilon_c(\delta_{a_1 c}\delta_{a_2 d} - \delta_{a_1 d}\delta_{a_2 c}) \ .$$

Further, all $n = 2$ semisimple Filippov (*i.e.*, Lie) algebras are rigid as Leibniz algebras.

For the $n = 3$ Euclidean simple FA A_4, the above is the deformation given in [40].

Theorem 2 (*A class of n-Leibniz central extensions of simple FAs*)
The n-Leibniz algebra central extensions of simple FA's that preserve the skewsymmetry of the $(n-1)$ first entries of the n-bracket (or of the fundamental objects) are all trivial for any $n > 2$.

For $n = 2$ the fundamental objects have only one algebra element and therefore there are no skewsymmetry restrictions. Our proof of the $n > 2$ theorem also extends to the $n = 2$ simple algebras A_3 ($so(3)$) and $A_{1,2}$ ($so(1,2)$); the case of arbitrary simple Lie algebras is covered in [46] and [47] (Prop. 3.2 and Cor. 3.7).

8. Final comments

We have outlined some properties of n-ary algebras and, in particular, of Filippov algebras. Although these structures are mathematically interesting in themselves, they have also appeared in physics as n-ary Poisson structures and, recently, in the mentioned Bagger-Lambert-Gustavsson model in the case of FAs.

Our theorems 1 and 2 above apply to a (natural) class of n-Leibniz deformations of FAs. Other possibilities will arise if the deformations are not restricted to n-Leibniz algebras with antisymmetric fundamental objects but, obviously, each type will require separate study.

Contractions of FAs have recently been introduced in [48].

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