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A degree of freedom and metric approach for non-singular Mueller matrices characterizing passive systems

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1. Introduction

The description of Mueller matrix set is always questionable, thought a part of the solution is already well known: non-singular non-depolarizing Mueller matrices are the positive multiples of Lorentz group matrices with positive left upper corner element and positive determinant (according to the used definition of Mueller matrix). Then, corresponding sub-group is given \([1][2]\) by \(\{ SO(3,1) \} \cup \{ G .SO(3,1) \} \) with \(G = \text{diag}(1,-1,-1,-1)\). Therefore, the characterization of matrices of \(SO(3,1)\), (the identity component of \(SO(3,1)\), the associated special orthogonal group) completely describes the two sub-groups. Extension of this result to the whole set of non-singular Mueller matrices is not straightforward since this set will be expected to form only a semigroup [3]. In a recent paper [2], regarding mathematical framework to define and analyze a general parametric form of an arbitrary non-singular Mueller matrix, we proposed to address this problem in a space with more than 4 dimensions in order to introduce a set of transformations with a group structure. We consider a group of transformations continuously connected to the identity transformation and preserving a \(N\)-vectors norm such as the proper Lorentz transformations with the 4-vectors of the Minkowski space.

2. Appropriate norm and dimension.

The problems with such approach are then: how do we determine \(N\), the space dimension and how do we chose the appropriate norm?

2.1. Appropriate norm

The proposed space should contain a Minkowskian subspace associated with the four first components for instance. Then, this subspace can be assimilated with the polarization space - it means \((x^0,x^1,x^2,x^3) = (S_0,S_1,S_2,S_3)\) - and the associated metric should be defined by the metric tensor \(g_{\mu\nu}\) with the signature \((1,-1,-1,-1)\). Considerations about properties of the polarizance vector or diattenuation, allow us [2] to demonstrate \(g_{44} = g_{55} = \ldots = g_{(N-1)(N-1)} = -1\). So, we are now considering the group \(O(N-1,1)\) and more precisely \(SO(N-1,1)\), its identity component.

2.2. Appropriate dimension

What is under interest is the number of degrees of freedom (DOF) associated with the polarization 4-dimensions subspace not with the full \(N\)-dimensions space. As \(SO(N-1,1)\), is a matrix Lie group with
N degrees of freedom, we can focus our attention on the N generators of the group rather than the infinite number of group elements. We prove that N=7 is the actual solution. Under this hypothesis, we have now a potential of 21 DOF. So how many DOF are actually present in the resulting Mueller matrix? We first divide these generators in 4 sets and consider the DOF contribution to Mueller matrix subspace of each set and the DOF added by the interaction between these sets under the matrix multiplication rule. The first set is composed by $J_{23}$, $J_{13}$, $J_{12}$, $J_{01}$, $J_{02}$ and $J_{03}$. The 6 generators give associated Mueller sub-matrices with a total of 6 DOF. In fact these Mueller matrices are the non-singular, non depolarizing Mueller matrices (product of $M_R$, the retardance matrix and $M_D$ the diattenuation one [4][2]), since these generators are translation in 7 dimensions of Lorentz group generators. The same approach with $J_{46}$, $J_{56}$, $J_{45}$ shows that no DOF can be directly added to the Mueller sub-space or by the block matrix multiplication rule either. We now consider the $J_3$ matrix defined by linear combination of 9 generators $J_{m4}$, $J_{m5}$, $J_{m6}$ with $m =1,2,3$. This set of generators gives an amount of only 6 DOF. The last set of 3 generators gives the matrix $J_4$ as a linear combination of generators $J_{06}$, $J_{05}$ and $J_{04}$. It is straightforward to demonstrate no DOF can be directly added to the Mueller sub-space by these generators. But it is clear that DOF can be added to the Mueller sub-space by the block matrix multiplication rule between $\exp(J_3)$ and $\exp(J_4)$. These matrix products add a diattenuation vector or a polarizance vector [4] with 3 DOF to the Mueller matrix subspace. Eventually, we obtain a total amount of 15 DOF for the resulting Mueller matrix subspace. The same approach proves we only have an amount of 14 DOF for the Mueller matrix subspace when dealing with a 6-dimensions space. Then $SO(6,1)$, is a physical admissible solution to solve the question of definition of arbitrary non-singular Mueller matrices. The previously proposed solution in [2], was just a subset of this one and it is possible [6] to identify missing matrices as ones with equal depolarization factor along the 3 principal axis with a non null polarizance or diattenuation vector. It is worth noticing that the relation between the both definitions of Mueller matrices and the different connected components of $O(3,1)$ introduced for the non-depolarizing matrices is not so straightforward for the case of depolarizing matrices since $SO(6,1)$ is a group of matrices with unit determinant but sub-matrices of this group can have negative determinant.

3. Conclusions

Starting from previous results about the non-depolarizing Mueller matrices, we generalized the method to any non-singular Mueller matrices. Considering the set of non-singular Mueller matrices as a set of linear operators on an embedded subspace of a more general space, we take metric and degree of freedom in consideration in order to specify this space and the group of associated linear transformations. Generators of this group are used to address the issue of the dimension of the full space and a 7 dimensions Minkowski space gives a physical admissible solution. Arbitrary non-singular Mueller matrices are then sub-matrices of $SO(6,1)$.

References