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Lagrangians and Hamiltonians with friction

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Abstract. Methods are presented for extending a promising formalism that incorporates dissipative forces into Quantum Mechanics. Hamiltonians and Lagrangians are constructed that have additional force and frictional terms while still being consistent with the rules of the formalism. The simple systems of a damped particle in a force field and a quadratically damped oscillator are discussed.

1. Introduction

Not long after Quantum Mechanics was formulated, in the 1930's, questions arose as to how to account for the physical properties of entropy or dissipation in this new branch of physics. Schrödinger himself realized that the equation named after him was not adequate to the task, seeing that it was reversible.

Since that time, a particularly well-reasoned approach that puts dissipative forces into the Lagrange-Hamiltonian mechanics on which Quantum Mechanics is based has come from Schuch [1-3]. His formalism is based on the hypothesis is that dissipative systems can be formulated in a way such that they follow the same variational principles as conservative systems. At the heart of the analysis is the realization that the gap between conservative mechanics and analogous formalisms having friction must be bridged by *non-canonical* (classical) transformations and not just a changing of coordinate space via unitary (quantum mechanical) transformations. The resulting dissipative Lagrangians ($L's$) and Hamiltonians $(H's)$ have the following convenient properties. The variation of the Langrangian produces the correct equations of motion. The corresponding Hamiltonians represent the total energy for the system. The Hamiltonians are a constant of the motion. The Heisenberg uncertainty does not vanish with time. These properties support the idea that the formalism is indeed a generalization of the variational foundation that underlies reversible mechanics but with dissipative effects included. The list of systems that have been studied so far includes the damped harmonic oscillator [1]. The intent herein is to show that this formalism allows for other forms to have additional assortments of frictional forces and potentials. Other details are found in [4].

2. Results

The process of deriving the new examples is guided by the following assumptions. Note that the term "real space" here is synonymous with physical space.

1. All $L's$ and $H's$ must have units of energy
2. Transformed space (i.e. Q-space) has no dissipation or explicit time dependence

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3. Transformed space is used to derive H from L
4. The variation of L, (i.e. $\delta L = 0$) in both real (i.e. q -space) and transformed space (Q -space) produce the correct equation of motion in real space.
5. As frictional and potential terms vanish, the correct simpler forms emerge.

Since there are an infinite number of frictional as well as conservative $L's$ and $H's$ that are associated with a given equation of motion [5, 6], the above assumptions greatly reduce the number of candidates. To obtain $L's$ and $H's$ that have the desired properties, the appropriate non-unitary transformations are needed to connect physical q -space, which has friction, to a transformed Q-space that does not have friction explicitly. As seen in the figure 1 below, the non-unitary nature of the transformation breaks the direct connection that the L in real space has with its respective H through the Legendre transform.

Q-space (transform space)

$$\begin{tikzcd} L(\dot{Q},Q) \arrow[r] & H = P\dot{Q} - L \arrow[r] & H(P,Q) \arrow[d] \\ L(\dot{q},q,t) \arrow[u] & H \neq p\dot{q} - L & H(p,q,t) \end{tikzcd}$$

q-space (real space)

Figure 1.

We see also that momentum in physical space and transformed space is defined as $p = m\dot{q}$ and $P = \frac{\partial L}{\partial \dot{Q}} = m\dot{Q}$ respectively. The canonical definition $p = \frac{\partial L}{\partial \dot{q}}$ no longer holds for real space frictional systems, at least those with explicit time dependence. In addition to the guidelines above it is known as illustrated in [5] that the frictional $L's$ and $H's$ have "expansion" or integrating factors of the form:

$$X = \exp\left(-\int \frac{\partial G}{\partial \dot{q}} dt \right) \tag{1}$$

where G is the term on the right hand side of the generic equation of motion, $\ddot{q} = G(\dot{q}, q, t)$.

Simultaneous to determining the non-unitary transform, the appropriate L in transformed space, or $L(\dot{Q},Q)$ must also be found. Each $L(\dot{Q},Q)$ has a kinetic $\dot{Q}^2$ term added to a generalized potential function with the form, $V(Q, \xi_0, \xi_1, \ldots \phi_0, \phi_1 \ldots)$, with Q as the transformed position and the ξ_i and ϕ_i the relevant coefficients of friction and conservative potential. The generalized potential has no explicit time dependence. The basic approach for creating the generalized potential for a given equation of motion is to include any and all pertinent terms that have the units of energy. For example if gravitational potential g is present, a term of the form $2gQ$ is included. Viscous friction implies a term such as $\frac{\gamma^2}{4}Q^2$. For quadratic friction, which is proportional to the square of velocity, there evidently are no such potential terms that can be found.

As an example, the frictional L and H for the damped harmonic oscillator are shown below. The first equation is the non-unitary transformation of the coordinate space, the second is the L in the transformed space. Once these are known, the dissipative Lagranians and Hamiltonians follow in a straightforward way. It is seen that the variational criterion $\delta L = 0$ produces the expected equation of motion in both q -space and Q -space. All other rules of the formalism are satisfied and H is a constant of the motion. For the damped harmonic oscillator:

$$Q = q\exp\left(\frac{\gamma t}{2}\right) \tag{2}$$

$$L(\dot{Q},Q) = \frac{m}{2}\left(\dot{Q}^2 - \left(\omega^2 - \frac{\gamma^2}{4}\right)Q^2\right) \tag{3}$$

$$H(P,Q) = \frac{m}{2}\left(\frac{P^2}{m^2} + \left(\omega^2 - \frac{\gamma^2}{4}\right)Q^2\right) \tag{4}$$

$$L(\dot{q},q,t) = \frac{m}{2}\left(\dot{q}^2 + \gamma\dot{q}q + \frac{\gamma^2}{2}q^2 - \omega^2 q^2\right)\exp(\gamma t) \tag{5}$$

$$H(p,q,t) = \frac{m}{2}\left(\frac{p^2}{m^2} + \frac{\gamma}{m}pq + \omega^2 q^2\right)\exp(\gamma t) \tag{6}$$

with (2) being the non-canonical connection between physical and transformed space and with $\exp(\gamma t)$ as the expansion factor.

The derivation of frictional $L's$ and $H's$ for a damped particle in a force field and a quadratically damped oscillator are now shown. The equations of motion for the two systems are:

$$\ddot{q} + \gamma\dot{q} + \alpha = 0 \tag{7}$$

where α is a constant force field and

$$\ddot{q} + b\dot{q}^2 + \omega^2 q = 0 \tag{8}$$

for the quadratically damped oscillator. Their corresponding q-transforms as well as $L(\dot{Q},Q)$ and $H(P,Q)$ are shown in table 1. The frictional $L's$ and $H's$ in physical space are shown in table 2.

Table 1.

Eqn of motion	$q - transform$	$L(\dot{Q},Q)$	$H(P,Q)$
$\ddot{q} + \gamma\dot{q} + \alpha = 0$	$Q = \left(q + \frac{\alpha t}{\gamma} - 2\frac{\alpha}{\gamma^2}\right)\exp\left(\frac{\gamma t}{2}\right) - 2\frac{\alpha}{\gamma^2}\cosh\left(\frac{\gamma t}{2}\right) + 4\frac{\alpha}{\gamma^2}$	$\frac{m}{2}\left(\dot{Q}^2 + \frac{\gamma^2}{4}Q^2 - 2\alpha Q\right)$	$\frac{m}{2}\left(\frac{P^2}{m^2} - \frac{\gamma^2}{4}Q^2 + 2\alpha Q\right)$
$\ddot{q} + b\dot{q}^2 + \omega^2 q = 0$	$Q = \frac{1}{b}(1 - \exp(-bq))\exp(bq)$	$\frac{m}{2}(\dot{Q}^2 - \frac{\omega^2}{b^2}(bQ+1)^2\ln(bQ+1) + \frac{\omega^2}{2b^2}(bQ+1)^2 - \frac{\omega^2}{2b^2})$	$\frac{m}{2}(\frac{P^2}{m^2} + \frac{\omega^2}{b^2}(bQ+1)^2\ln(bQ+1) - \frac{\omega^2}{2b^2}(bQ+1)^2 + \frac{\omega^2}{2b^2})$

Table 2.

Eqn of motion	$L(\dot{q},q,t))$	$H(p,q,t))$
$\ddot{q}+\gamma\dot{q}+\alpha=0$	$\frac{m}{2}((\dot{q}^2+\gamma\dot{q}q+\frac{\gamma^2}{2}q^2$ $-2\alpha q-\frac{\alpha}{\gamma}\dot{q}+2\frac{\alpha^2}{\gamma^2}-2\frac{\alpha^2}{\gamma}t+\alpha\dot{q}t$ $+\alpha\gamma qt+\frac{\alpha^2t^2}{2})\exp(\gamma t)$ $+\frac{\alpha}{\gamma}\dot{q}-3\frac{\alpha^2}{\gamma^2}+\frac{\alpha^2}{\gamma^2}\cosh(\gamma t))$	$\frac{m}{2}((\frac{p^2}{m^2}+\frac{\gamma}{m}pq$ $+\alpha q-\frac{\alpha}{m\gamma}p-2\frac{\alpha^2}{\gamma^2}+\frac{\alpha^2}{\gamma}t+\frac{\alpha}{m}pt)\exp(\gamma t)$ $+\alpha q+\frac{\alpha}{m\gamma}p+\frac{2\alpha^2}{\gamma^2}+\frac{\alpha^2t}{\gamma})$
$\ddot{q}+b\dot{q}^2+\omega^2q=0$	$\frac{m}{2}\left(\left(\dot{q}^2-\frac{\omega^2}{b}q+\frac{\omega^2}{2b^2}\right)\exp(2bq)-\frac{\omega^2}{2b^2}\right)$	$\frac{m}{2}\left(\left(\frac{p^2}{m^2}+\frac{\omega^2}{b}q-\frac{\omega^2}{2b^2}\right)\exp(2bq)+\frac{\omega^2}{2b^2}\right)$

What follows is an explanation in detail as to how the rules of the formalism and related guidelines were used to produce the above examples.

Example 1: The damped particle in a force field. The ansatz q-transform is taken to be that for the free damped particle as shown in [4] which is then combined with an as yet unspecified function of time:

$$Q = q\exp\left(\frac{\gamma}{2}t\right)+f(t). \tag{9}$$

The use of $f(t)$ is justified in [5]. The initial $L(\dot{Q},Q)$ is formed from the generalized potential terms that correspond to viscous friction and a constant field,

$$L(\dot{Q},Q)=\frac{m}{2}\left(\dot{Q}^2+\frac{\gamma^2}{4}Q^2-2\alpha Q\right). \tag{10}$$

The variation, $\delta L(\dot{Q},Q)=0$ is calculated in the same way as for a conservative system:

$$\delta L(\dot{Q},Q)=\frac{m}{2}\left(2\ddot{Q}-\frac{\gamma^2}{2}Q-2\alpha\right)=0. \tag{11}$$

Equation (9) is differentiated to find $\dot{Q}$ and $\ddot{Q}$ which are substituted into (11) to convert it to physical space:

$$(\ddot{q}+\gamma\dot{q})\exp\left(\frac{\gamma}{2}t\right)+\ddot{f}-\frac{\gamma^2}{4}f+\alpha=0. \tag{12}$$

The function $f(t)$ is determined by setting the differential equation in f in equation (12) to be equal to the term that is needed to complete the equation of motion:

$$\ddot{f}-\frac{\gamma^2}{4}f+\alpha=\alpha\exp\left(\frac{\gamma}{2}t\right). \tag{13}$$

The general solution for $f(t)$ is:

$$f(t)=c_1\exp\left(\frac{\gamma}{2}t\right)+c_2\exp\left(\frac{-\gamma}{2}t\right)+\frac{\alpha t}{\gamma}\exp\left(\frac{\gamma}{2}t\right)+4\left(\frac{\alpha}{\gamma^2}\right). \tag{14}$$

By using power series expansions of $c_1 \exp\left(\frac{\gamma}{2}t\right)$ and $c_2 \exp\left(\frac{-\gamma}{2}t\right)$, we can know how to set the two constants c_1 and c_2 so that the frictional L and H in q-space properly turn into the expected frictionless L and H as the viscous coefficient γ vanishes. We get

$$c_1 = -3\frac{\alpha}{\gamma^2}, \qquad c_2 = -\frac{\alpha}{\gamma^2}. \tag{15a, 15b}$$

Therefore $f(t)$ is:

$$f(t) = -3\frac{\alpha}{\gamma^2}\exp\left(\frac{\gamma}{2}t\right) - \frac{\alpha}{\gamma^2}\exp\left(\frac{-\gamma}{2}t\right) + \frac{\alpha t}{\gamma}\exp\left(\frac{\gamma}{2}t\right) + 4\left(\frac{\alpha}{\gamma^2}\right). \tag{16}$$

By adding $f(t)$ into equation (9) and using the result to convert the corresponding $L(\dot{Q},Q)$ and $H(P,Q)$ to real space, $L(\dot{q},q,t)$ and $H(p,q,t)$ can be determined.

Example 2: The quadratically damped oscillator. The ansatz q-transform is taken to be the free quadratically damped particle which is derived in [4]:

$$Q = \frac{1}{b}(1 - \exp(-bq))\exp(bq). \tag{17}$$

The initial $L(\dot{Q},Q)$ is taken to be:

$$L(\dot{Q},Q) = \frac{m}{2}\left(\dot{Q}^2\right). \tag{18}$$

There could also be a $\omega^2 Q^2$ term but it will be more convenient to leave this out. As a result:

$$\delta L(\dot{Q},Q) = \frac{m}{2}\left(2\ddot{Q}\right) = 0. \tag{19}$$

After converting the differential to physical space:

$$\left(\ddot{q} + b\dot{q}^2\right)\exp(bq) = 0. \tag{20}$$

It is clear that addition of $\omega^2 q \exp(bq)$ is needed to complete the equation of motion. Since there is no explicit time dependence the q-transform can be inverted so that the needed term as a function of Q is:

$$\omega^2 q \exp(bq) = \frac{\omega^2}{b}\ln(bQ + 1)(bQ + 1). \tag{21}$$

The right hands side of (21) can be integrated to find the needed term in $L(\dot{Q},Q)$ as seen as the complicated expression in table 1. By expanding $L(\dot{Q},Q)$ in power series it is seen that the $\omega^2 Q^2$ term is part of the series. After $L(\dot{Q},Q)$ is converted to real space using the q-transform, it is seen that the term $-\frac{\omega^2}{2b^2}$ is needed to allow $L(\dot{q},q,t)$ and $H(p,q,t)$ to correctly turn into simpler forms as ω or b vanish.

3. Discussion and conclusion

Additional combinations of friction and potential can be included into a variational formalism for mechanics that includes dissipative forces. The resulting Hamiltonians open up the possibility of modeling dissipative quantum systems. The example containing both a force field and viscous friction is noteworthy in that it can represent a steady state system as the terminal velocity of the particle in the field is reached. A frictional Hamiltonian like this may be useful, for instance, in the modeling phenomena such as lasers as is discussed in [7]. Since the dissipative terms of the Hamiltonians discussed herein are given as functions of velocity, they are in essence macroscopic and phenomenological in nature. Frictional forces of this type represent the average interaction of a larger environment on a particle or system of particles without all the microscopic details. Nevertheless, such simplifications, if justifiable, can be useful in capturing the larger aspects of a problem while reducing the calculational complexity.

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