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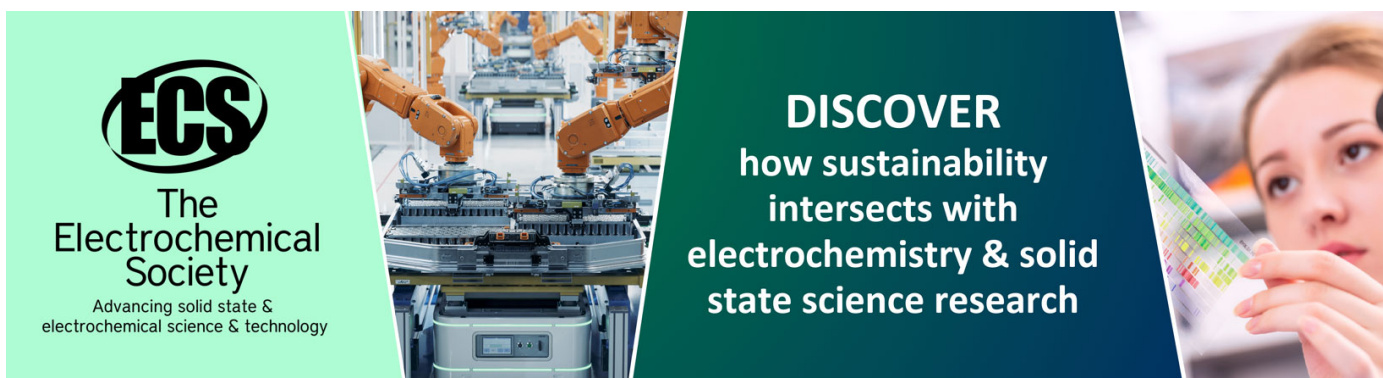
Characterization for entropy and solution of set function on lattice

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Characterization for entropy and solution of set function on lattice

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Abstract. We propose a definition for the entropy of a monotone set function defined on a lattice which are not necessarily the whole power set, but satisfy the condition of regularity. Our definition encompasses the classical definition of Shannon for probability measures, as well as the definition of Marichal for classical fuzzy measures and may have applicability to most fuzzy measures which appear in applications. We give also an axiomatization of this entropy. This axiomatization is in the spirit of Faddeev's axiomatization for the classical Shannon entropy. After that, using same idea we introduce a generalization of the Shapley value for a set function defined on a lattice and give two types of necessary and sufficient conditions.

1. Introduction

The classical definition of the Shannon entropy [11] for probability measures is at the core of information theory. Therefore, many attempts for defining an entropy for a set function more general than the classical probability measures have been done, in particular for the monotone set function, what is called the fuzzy measure [13] or the capacity. A first attempt by Yager [15] and Marichal and Roubens [10] proposed a definition having suitable properties, and which can be considered as the generalization of the Shannon entropy. Another attempt was also done by Dukhovny [2], in a different spirit. All these works considered a finite universal set, and the power set as underlying set system. Whereas we consider more general fuzzy measure, defined on lattices or set systems satisfying a kind of regularity, i.e., that all maximal chains have the same length, which are not necessary the whole power set [5]. Our definition encompasses the classical definition of Shannon, as well as the definition of Marichal [10], and may have applicability to most monotone set functions which appear in applications. Moreover our characterization of this entropy is in the spirit of Faddeev's axiomatization for the classical Shannon entropy [6]. In this paper, we introduce a definition for the entropy of the fuzzy measure defined on lattices, which is a yet more general case of monotone set functions. A set system is merely a collection of subsets of some universal set, containing the empty set and the universal set itself. The difference with usual fuzzy measures on power set is that the fuzzy measure, with which we are concerned, is not defined for every subset (or coalition, in a game theoretic perspective). We define the entropy for such fuzzy measures on set systems, provided that the set system satisfies a regularity condition. Moreover, using join-irreducible elements, there is a close connection between set systems and lattices, so that our definition can be applied to monotone set function on lattices as well. We give also a characterization for this generalized entropy.

Using the same idea, we can define a solution of game defined on a lattice. The second aim of the paper is to provide a definition of a solution of game on lattice, which encompasses the Shapley value. We give two types of axiomatic characterizations of a solution for game on a lattice.

2. Preliminaries

Throughout this paper, we consider a finite universal set $N = \{1, 2, \dots, n\}$, $n \geq 1$, and 2^N denotes the power set of N .

Definition 1 (set system) Let $\mathfrak{N}$ be a subset of 2^N . If $\mathfrak{N}$ contains $\emptyset$ and N , then we call $(N, \mathfrak{N})$ (or simply $\mathfrak{N}$ if no confusion occurs) a set system.

Let $A, B \in \mathfrak{N}$. We say that A is covered by B , and write $A \prec B$ or $B \succ A$, if $A \subsetneq B$ and $A \subseteq C \subsetneq B$ together with $C \in \mathfrak{N}$ imply $C = A$.

Definition 2 (maximal chain) Let $\mathfrak{N}$ be a set system. We call $\mathfrak{C}$ a maximal chain of $\mathfrak{N}$ if $\mathfrak{C} = (C_0, C_1, \dots, C_m)$ satisfies $\emptyset = C_0 \prec C_1 \prec \dots \prec C_m = N$, $C_i \in \mathfrak{N}$, $i = 0, \dots, m$.

The length of the maximal chain $\mathfrak{C} = (C_0, C_1, \dots, C_m)$ is m . We denote the set of all maximal chains of $\mathfrak{N}$ by $\mathcal{M}(\mathfrak{N})$.

Definition 3 (totally ordered set system, chain) We say that $(N, \mathfrak{N})$ is a totally ordered set system or a chain if for any $A, B \in \mathfrak{N}$, either $A \subseteq B$ or $A \supsetneq B$.

If $(N, \mathfrak{N})$ is a totally ordered set system, then it has only one maximal chain.

Definition 4 (regular set system) We say that $(N, \mathfrak{N})$ is a regular set system if for any $A, B \in \mathfrak{N}$ satisfying $A \succ B$, $|A \setminus B| = 1$ holds.

Definition 5 (fuzzy measure on a set system) Let $(N, \mathfrak{N})$ be a set system. A function $v : \mathfrak{N} \rightarrow [0, 1]$ is a fuzzy measure on $(N, \mathfrak{N})$ if it satisfies $v(\emptyset) = 0$, $v(N) = 1$ and for any $A, B \in \mathfrak{N}$, $v(A) \leq v(B)$ whenever $A \subseteq B$.

For clarifying the domain of v , we denote often the triplet $(N, \mathfrak{N}, v)$ instead of simply v . we denotes it by the triplet $(N, \mathfrak{N}, v)$. $\mathcal{F}(N, \mathfrak{N})$ denotes the set of all fuzzy measure defined on $(N, \mathfrak{N})$.

For $v \in \mathcal{F}(N, \mathfrak{N})$ and $\mathfrak{C} := (C_0, C_1, \dots, C_m) \in \mathcal{M}(\mathfrak{N})$, define $p^{v, \mathfrak{C}}$ by

$$\begin{aligned} p^{v, \mathfrak{C}} &:= (p_1^{v, \mathfrak{C}}, p_2^{v, \mathfrak{C}}, \dots, p_m^{v, \mathfrak{C}}) \\ &= (v(C_1) - v(C_0), v(C_2) - v(C_1), \dots, v(C_m) - v(C_{m-1})). \end{aligned}$$

Note that $p^{v, \mathfrak{C}}$ satisfies $p_i^{v, \mathfrak{C}} \geq 0$, $i = 1, \dots, m$ and $\sum_{i=1}^m p_i^{v, \mathfrak{C}} = 1$.

We turn now to definitions of the entropy. We first recall the classical definition of Shannon.

Definition 6 (Shannon entropy[11]) For a probability measure $p = (p_1, \dots, p_n)$ on N , the Shannon entropy of p is defined by

$$H_S(p) = H_S(p_1, \dots, p_n) := - \sum_{i=1}^n p_i \log p_i,$$

where $p_i := p(\{i\})$ and $\log$ denoting the base 2 logarithm, and by convention $0 \log 0 := 0$.

Marichal has generalized the Shannon entropy for applying to the fuzzy measure on 2^N .

Definition 7 (Marichal's entropy [10]) For $v \in \mathcal{F}(N, 2^N)$, Marichal's entropy of v is defined by

$$H_M(v) = H_M(N, 2^N, v) := - \sum_{A \subseteq N \setminus \{i\}} \gamma_{|A|}^n (v(A \cup \{i\}) - v(A)) \log(v(A \cup \{i\}) - v(A)),$$

where

$$\gamma_k^n := \frac{(n - k - 1)!k!}{n!}.$$

The axiomatization of Marichal's entropy is proposed by Kojadinovic et al. [9]. Their axiomatization is however rather complicated, due to the presence of a recursive axiom, whose meaning is hard to grasp.

Using the concept of the maximal chain, we have proposed a generalization of the definition of the entropy for fuzzy measures defined on regular set systems.

Definition 8 (entropy of fuzzy measure on set system [5]) Let $(N, \mathfrak{N})$ be a regular set system. For $v \in \mathcal{F}(N, \mathfrak{N})$, the entropy of v is defined by

$$H_{HG}(v) = H_{HG}(N, \mathfrak{N}, v) := \frac{1}{|\mathcal{M}(\mathfrak{N})|} \sum_{\mathfrak{C} \in \mathcal{M}(\mathfrak{N})} H_S(p^{v, \mathfrak{C}}).$$

We discuss the domains of H . Let Σ_n be a all regular set of $N = \{1, 2, \dots, n\}$. The domain of $\mathcal{F}_{\mathcal{R}} := \bigcup_{n=1}^{\infty} \bigcup_{\mathfrak{N} \in \Sigma_n} \mathcal{F}(N, \mathfrak{N})$, that is, all of the fuzzy measures defined on regular set systems.

We introduce further concepts about fuzzy measures, which will be useful for stating axioms of the entropy.

Definition 9 (dual measure) For $v \in \mathcal{F}_{\mathcal{R}}$, the dual measure of v is defined on $\mathfrak{N}^d := \{A \in 2^N \mid A^c \in \mathfrak{N}\}$ by $v^d(A) := 1 - v(A^c)$ for any $A \in \mathfrak{N}^d$, where $A^c := N \setminus A$.

Definition 10 (permutation of v) Let π be a permutation on N . For $v \in \mathcal{F}_{\mathcal{R}}$, the permutation of v by π is defined on $\pi(\mathfrak{N}) := \{\pi(A) \in 2^N \mid A \in \mathfrak{N}\}$ by $\pi \circ v(A) := v(\pi^{-1}(A))$.

Let us consider a chain of length 2 as a set system, denoted by $\mathbf{2}$ (e.g., $\{\emptyset, \{1\}, \{1, 2\}\}$), and a fuzzy measure $v^{\mathbf{2}}$ on it. We denote by the triplet $(0, u, 1)$ the values of $v^{\mathbf{2}}$ along the chain and we suppose $\mathbf{2} := \{\emptyset, \{1\}, \{1, 2\}\}$ unless otherwise noted.

Definition 11 (embedding of $v^{\mathbf{2}}$) Let $(N, \mathfrak{C})$ be a totally ordered regular set system such that $\mathfrak{C} := \{C_0, \dots, C_n\}$, $C_{i-1} \prec C_i$, $i = 1, \dots, n$. For $v \in \mathcal{F}(N, \mathfrak{C})$, $v^{\mathbf{2}} = (0, u, 1)$ and $C_k \in \mathfrak{C}$, v^{C_k} is called the embedding of $v^{\mathbf{2}}$ into v at C_k and defined on the totally ordered regular set system $(N^{C_k}, \mathfrak{C}^{C_k})$ by

$$v^{C_k}(A) := \begin{cases} v(A), & \text{if } A = C_j, j < k, \\ v(C_{k-1}) + u \cdot (v(C_k) - v(C_{k-1})), & \text{if } A = C'_k \\ v(C_{j-1}), & \text{if } A = C'_j, j > k, \end{cases}$$

where $\{i_k\} := C_k \setminus C_{k-1}$, $i'_k \neq i''_k$, $(N \setminus \{i_k\}) \cap \{i'_k, i''_k\} = \emptyset$, $N^{C_k} := (N \setminus \{i_k\}) \cup \{i'_k, i''_k\}$, $C'_k := (C_k \setminus \{i_k\}) \cup \{i'_k\}$, $C'_j := (C_{j-1} \setminus \{i_k\}) \cup \{i'_k, i''_k\}$ for $j > k$, and $\mathfrak{C}^{C_k} := \{C_0, \dots, C_{k-1}, C'_k, C'_{k+1}, \dots, C'_{n+1}\}$.

Definition 12 (join-irreducible element) An element $x \in (L, \leq)$ is join-irreducible if for all $a, b \in L$, $x \neq \perp$ and $x = a \vee b$ implies $x = a$ or $x = b$.

$\mathcal{J}(L)$ denotes the set of all join-irreducible elements of L . The mapping η for any $a \in L$, defined by

$$\eta(a) := \{x \in \mathcal{J}(L) \mid x \leq a\}$$

is a lattice-isomorphism of L onto $\eta(L) := \{\eta(a) \mid a \in L\}$, that is, $(L, \leq) \cong (\eta(L), \subseteq)$. In this paper, though we consider only set functions defined on set systems, using the translation η , we can treat set functions defined on lattice $(L, \leq)$ as one defined on set system $(\mathcal{J}(L), \eta(L))$.

3. Axiomatization of the entropy of fuzzy measures

We introduce five axioms for the entropy of fuzzy measures.

Axiom 1 (continuity) The function $f(u) := H(0, u, 1) = H(v^2, \{1, 2\}, \mathbf{2})$ is continuous on $[0, 1]$, and there exists $u_0 \in [0, 1]$ such that $f(u_0) > 0$.

Axiom 2 (dual invariance) For any $v^2 = (0, u, 1) \in \mathcal{F}(\{1, 2\}, \mathbf{2})$,

$$H(0, u, 1) = H(0, 1 - u, 1).$$

Axiom 3 (increase by embedding) Let $(N, \mathfrak{C})$ be a totally ordered regular set system. For any $v \in \mathcal{F}(N, \mathfrak{C})$, any $C_k \in \mathfrak{C}$ and any $v^2 = (0, u, 1)$, the entropy of v^{C_k} is

$$H(v^{C_k}) = H(v) + (v(C_k) - v(C_{k-1})) \cdot H(0, u, 1).$$

Axiom 4 (convexity) Let $(N, \mathfrak{N})$, $(N, \mathfrak{N}_1), \dots, (N, \mathfrak{N}_k)$ be regular set systems satisfying $\mathcal{M}(\mathfrak{N}) = \bigcup_{j=1}^k \mathcal{M}(\mathfrak{N}_j)$, and $\mathcal{M}(\mathfrak{N}_i) \cap \mathcal{M}(\mathfrak{N}_j) = \emptyset$ for all $i \neq j$. Then there exists $(\alpha_1, \dots, \alpha_k) \in]0, 1[^k$ with $\sum_{j=1}^k \alpha_j = 1$ such that for any $v \in \mathcal{F}(N, \mathfrak{N})$, it holds that

$$H(v) = \alpha_1 H(v|_{\mathfrak{N}_1}) + \dots + \alpha_k H(v|_{\mathfrak{N}_k}).$$

Axiom 5 (permutation invariance) For any $v \in \mathcal{F}_{\mathcal{R}}$ and any permutation π on N satisfying $\pi(\mathfrak{N}) = \mathfrak{N}$, it holds that $H(v) = H(\pi \circ v)$.

Theorem 13 ([6]) Let $(N, \mathfrak{N})$ be a regular set system and $(N, \mathfrak{N}, v)$ a fuzzy measure. Then there exists the unique function satisfying Axioms 1, 2, 3, 4 and 5, and it is given by H_{HG} .

Now we recall the Tsallis entropy. The Tsallis entropy was introduced in 1988 ([14]), and it has been studied intensively by many authors.

Definition 14 Let q be a positive real number. For a probability measure $p = (p_1, \dots, p_n)$ on N , the Tsallis entropy of p is defined by

$$H_T(p) = H_T(p_1, \dots, p_n) = - \sum_{i=1}^n p_i^q \ln_q p_i,$$

where $p_i := p(\{i\})$ and

$$\ln_q(x) := \frac{x^{1-q} - 1}{1 - q}.$$

We can define the Tsallis entropy of the fuzzy measure defined on set systems, using our framework.

Definition 15 Let $(N, \mathfrak{N})$ be a regular set system. For $v \in \mathcal{F}(N, \mathfrak{N})$ the entropy of v is defined by

$$H_T(v) = H_T(N, \mathfrak{N}, v) := \frac{\lambda_q}{|\mathcal{M}(\mathfrak{N})|} \sum_{\mathfrak{C} \in \mathcal{M}_n(\mathfrak{N})} H_T(p^{v, \mathfrak{C}}),$$

where λ_p is a positive constant that depends on q .

Modifying Axiom 3 as follows, we obtain an axiomatization of (H_T) (see also [3]).

Axiom 3' Let $(N, \mathfrak{C})$ be a totally ordered regular set system. For any $v \in \mathcal{F}(N, \mathfrak{C})$, any $C_k \in \mathfrak{C}$ and any $v^2 = (0, u, 1)$, the entropy of v^{C_k} is

$$H(v^{C_k}) = H(v) + (v(C_k) - v(C_{k-1}))^q \cdot H(0, u, 1).$$

Theorem 16 Let $(N, \mathfrak{N})$ be a regular set system and $(N, \mathfrak{N}, v)$ a fuzzy measure. Then there exists the unique function satisfying Axioms 1, 2, 3', 4 and 5, and it is given by H_T .

4. Solution of cooperative game

We call a function $v : \mathfrak{N} \rightarrow \mathbb{R}$ a *characteristic function* of the cooperative game if it satisfies $v(\emptyset) = 0$. Now we consider N is a set consisting of n players. Then a subset of N is called a coalition, and the characteristic function v means the profit by coalitions of n players. We denote the characteristic function by the triplet $(N, \mathfrak{N}, v)$ in the same manner as the fuzzy measure and we call it *generalized cooperative game* or simply *game*. $(N, 2^N, v)$ is a classical cooperative game. The solution is the function from the whole set of the game v to n -dimension real which measures each player's contribution or share-out. The Shapley value is the most important concept as a solution of the game, and they are characterized by natural axiomatizations [12]. We have generalized the Shapley value for applying games defined on $\mathfrak{N} \subseteq 2^N$ which satisfies a kind of regularity, not only defined on 2^N [5][7]. Our generalization make the Shapley value applicable to more general games, for example multi-choice game [8], bi-capacity [4], games on anti-matroid [1]. We have also shown an axiom system which consists of six axioms characterizing this generalized the Shapley value.

We denote the set of all games defined on the regular set system $(N, \mathfrak{N})$ by $\mathcal{G}(N, \mathfrak{N})$. Let Σ_n be a all regular set of $N = \{1, 2, \dots, n\}$ and define $\mathcal{G}_{\mathcal{R}} := \bigcup_{n=1}^{\infty} \bigcup_{\mathfrak{N} \in \Sigma_n} \mathcal{G}(N, \mathfrak{N})$, that is, all of the games defined on regular set systems, and $\mathcal{G}_{\mathcal{P}} := \bigcup_{n=1}^{\infty} \mathcal{G}(N, 2^N)$, that is, all of the games defined on power sets.

Definition 17 (original Shapley value [12]) For $v \in \mathcal{G}_{\mathcal{P}}$, the Shapley value of v , $\Phi_{\text{sh}}(v) = \Phi_{\text{sh}}(N, 2^N, v) := (\phi_{\text{sh}}^1(v), \dots, \phi_{\text{sh}}^n(v)) \in [0, 1]^n$ is defined by

$$\phi_{\text{sh}}^i(v) = \phi_{\text{sh}}^i(N, 2^N, v) := \sum_{E \subseteq N \setminus \{i\}} \gamma_{|E|}^n(v(E \cup \{i\}) - v(E)), \quad i = 1, \dots, n.$$

We denote $\Phi(N, \mathfrak{N}, v)$ and $\phi^i(N, \mathfrak{N}, v)$ instead of $\Phi(v)$ and $\phi^i(v)$ for clarifying the domain of v .

It is known that the Shapley value is represented with maximal chains as follows.

$$\phi_{\text{sh}}^i(v, N, 2^N) = \frac{1}{|\mathcal{M}(2^N)|} \sum_{\mathfrak{C} \in \mathcal{M}(2^N)} (v(\mathfrak{C}_*(i) \cup \{i\}) - v(\mathfrak{C}_*(i))), \quad i = 1, \dots, n,$$

where $\mathfrak{C}_*(i) := \sup\{C \in \mathfrak{C} \mid C \not\ni i\}$.

Faigle and Kern had defined a value for applying the generalized cooperative game using the concept of the maximal chain. We extend their value for applying to more general cases.

Definition 18 (Generalization of Shapley value) For $v \in \mathcal{G}_{\mathcal{R}}$, the solution $\Psi(v) = \Psi(N, \mathfrak{N}, v) := (\psi^1(v), \dots, \psi^n(v)) \in \mathbb{R}^n$ of $(N, \mathfrak{N}, v)$ is defined by

$$\psi^i(v, N, \mathfrak{N}) := \frac{1}{|\mathcal{M}_n(\mathfrak{N})|} \sum_{\mathfrak{C} \in \mathcal{M}_n(\mathfrak{N})} (v(\mathfrak{C}_*(i) \cup \{i\}) - v(\mathfrak{C}_*(i))), \quad i = 1, \dots, n.$$

The original Shapley value is characterized by several reasonable axioms.

Axiom 6 (efficiency) For any $v \in \mathcal{G}_{\mathcal{P}}$,

$$\sum_{i \in N} \phi^i(v) = v(N).$$

Axiom 7 (0-valued for null player) Fix $v \in \mathcal{G}_{\mathcal{P}}$. For any null player $i \in N$, i.e., $v(A \cup \{i\}) = v(A)$ for any $A \in N \setminus \{i\}$, $\phi^i(v) = 0$ holds.

Axiom 8 (symmetry) If $i, j \in N$ are symmetry of $v \in \mathcal{G}_{\mathcal{P}}$, i.e., $v(A \cup \{i\}) = v(A \cup \{j\})$ for any $A \in N \setminus \{i, j\}$, $\phi^i(v) = \phi^j(v)$ holds.

Axiom 9 (linearity) For any $v, w \in \mathcal{G}_{\mathcal{P}}$, $\Phi(v + w) = \Phi(v) + \Phi(w)$ holds.

Theorem 19 Let $(N, 2^N, v)$ be a game. Then there exists the unique function $\Phi : \mathcal{G}_{\mathcal{P}} \rightarrow \mathbb{R}^n$ satisfying Axioms 6, 7, 8 and 9, and it is given by Φ_{sh} .

For yielding to our generalized value, we generalize axioms the above and add one more axiom. We denote the set of all game defined on n -length chain $(N, \mathfrak{C})$ by $\mathcal{G}(N, \mathfrak{C})$ and define $\mathcal{G}_{\mathcal{C}} := \bigcup_{n=1}^{\infty} \mathcal{G}(N, \mathfrak{C})$, that is, all of the games defined on regular set systems which are chains.

Axiom 6' (efficiency) For any $v \in \mathcal{G}_{\mathcal{P}} \cup \mathcal{G}_{\mathcal{C}}$

$$\sum_{i \in N} \phi^i(v) = v(N).$$

Definition 20 (null-player of v on the set system) i is called null-player if $i \in N$ satisfies that $v(A \cup \{i\}) = v(A)$ whenever $A \in \mathfrak{N}$, $A \cup \{i\} \in \mathfrak{N}$, $\phi^i(v) = 0$.

Axiom 7' (0-valued for null-player) Fix $v \in \mathcal{G}_{\mathcal{P}} \cup \mathcal{G}_{\mathcal{C}}$. For any null-player $i \in N$, $\phi^i(v) = 0$ holds.

Axiom 8' (symmetry) Let σ be a permutation on N . For any $v \in \mathcal{G}_{\mathcal{P}} \cup \mathcal{G}_{\mathcal{C}}$,

$$\phi^i(N, 2^N, v) = \phi^{\sigma(i)}(N, 2^N, \sigma \circ v), \quad \phi^i(N, \mathfrak{C}, v) = \phi^{\sigma(i)}(N, \sigma(\mathfrak{C}), \sigma \circ v),$$

where $\sigma(\mathfrak{C}) := \{\sigma(A) \mid A \in \mathfrak{C}\}$, $\sigma \circ v(S) := v(\sigma^{-1}(S))$, $S \in \sigma(\mathfrak{C})$.

Axiom 9' (linearity) For any $v, w \in \mathcal{G}_{\mathcal{P}} \cup \mathcal{G}_{\mathcal{C}}$, $\Phi(v + w) = \Phi(v) + \Phi(w)$ holds.

For yielding to our generalized value, we generalize axioms the above and add one more axiom.

Axiom 10 (convexity) Let $(N, \mathfrak{N})$, $(N, \mathfrak{N}_1)$, $(N, \mathfrak{N}_2)$, $\dots$, $(N, \mathfrak{N}_m)$ be regular set systems satisfying $\mathcal{M}(\mathfrak{N}) = \mathcal{M}(\mathfrak{N}_1) \cup \dots \cup \mathcal{M}(\mathfrak{N}_m)$ and $\mathcal{M}(\mathfrak{N}_1) \cap \dots \cap \mathcal{M}(\mathfrak{N}_m) = \emptyset$ and v be a game on $\mathfrak{N}$. Then there exists $\alpha_1, \dots, \alpha_m \in]0, 1[$, with $\sum_{k=1}^m \alpha_k = 1$ such that for every game $v \in \mathcal{G}(N, \mathfrak{N})$,

$$\Phi(v) = \alpha_1 \Phi(v|_{\mathfrak{N}_1}) + \dots + \alpha_m \Phi(v|_{\mathfrak{N}_m}).$$

Theorem 21 *Let $(N, \mathfrak{N})$ be a regular set system and $(N, \mathfrak{N}, v)$ a game. Then there exists the unique function satisfying Axioms 6', 7', 8', 9' and 10, and it is given by Ψ .*

Our new axioms 6', 7', 8' and 9' include axioms 6, 7, 8 and 9, respectively, and assertion of the Theorem 21 includes the theorem 19, so that our new system of axioms is suitable for a solution of the cooperative game.

We treat cooperative games defined on regular set systems. Most cooperative games which appear in applications can be regarded as games on regular set systems using by a kind of translations (See [5]).

Proof of Theorem 21 Assume that Φ satisfies axioms 6', 7', 8', 9' and 10.

(i) Case $\mathfrak{N}$ is a chain. Define $v_j \in \mathcal{G}_C$ as

$$v_j(S) := \begin{cases} 1, & |S| \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

By axioms 6' and 7', for any $\lambda \in \mathbb{R}$, it holds that

$$\phi^i(\lambda v_j) = \begin{cases} \lambda, & \{i\} = C_j \setminus C_{j-1}, \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$\phi^i(\lambda v_j) = \lambda v_j(\mathfrak{C}_*(i) \cup \{i\}) - \lambda v_j(\mathfrak{C}_*(i)).$$

$v \in \mathcal{G}_C$ is identified by $v(C_1), \dots, v(C_n)$, so that we can regard $v \in \mathcal{G}_C$ as the element of n -dimension vector space. Since the linear independency of $v_j, j = 1, \dots, n$, for $v \in \mathcal{G}_C$, there exist $\lambda_j, j = 1, \dots, n$ with

$$v = \sum_{j=1}^n \lambda_j v_j$$

such that

$$v(C_j) = \sum_{j=i}^n \lambda_j.$$

By axiom 9', for any $v \in \mathcal{G}_C$, it holds that

$$\begin{aligned} \phi^i(v) &= \phi^i\left(\sum_{j=1}^n \lambda_j v_j\right) = \sum_{j=1}^n \phi^i(\lambda_j v_j) = \sum_{j=1}^n \{\lambda_j v_j(\mathfrak{C}_*(i) \cup \{i\}) - \lambda_j v_j(\mathfrak{C}_*(i))\} \\ &= \sum_{j=i}^n \{\lambda_j v_j(\mathfrak{C}_*(i) \cup \{i\}) - \lambda_j v_j(\mathfrak{C}_*(i))\} = v(\mathfrak{C}_*(i) \cup \{i\}) - v(\mathfrak{C}_*(i)) \end{aligned}$$

and $\Phi = \Psi$.

(ii) Case $\mathfrak{N} = 2^N$. By axiom 10, there exist $\alpha_{\mathfrak{C}} \in [0, 1], \mathfrak{C} \in \mathcal{M}(2^N)$ with $\sum_{\mathfrak{C} \in \mathcal{M}(2^N)} \alpha_{\mathfrak{C}} = 1$ such that for any $v \in \mathcal{G}_P$

$$\phi^i(v) = \sum_{\mathfrak{C} \in \mathcal{M}(2^N)} \alpha_{\mathfrak{C}} \phi^i(v|_{\mathfrak{C}}) \quad (1)$$

holds.

Define $\mathfrak{N}_{\{i\}} := \{S \in 2^N \mid S \subseteq \{i\} \text{ or } S \supsetneq \{i\}\}, i = 1, \dots, n$, then by axiom 10 there exist $\beta_{1,1}, \dots, \beta_{1,n}$ with $\beta_{1,1} + \dots + \beta_{1,n} = 1$ such that for any $v \in \mathcal{G}_P$,

$$\phi^i(v) = \beta_{1,1} \phi^i(v|_{\mathfrak{N}_{\{1\}}}) + \dots + \beta_{1,n} \phi^i(v|_{\mathfrak{N}_{\{n\}}})$$

holds.

Assume that $\beta_{1,1}, \dots, \beta_{1,n}$ are not unique and there exists another representation

$$\phi^i(v) = \beta'_{1,1}\phi^i(v|_{\mathfrak{N}_{\{1\}}}) + \dots + \beta'_{1,n}\phi^i(v|_{\mathfrak{N}_{\{n\}}}).$$

Defining $v_j \in \mathcal{G}_{\mathcal{P}}$ as

$$v_j(S) := \begin{cases} 1, & |S| \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

players except i are null-players of $v_1|_{\mathfrak{N}_{\{i\}}}$, so that for any $j \neq i$, $\phi^i(v|_{\mathfrak{N}_{\{j\}}}) = 0$ holds and

$$\phi^i(v_1) = \beta_{1,i}\phi^1(v_1|_{\mathfrak{N}_{\{i\}}}), \quad \phi^i(v_1) = \beta'_{1,i}\phi^1(v_1|_{\mathfrak{N}_{\{i\}}}).$$

Hence $\beta_{1,i} = \beta'_{1,i}$, so that $\beta_{1,i}$ is unique. Moreover let σ be a permutation of players i_1 and i_2 , then by axiom 8' for any $i_1, i_2 \in N$,

$$\begin{aligned} \phi^{i_1}(N, \mathfrak{N}_{\{i_1\}}, v_1|_{\{i_1\}}) &= \phi^{\sigma(i_1)}(N, \sigma(\mathfrak{N}_{\{i_1\}}), \sigma \circ v_1|_{\{i_1\}}) \\ &= \phi^{i_2}(N, \mathfrak{N}_{\{i_2\}}, v_1|_{\{i_2\}}) \end{aligned}$$

and since any players i_1 and $i_2 \in N$ are symmetry of v_1 , for any $i = 1, \dots, n$,

$$\phi^i(v_1) = \beta_{1,i}\phi^1(v_1|_{\mathfrak{N}_{\{1\}}}) = \phi^1(v_1),$$

so that we obtain $\beta_{1,1} = \dots = \beta_{1,n} = 1/n$. Next define $\mathfrak{N}_{\{1,i\}} := \{S \in \mathfrak{N}_{\{1\}} \mid S \subseteq \{1, i\} \text{ or } S \supseteq \{1, i\}\}$, $i = 2, \dots, n$, then by axiom 10 there exist $\beta_{2,2}, \dots, \beta_{2,n}$ with $\beta_{2,2} + \dots + \beta_{2,n} = 1$, such that for any $v|_{\mathfrak{N}_{\{1\}}} \in \mathcal{G}(N, \mathfrak{N}_{\{1\}})$, it holds that

$$\phi^i(v|_{\mathfrak{N}_{\{1\}}}) = \beta_{2,2}\phi^i(v|_{\mathfrak{N}_{\{1,2\}}}) + \dots + \beta_{2,n}\phi^i(v|_{\mathfrak{N}_{\{1,n\}}}).$$

Players except i are null-players of $v_2|_{\mathfrak{N}_{\{2,i\}}}$, so that for any $j \neq i$, $\phi^i(v|_{\mathfrak{N}_{\{1,j\}}}) = 0$ holds and since for $i = 2, \dots, n$, we have $\phi^i(v_2|_{\mathfrak{N}_{\{1\}}}) = \beta_{2,i}\phi^i(v_2|_{\mathfrak{N}_{\{1,i\}}})$, $\beta_{2,i}$ is unique, and by axiom 8' we have $\beta_{2,2} = \dots = \beta_{2,n} = 1/(n-1)$. Similarly, define $\mathfrak{N}_{\{1,\dots,k,i\}} := \{S \in \mathfrak{N}_{\{1,\dots,k\}} \mid S \subseteq \{1, \dots, k, i\} \text{ or } S \supseteq \{1, \dots, k, i\}\}$, then we have generally for $i = k+1, \dots, n$, by axiom 10 there exist $\beta_{k+1,k+1}, \dots, \beta_{k+1,n}$ with $\beta_{k+1,k+1} + \dots + \beta_{k+1,n} = 1$ such that for any $v|_{\mathfrak{N}_{\{1,\dots,k\}}} \in \mathcal{G}(N, \mathfrak{N}_{\{1,\dots,k\}})$,

$$\phi^i(v|_{\mathfrak{N}_{\{1,\dots,k\}}}) = \beta_{k+1,k+1}\phi^i(v|_{\mathfrak{N}_{\{1,\dots,k,k+1\}}}) + \dots + \beta_{k+1,n}\phi^i(v|_{\mathfrak{N}_{\{1,\dots,k,n\}}})$$

holds.

Players except i are null-players of $v_{k+1}|_{\{1,\dots,k,i\}}$, so that for any $j \neq i$, $\phi^i(v|_{\mathfrak{N}_{\{1,\dots,k,j\}}}) = 0$ holds and since for $i = k+1, \dots, n$, we have

$$\begin{aligned} \phi^i(v_{k+1}|_{\mathfrak{N}_{\{1,\dots,k\}}}) &= \beta_{k+1,k+1}\phi^i(v_{k+1}|_{\mathfrak{N}_{\{1,\dots,k,k+1\}}}) + \dots + \beta_{k+1,n}\phi^i(v_{k+1}|_{\mathfrak{N}_{\{1,\dots,k,n\}}}) \\ &= \beta_{k+1,i}\phi^i(v_{k+1}|_{\mathfrak{N}_{\{1,\dots,k,i\}}}), \end{aligned}$$

$\beta_{k+1,i}$ is unique, and by axiom 8' we have $\beta_{k+1,k+1} = \dots = \beta_{k+1,n} = 1/(n-k)$.

For other terms, decomposing the set system to chains, we obtain

$$\begin{aligned}
 \phi^i(v) &= \beta_{1,1}\phi^i(v|\mathfrak{N}_{\{1\}}) + \cdots + \beta_{1,n}\phi^i(v|\mathfrak{N}_{\{n\}}) \\
 &= \beta_{1,1}[\beta_{2,2}\phi^i(v|\mathfrak{N}_{\{1,2\}}) + \cdots + \beta_{2,n}\phi^i(v|\mathfrak{N}_{\{1,n\}})] + \beta_{1,2}\phi^i(v|\mathfrak{N}_{\{2\}}) + \cdots + \beta_{1,n}\phi^i(v|\mathfrak{N}_{\{n\}}) \\
 &\vdots \\
 &= \beta_{1,1} \cdots \beta_{n-1,n-1}\phi^i(v|\mathfrak{N}_{\{1,\dots,n-1\}}) + \beta_{1,1} \cdots \beta_{n-1,n-2}\beta_{n-1,n}\phi^i(v|\mathfrak{N}_{\{1,\dots,n-2,n\}}) \\
 &\quad + \sum_{\mathfrak{C} \in \mathcal{M}(2^N) \setminus (\mathfrak{C}_1 \cup \mathfrak{C}_2)} \beta\phi^i(v|\mathfrak{C}) \\
 &= \frac{1}{n!}\phi^i(v|\mathfrak{C}_1) + \frac{1}{n!}\phi^i(v|\mathfrak{C}_2) + \sum_{\mathfrak{C} \in \mathcal{M}(2^N) \setminus (\mathfrak{C}_1 \cup \mathfrak{C}_2)} \beta\phi^i(v|\mathfrak{C}),
 \end{aligned}$$

where $\mathfrak{C}_1 := \{\emptyset, \{1\}, \{1, 2\}, \dots, N\}$, $\mathfrak{C}_2 = \{\emptyset, \{1\}, \{1, 2\}, \dots, \{1, \dots, n-2\}, \{1, \dots, n-2, n\}, N\}$.

Therefore for (1) $\alpha_{\mathfrak{C}_1} = \alpha_{\mathfrak{C}_2} = 1/n!$ are obtained. Similary $\alpha_{\mathfrak{C}}, \mathfrak{C} \in \mathcal{M}(2^N) \setminus \{\mathfrak{C}_1, \mathfrak{C}_2\}$ is obtained uniquely as $1/n!$.

Consequently, it holds that

$$\phi^i(v) = \frac{1}{|\mathcal{M}(2^N)|} \phi^i(v|\mathfrak{C}), \quad (2)$$

and $\Phi = \Psi$.

(iii) Case $\mathfrak{N}$ is a general set system. Let $\mathfrak{N} \subsetneq 2^N$. Since

$$\mathcal{M}(2^N) = \mathcal{M}(\mathfrak{N}) \cup \bigcup_{\mathfrak{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})} \mathfrak{C},$$

by axiom 10 there exist $\alpha, \beta_{\mathfrak{C}}, \mathfrak{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})$ with

$$\alpha + \sum_{\mathfrak{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})} \beta_{\mathfrak{C}} = 1 \quad (3)$$

such that for any $v \in \mathcal{G}_{\mathcal{P}}$

$$\begin{aligned}
 \phi^i(v) &= \alpha\phi^i(v|\mathfrak{N}) + \sum_{\substack{\mathfrak{C} \in \\ \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})}} \beta_{\mathfrak{C}}\phi^i(v|\mathfrak{C}) \\
 &= \alpha \left(\sum_{\mathfrak{C} \in \mathcal{M}(\mathfrak{N})} \alpha_{\mathfrak{C}}\phi^i(v|\mathfrak{C}) \right) + \sum_{\mathfrak{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})} \beta_{\mathfrak{C}}\phi^i(v|\mathfrak{C}) = \sum_{\mathfrak{C} \in \mathcal{M}(2^N)} \beta_{\mathfrak{C}}\phi^i(v|\mathfrak{C}),
 \end{aligned} \quad (4)$$

where for $\mathfrak{C} \in \mathcal{M}(\mathfrak{N})$ $\beta_{\mathfrak{C}} := \alpha\alpha_{\mathfrak{C}}$, holds. By (2) for $\mathfrak{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})$ $\beta_{\mathfrak{C}} = 1/n!$ and for $\mathfrak{C} \in \mathcal{M}(\mathfrak{N})$ $\beta_{\mathfrak{C}} = \alpha\alpha_{\mathfrak{C}} = 1/n!$, and since (3)

$$\alpha = 1 - \frac{1}{n!} \cdot |\mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})| = 1 - \frac{n! - |\mathcal{M}(\mathfrak{N})|}{n!} = \frac{|\mathcal{M}(\mathfrak{N})|}{n!}, \quad (5)$$

substituting (5) for (4), we obtain

$$\phi^i(v|\mathfrak{N}) = \frac{1}{\alpha} \left(\phi^i(v) - \sum_{\mathfrak{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})} \frac{1}{n!} \phi^i(v|\mathfrak{C}) \right) = \frac{1}{|\mathcal{M}(\mathfrak{N})|} \sum_{\mathfrak{C} \in \mathcal{M}(\mathfrak{N})} \phi^i(v|\mathfrak{C}).$$

□

On the other hand, we can give another different type of axiomatization for the generalization of Shapley value, which can be characterized with the same concept of the entropy.

Axiom 11 (continuity) The function the function $f(s, t) := \phi_1(0, s, t)$ is continuous for s on $\mathbb{R}$.

Axiom 12 (efficiency) For any game $(0, s, t)$ on **2**, $\phi_1(0, s, t) + \phi_2(0, s, t) = v(N) = t$.

Axiom 13 (dual invariance) For any $(0, s, t)$, $\Phi(0, s, t) = \Phi(0, s, t)^d$ holds.

Axiom 14 (embedding efficiency) For any $v \in \mathcal{G}_{\mathcal{C}}$, any $(0, s, 1)$ and any $C_k \in \mathfrak{C}$, $\phi_i(v^{C_k}) = \phi_i(v)$ for any $i \neq i'_k, i''_k$, $\phi_{i'_k}(v^{C_k}) = \phi_{i_k}(v) \cdot \phi_1(0, s, 1)$ and $\phi_{i''_k}(v^{C_k}) = \phi_{i_k}(v) \cdot \phi_2(0, s, 1)$ hold, where $\{i_k\} := C_k \setminus C_{k-1} = \{i'_k, i''_k\}$.

Axiom 15 (permutation invariance) For any $v \in \mathcal{G}_{\mathcal{R}}$ and any permutation π on N satisfying $\pi(\mathfrak{N}) = \mathfrak{N}$, $\phi_i(v) = \phi_{\pi(i)}(\pi \circ v)$, $i = 1, \dots, n$ holds.

We obtain the following theorem:

Theorem 22 ([7]) Let $(N, \mathfrak{N})$ be a regular set system and $(N, \mathfrak{N}, v)$ a game. Then there exists the unique function satisfying Axioms 11, 12, 13, 14 and 15, and it is given by Ψ .

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