Journal of Physics: Conference Series

A geometric approach to higher-order Riccati chain: Darboux polynomials and constants of the motion

To cite this article: José F Cariñena *et al* 2009 *J. Phys.: Conf. Ser.* **175** 012009

A geometric approach to higher-order Riccati chain: Darboux polynomials and constants of the motion

José F. Cariñena[1], **Partha Guha**[§2] **and Manuel F. Rañada**[3]

Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain
[§] Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, D-04103 Leipzig, Germany
[§] S.N. Bose National Centre for Basic Sciences, JD Block
Sector-3, Salt Lake, Calcutta-700098, India

E-mail: [1]jfc@unizar.es, [2]partha@bose.res.in, [3]mfran@unizar.es

Abstract. The properties of higher-order Riccati equations are investigated. The second-order equation is a Lagrangian system and can be studied by using the symplectic formalism. The second-, third- and fourth-order cases are studied by proving the existence of Darboux functions. The corresponding cofactors are obtained and some related properties are discussed. The existence of generators of t-dependent constants of motion is also proved and then the expressions of the associated time-dependent first integrals are explicitly obtained. The connection of these time-dependent first integrals with the so-called master symmetries, characterizing some particular Hamiltonian systems, is also discussed. Finally the general n-th-order case is analyzed.

1. Introduction

Symmetry analysis and the corresponding reduction process is a fundamental approach in the solution of many physical problems. The introduction by Lie of the concept of the now called Lie groups and algebras has played a relevant rôle in the solution of systems of differential equations. The knowledge of infinitesimal symmetries for a given system of differential equations allows us to introduced adapted coordinates which reduce the system to one of a simpler form, and in particular differential equations to lower order ones. For instance linear second order differential equations are invariant under dilations and the corresponding reduction equation turns out to be a first order Riccati equation. This is one of the reasons for the relevance in physics of such an equation.

The second-order Riccati equation has been studied in [1] from a geometric perspective and it has been proved to admit two alternative Lagrangian formulations, both Lagrangians being of a non-natural class (neither potential nor kinetic term). Our aim is to use analogous geometric techniques to deal with the higher-order terms of the so called Riccati sequence [2, 3, 4]. The symmetry properties of the Riccati sequence (also known as Riccati hierarchy or chain) has been recently studied by Euler *et al* [5]. They showed that the members of the Riccati hierarchy admit the maximal number of Lie point symmetries for an equation of order no lower than two. The first two members of the hierarchy are special cases of the standard Riccati equation and the second-order Riccati equation (SORE) (also known as Painlevé-Ince equation).

With this aim we find some Darboux polynomials (see [6, 7] for introduction) which may be helpful in the reduction or even solution of the given problem. In 1878 Darboux [8] showed how one can construct the first integrals of planar polynomial ordinary differential equations (ODEs) possessing sufficient invariant algebraic curves. Nowadays Darboux method of integrability is considered to be one of the best methods for finding first integrals of polynomial ODEs. Recall that Prelle and Singer [9] proposed a few years ago a method for solving first-order ODEs that presents their solutions in terms of elementary functions when such a solution exists. Recently Duarte *et al.* [10] modified the Prelle-Singer technique and applied it to second-order ODEs. The method was based on a conjecture that if an elementary solution exists for the given second-order ODE then there exists at least one elementary first integral $I(t, x, \dot{x})$ whose derivatives are all rational functions of t, x and $\dot{x}$. More recently Chandrasekar *et al.* [11] discussed a method for solving n-th-order scalar ODEs by extending some ideas based on the Prelle-Singer method for second-order ODEs; however there is not so much work done on higher-order operators.

Our first objective is the study of higher-order Riccati equations, as $\ddot{x}+3k\,x\,\dot{x}+k^2x^3 = 0$, which was recently studied in [1] within a Lagrangian formulation, from a more geometric perspective and using their relations with the theory of Darboux and the extended Prelle-Singer methods. We give a new development of a deeper analysis of these equations using presymplectic forms, Darboux polynomials and Jacobi multipliers. This suggests us how to deal with higher-order Riccati equations in a similar approach.

The plan of this article is as follows: In Sec. 2 we recall the Riccati chain and its relation with the series of differential equations $y^{(n)} = 0$ from which the elements of the Riccati sequence appear by using the Lie reduction method. Section 3 is devoted to review some results of [1] for the second-order Riccati equation and other mathematical tools as Darboux polynomials and Jacobi last multipliers which enable us with a better understanding of the Lagrangian theory developed in [1]. We develop in Section 4 an analogous formalism for the third-order Riccati equation (TORE). In particular, we find a Darboux polynomial and a presymplectic form for the TORE. Section 5 is devoted to an analogous formalism for the fourth-order Riccati equation. Finally the generic n-th-order case is analysed in Section 6 and the results are summarized in the last section.

2. The Riccati chain

The Riccati sequence can be introduced as follows. Let us define, for a given real number $k \in \mathbb{R}$, the differential operator

$$\mathbb{D}_k = \frac{d}{dt} + k\,x\,, \tag{1}$$

and then consider the sequence obtained by applying such a differential operator to the function $x(t)$ in an iterative way so that we obtain

$$
\begin{array}{ll}
n=0 & \mathbb{D}_k^0 x = x \\
n=1 & \mathbb{D}_k x = \left(\dfrac{d}{dt} + kx\right) x = \dot{x} + k\,x^2 \\
n=2 & \mathbb{D}_k^2 x = \left(\dfrac{d}{dt} + kx\right)^2 x = \ddot{x} + 3k\,x\,\dot{x} + k^2\,x^3 \\
n=3 & \mathbb{D}_k^3 x = \left(\dfrac{d}{dt} + kx\right)^3 x = \dddot{x} + 4\,k\,x\ddot{x} + 6k^2\,x^2\dot{x} + 3k\,\dot{x}^2 + k^3\,x^4 \\
n=4 & \mathbb{D}_k^4 x = \left(\dfrac{d}{dt} + kx\right)^3 x = \ddddot{x} + 5kx\,\dddot{x} + 10k\,\dot{x}\ddot{x} + 15k^2\,x\,\dot{x}^2 + 10k^2x^2\ddot{x} \\
 & \qquad + \; 10k^3x^3\dot{x} + k^4\,x^5
\end{array}
\tag{2}
$$

and analogous expressions for higher values of n.

The equation

$$R^{(j)}(x,\dots,x^{(j)}) = \mathbb{D}_k^j x = 0\,, \quad j = 0,1,\dots,$$

is a Riccati equation of order j. Note that $R^{(0)}(x) = x$, the particular case $R^{(1)}(x,\dot{x}) = 0$, obtained for $j = 1$, is the standard Riccati equation with constant coefficients in its reduced form:

$$\dot{x} + k\,x^2 = 0\,, \tag{3}$$

and the particular case $R^{(2)}(x,\dot{x},\ddot{x}) = 0$, obtained for $j = 2$, is the second-order Riccati equation

$$\ddot{x} + 3k\,x\,\dot{x} + k^2\,x^3 = 0\,, \tag{4}$$

recently studied in [1]. Our aim here is to first study the Riccati equation of third-order

$$\dddot{x} + 4\,k\,x\ddot{x} + 6k^2\,x^2\dot{x} + 3k\,\dot{x}^2 + k^3\,x^4 = 0\,, \tag{5}$$

which appears in the same way as the first- (3) and second-order Riccati (4) equations; then the study proceeds with other higher-order equations. Of course, for any j we have $R^{(j+1)} = \mathbb{D}_k R^{(j)}$, by definition.

The relevance of usual first-order Riccati equation in classical and quantum physics is mainly due to its appearance in the Lie reduction process when taking into account invariance under infinitesimal dilations of linear second-order differential equations [12] and it was proved in [1] that the second-order Riccati equations appear in a similar way from linear third-order differential equations and in general the j-order Riccati equation appears as a reduction of a linear $(j+1)$-order differential equation. More specifically, given a differential equation $y^{(n)} = 0$, its invariance under dilations suggests, according to Lie recipe, to look for a new variable z such that the dilation vector field $y\,\partial/\partial y$ becomes $(1/k)\partial/\partial z$, the factor k being purely conventional. Then $y = e^{kz}$, up to an irrelevant factor, and with this change of variable, $y^{(n)} = 0$, for $n > 1$, becomes $R^{(n-1)}(x) = 0$ with $x = \dot{z}$, because, when $D = d/dt$, we have that $e^{-kz}De^{kz} = D + k\,x$, and then we can prove by complete induction that $y^{(n)} = k\,e^{kz}R^{(n-1)}(x)$. In fact, the property is true for $n = 1$, because $y' = k\,\dot{z}\,e^{kz} = e^{kz}\,k\,x = k\,e^{kz}\,R^{(0)}(x)$, and if the property is true for n, it also holds for $n+1$, because

$$D^{n+1}y = D(y^{(n)}) = D(k\,e^{kz}R^{(n-1)}(x)) = k\,e^{kz}(D + k\,x)R^{(n-1)}(x) = k\,e^{kz}\,R^{(n)}(x).$$

Therefore, Lie recipe applied to the invariant under dilation equation $y^{(n)} = 0$ transforms such equation into $R^{(n-1)}(x) = 0$. Of course the most general Riccati equation of order n can be obtained from a linear combination $\sum_{j=0}^{n+1} a_j(t)y^{(j)}$ which gives rise to

$$a_0(t) + k\sum_{j=0}^{n} a_{j+1}(t)\, R^{(j)}(x,\ldots,x^{(j)}) = 0\,.$$

There is an identity generalising the chain rule to higher derivatives which provides us the element of order n of the Riccati chain. This identity, usually known as Faá di Bruno formula [13, 14], is given by

$$\frac{d^n}{dt^n} g(f(t)) = \sum \frac{n!}{b_1!\,\cdots\, b_n!} g^{(b_1+\cdots+b_n)}(f(t)) \left(\frac{\dot f(t)}{1!}\right)^{b_1} \cdots \left(\frac{f^{(n)}(t)}{n!}\right)^{b_n}, \tag{6}$$

where the sum is over all natural numbers such that $b_1 + 2b_2 + \cdots + n\, b_n = n$.

In our case $f(t) = z(t)$ and $g(t) = e^{kt}$, and taking into account that $g^{(l)} = k^l\, g$ and that $\dot z = x$:

$$y^{(n)} = \frac{d^n}{dt^n} e^z = e^z \sum_{b_1+\cdots+nb_n=n} \frac{n!}{b_1!\,\cdots\, b_n!}\, k^{b_1+\cdots+b_n} x^{b_1} \left(\frac{\dot x}{2!}\right)^{b_2} \cdots \left(\frac{x^{(n-1)}}{n!}\right)^{b_n}, \tag{7}$$

which can be used to find the nth-element of the Riccati sequence:

$$\sum_{b_1+\cdots+nb_n=n} \frac{n!}{b_1!\,\cdots\, b_n!}\, k^{b_1+\cdots+b_n} x^{b_1} \left(\frac{\dot x}{2!}\right)^{b_2} \cdots \left(\frac{x^{(n-1)}}{n!}\right)^{b_n} = 0\,. \tag{8}$$

Note that the coefficient of the highest-order derivative in (8), which corresponds to $b_n = 1$, is 1.

3. The Lagrangian form of the second-order Riccati equation

We denote $\widetilde{R}^{(j)}(x, v^{(1)},\ldots,v^{(j)})$ the expression obtained from $R^{(j)}$ by a simple substitution of $v = v^{(1)}$ by $\dot x$, $a = v^{(2)}$ by $\ddot x$, and so on. For instance,

$$\widetilde{R}^{(0)}(x) = x,\quad \widetilde{R}^{(1)}(x,v) = v + kx^2,\quad \widetilde{R}^{(2)}(x,v,a) = a + 3kxv + k^2x^3,$$

$$\widetilde{R}^{(3)}(x,v,a,w) = w + 4kxa + 6k^2x^2\, v + 3kv^2 + k^3x^4,$$

$$\widetilde{R}^{(4)}(x,v,a,w,u) = u + 5kxw + 10kva + 15k^2xv^2 + 10k^2x^2a + 10k^3x^3v + k^4x^5\,.$$

In the geometric approach to differential equations the element of order j in our Riccati sequence is given by the zero level set of the function $\widetilde{R}^{(j)}$ defined in $T^j\mathbb{R}$ (the j-th-order tangent bundle, see e.g. [15, 16]), which can also be seen as a section for the natural map $\tau_{j,j-1} : T^j\mathbb{R} \to T^{j-1}\mathbb{R}$.

Journal of Physics: Conference Series **175** (2009) 012009 doi:10.1088/1742-6596/175/1/012009

There is another geometric interpretation of a differential equation of order j as given by a vector field in $T^{j-1}\mathbb{R}$. For instance, recall that an autonomous system of second-order differential equations like $\ddot{x} = F(x, \dot{x})$ has associated the system of first-order differential equations

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & v \\ \dfrac{dv}{dt} & = & F(x,v) \end{array} \right. , \tag{9}$$

whose solutions are the integral curves of the vector field in $T\mathbb{R}$

$$\Gamma_F^{(2)} = v\,\frac{\partial}{\partial x} + F(x,v)\frac{\partial}{\partial v}\,.$$

In particular, for the second-order Riccati equation (4), we have

$$\Gamma^{(2)} = v\,\frac{\partial}{\partial x} - (3kxv + k^2x^3)\frac{\partial}{\partial v}\,. \tag{10}$$

In the alternative geometric approach, the second-order differential equation can be seen as a section for $\tau_{2,1} : T^2\mathbb{R} \to T\mathbb{R}$, $\sigma(x, v) = (x, v, F(x, v))$, which in the second-order Riccati equation (4) reduces to $\sigma(x, v) = (x, v, -(3kxv + k^2x^3))$.

Similarly, an autonomous third-order differential equation $\dddot{x}= F(x, \dot{x}, \ddot{x})$ is studied either by means of the system of first-order differential equations

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & v \\ \dfrac{dv}{dt} & = & a \\ \dfrac{da}{dt} & = & F(x,v,a) \end{array} \right. \tag{11}$$

whose solutions are the integral curves of the vector field in $T^2\mathbb{R}$

$$\Gamma_F^{(3)} = v\frac{\partial}{\partial x} + a\frac{\partial}{\partial v} + F(x,v,a)\frac{\partial}{\partial a}\,,$$

or by the section σ for $\tau_{3,2} : T^3\mathbb{R} \to T^2\mathbb{R}$ given by $\sigma(x, v, a) = (x, v, a, F(x, v, a))$.

In particular, (5) is described either by the vector field

$$\Gamma^{(3)} = v\frac{\partial}{\partial x} + a\frac{\partial}{\partial v} - (4\,k\,x\,a + 6k^2\,x^2\,v + 3k\,v^2 + k^3\,x^4)\frac{\partial}{\partial a}\,, \tag{12}$$

or by the section $\sigma(x, v, a) = (x, v, a, -(4\,k\,x\,a + 6k^2\,x^2\,v + 3k\,v^2 + k^3\,x^4))$.

In the general case of a generic $\Gamma^{(j)}$ the connection between the vector field and the section describing the system is given by the $\mathbf{T}^{(j-1)}$ operator introduced in [15] or the canonical immersion $i_{j-1,1}$ used in [16].

In the search for constants of the motion allowing us to reduce a given system, the Jacobi multipliers [17] play a relevant rôle (see later on for the definition of Jacobi multiplier). A very useful ingredient for the determination of a Jacobi multiplier for a vector field, or even a Lagrangian in the case of a second-order differential equation, is the concept of Darboux polynomial we recall next.

Definition 1 *Let U be an open subset of $\mathbb{R}^n$. We say that a polynomial function $\mathcal{D} : U \to \mathbb{R}$ is a Darboux polynomial for the polynomial vector field X if there is a polynomial function f defined in U such that $X\mathcal{D} = f\mathcal{D}$. The function f is said to be the cofactor corresponding to such Darboux polynomial.*

Note that when $\mathcal{D}$ is a Darboux polynomial for the polynomial vector field X, the zero level set of $\mathcal{D}$ is an algebraic curve invariant under X, and that given two Darboux polynomials for X, $\mathcal{D}_i : U_i \to \mathbb{R}$, $i = 1, 2$, with cofactors f_1 and f_2 respectively, its product $\mathcal{D}_1\mathcal{D}_2 : U_1 \cap U_2 \to \mathbb{R}$ is a Darboux polynomial with cofactor $f_1 + f_2$. Consequently, the power $\mathcal{D}^n$, $n \in \mathbb{N}$, of a Darboux polynomial $\mathcal{D}$ with cofactor f is a Darboux polynomial with cofactor $n\, f$. Moreover, for any integer number p, $X\mathcal{D}^p = p\, f\, \mathcal{D}^p$, and $X(\mathcal{D}_1/\mathcal{D}_2) = (f_1 - f_2)(\mathcal{D}_1/\mathcal{D}_2)$. Consequently, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are Darboux polynomials with the same cofactor, the quotient $\mathcal{D}_1/\mathcal{D}_2$ is a first integral.

The rational functions $\mathcal{F}$ such that $X\mathcal{F} = f\,\mathcal{F}$, with f a polynomial function are also known as second-integrals for X, because they are first-integrals when $f = 0$. More generally, a set of C^1-functions $I_i = I_i(x)$, $i = 1, \ldots, r$, is considered to be a set of higher order integrals if there exist r^2 C^1-functions β_{ij} such that

$$XI_i = \sum_{j=1}^{r} \beta_{ij} I_j, \qquad i = 1, \cdots, r.$$

In this case, if $dI_1 \wedge \cdots \wedge dI_r \neq 0$ the set $\bigcap_{k=1}^{r} I_k^{-1}(0)$ is an invariant $(n-r)$-dimensional manifold. If for an index i we have that $\beta_{i1} = \beta_{i2} = \cdots = \beta_{ir} = 0$, the function I_i is called first integral, otherwise, if some β_{ij} are not zero, the system can be used to define second, third or higher order integrals.

In this paper we often use products of powers of Darboux polynomials as playing a rôle similar to that of the Lagrangian. In fact, it can be seen that for second-order Riccati equations the Lagrangian is the inverse of a Darboux polynomial, because as it has been proved in [1], a general second-order Riccati equation admits a Lagrangian formulation with a Lagrangian of a non-mechanical type

$$L(x, v) = \frac{1}{v + k\, U(x, t)}\,,$$

where $U(x, t) = c_0(t) + c_1(t)\, x^2$ for a convenient choice of functions $c_i(t)$. For instance, (4) is described by the Lagrangian function

$$L_1(x, v) = \frac{1}{v + k\, x^2} = (\widetilde{R}^{(1)}(x, v))^{-1}\,. \tag{13}$$

Note that $\widetilde{R}^{(1)}(x, v)$ is a Darboux polynomial for $\Gamma^{(2)}$ with cofactor $-kx$, because $(\Gamma^{(2)}\widetilde{R}^{(1)})(x, v) = -kxv - k^2x^2 = -kx\, R^{(1)}(x, v)$, and when $L_1(x, v) = (R^{(1)}(x, v))^{-1}$, the vector field $\Gamma^{(2)}$ given by (10) is such that $\Gamma^{(2)}L_1 = k\, x\, L_1$. Moreover, the associated Liouville 1-form θ_{L_1} and the corresponding symplectic 2-form $\omega_{L_1} = -d\theta_{L_1}$,

$$\theta_{L_1} \;=\; \frac{\partial L_1}{\partial v}\, dx = -\frac{1}{(v + kx^2)^2}\; dx = -{L_1}^2\, dx\,, \tag{14}$$

$$\omega_{L_1} \;=\; -d\theta_{L_1} = \frac{2\, dx \wedge dv}{(v + kx^2)^3} = d{L_1}^2 \wedge dx\,, \tag{15}$$

together with the corresponding energy function

$$E_{L_1}(x,v) = v\,\frac{\partial L_1}{\partial v} - L_1 = \frac{-(2v+kx^2)}{(v+kx^2)^2}\,, \tag{16}$$

show that $\Gamma^{(2)}$ is such that $i(\Gamma^{(2)})\omega_{L_1} = dE_{L_1}$, i.e. $\Gamma^{(2)}$ is the dynamical vector field defined by the Lagrangian L_1.

We recall that given a vector field X in an oriented manifold (M,Ω), a function R such that $R\,i(X)\Omega$ is closed is said to be a Jacobi multiplier for X. Note that this means that $R\,X$ is a divergenceless vector field and then

$$\mathcal{L}_{RX}\Omega = \mathrm{div}\,(RX)\;\Omega = [X(R) + R\,\mathrm{div}\,X]\Omega = 0\,,$$

and therefore we see that R is a last multiplier for X if and only if

$$X(R) + R\,\mathrm{div}\,X = 0\,. \tag{17}$$

Note that if R is a never vanishing Jacobi multiplier, then $f\,R$ is a Jacobi multiplier too if and only if f is a constant of motion, i.e. $Xf = 0$, because if (17) is true, then for any function f,

$$X(f\,R) + f\,R\,\mathrm{div}\,X = (Xf)\,R + f(X(R) + R\,\mathrm{div}\,X) = (Xf)\,R\,,$$

and consequently $f\,R$ is a multiplier too if and only if $Xf = 0$.

The remarkable point is that if $\mathcal{D}_1,\ldots,\mathcal{D}_k$ are Darboux polynomials with corresponding cofactors f_i, $i=1,\ldots,k$, one can look for multiplier factors of the form

$$R = \prod_{i=1}^{k} \mathcal{D}_i^{\nu_i} \tag{18}$$

and then

$$\frac{X(R)}{R} = \sum_{i=1}^{k} \nu_i\,\frac{X(\mathcal{D}_i)}{\mathcal{D}_i} = \sum_{i=1}^{k} \nu_i\,f_i\,,$$

and therefore, if the coefficients ν_i can be chosen such that

$$\sum_{i=1}^{k} \nu_i\,f_i = -\mathrm{div}\,X \tag{19}$$

holds, then we arrive to

$$\frac{X(R)}{R} = \sum_{i=1}^{k} \nu_i\,f_i = -\mathrm{div}\,X\,,$$

and consequently R is a Jacobi last multiplier for X.

In the case of $X = \Gamma^{(2)}$ we are considering we see that as $\mathrm{div}\,\Gamma^{(2)} = -3kx$, a Jacobi multiplier obtained as a power of the Darboux polynomial $\mathcal{D}_1 = \widetilde{R}^{(1)}$ is $R(x,v) = (\widetilde{R}^{(1)}(x,v))^{-3}$.

The main point is that if R is a Jacobi multiplier for a vector field which corresponds to a second-order differential equation, then there is an essentially unique Lagrangian (up to addition of a gauge term) such that $R = \partial^2 L/\partial v^2$ [18, 19].

Actually, the inverse problem for the existence of a Lagrangian was analysed by Helmholtz and he found a set of conditions that a multiplier matrix $g_{ij}(x,\dot{x})$ must satisfy in order for a given system of second-order equations

$$\ddot{x}^j = F^j(x,\dot{x})\,,\quad j=1,2,\ldots,n,$$

or the corresponding system

$$\begin{cases} \dfrac{dx^i}{d\tau} &=& v^i \\ \dfrac{dv^i}{dt} &=& F^i(x,v) \end{cases}$$

which provides us the integral curves for the vector field

$$X = v^i\frac{\partial}{\partial x^i} + F^i(x,v)\frac{\partial}{\partial v^i}\,,$$

when written of the form

$$g_{ij}(\ddot{x}^j - F^j(x,\dot{x})) = 0\,,\quad i,j=1,2,\ldots,n,$$

to be the set of Euler–Lagrange equations for a certain Lagrangian L [20, 21, 22, 23] (the summation convention on repeated indices is assumed). If a matrix solution g_{ij} is obtained then it can be identified with the Hessian matrix of L, that is $g_{ij} = \partial L/\partial v^i\partial v^j$, and a Lagrangian L can be obtained by direct integration of the g_{ij} functions. The two first conditions just impose regularity and symmetry of the matrix g_{ij}; the two other ones are equations introducing relations between the derivatives of g_{ij} and the derivatives of the functions F^i. Here we only write the fourth set of conditions that determine the time-evolution of the g_{ij}

$$X(g_{ij}) = g_{ik}A^k{}_j + g_{jk}A^k{}_i\,,\quad A^i{}_j = -\frac{1}{2}\frac{\partial F^i}{\partial v^j}\,.$$

When the system is one-dimensional we have $i=j=k=1$ and then the three first set of conditions become trivial and the fourth one reduces to one single first-order P.D.E.

$$X(g) + g\,\frac{\partial F}{\partial v} \equiv v\,\frac{\partial g}{\partial x} + F\,\frac{\partial g}{\partial v} + g\,\frac{\partial F}{\partial v} = 0\,, \tag{20}$$

which is nothing but the equation (17) defining the Jacobi multipliers, because $\mathrm{divX} = \partial F/\partial v$.

Therefore, the generalised inverse problem reduces to find the function g as a solution of this equation, i.e. a Jacobi multiplier. Once a solution g is known a Lagrangian L is obtained by integrating the function g two times with respect to velocities. The funtion L so obtained from g is unique up to addition of a gauge term. For instance in the case of the equation (4) we find, up to a gauge term, the Lagrangian (13).

It is also noteworthy that if a system with one degree of freedom admits two different regular Lagrangins then the function F definida por

$$\frac{\partial^2 L_1}{\partial v^2} = F\,\frac{\partial L_2}{\partial v^2}$$

is a constant of the motion, because such a function F is the quotient of two different Jacobi multipliers. This result is usually attributed to Currie and Saletan [24], but actually data back to Jacobi's time.

Note however that $\mathcal{D}_2(x,v) = 2v - kx^2$ is also a Darboux polynomial with cofactor $f_2(x,v) = -2kx$, because $(\Gamma^{(2)}\mathcal{D}_2)(x,v) = v(2kx) - 2(3kxv + k^2x^3) = -2kx(2v + kx^2)$. One can therefore look for a new Jacobi multiplier that is a power of $\mathcal{D}_2$ and we obtain $R'(x,v) = (2v + kx^2)^{-3/2}$, defining the new Lagrangian function

$$L'_1(x,v) = \sqrt{2v + kx^2}$$

Note that the quotient of both Jacobi multipliers is but a function of the energy, which is a constant of the motion.

A function T that satisfies the following property

$$\frac{d}{dt}T \neq 0 \quad ,\ldots, \quad \frac{d^m}{dt^m}T \neq 0\,, \quad \frac{d^{m+1}}{dt^{m+1}}T = 0\,,$$

is called a generator of integrals of motion of degree m. Notice that this means that the function T is a non-constant function generating a constant of motion by time derivation and also that T can be used for generating a t-dependent constant of the motion. This function, to be denoted by J_t, turns out to be a polynomial of order m in the time t. In the Hamiltonian (symplectic) case this property is directly related with the existence of "master symmetries" [25, 26, 27, 28]. If the dynamics is represented by a Hamiltonian vector field Γ_H, then a vector field Z that satisfies the following two properties

$$[Z, \Gamma_H] \;=\; \widetilde{Z} \neq 0\,, \qquad [\,\widetilde{Z}\,, \Gamma_H] = 0\,,$$

is called a "master symmetry" of degree $m = 1$ for Γ_H. If Z is such that

$$[Z, \Gamma_H] = \widetilde{Z} \neq 0\,,\ [\,\widetilde{Z}\,, \Gamma_H] \neq 0 \quad \text{and} \quad [\,[\,\widetilde{Z}\,, \Gamma_H]\,, \Gamma_H] = 0\,,$$

then it is called a "master symmetry" of degree $m = 2$, and similarly for $m = 3, 4, \ldots$.

In the second-order Riccati case (4), we have that the Lagrangian $T_1 = L_1$ is a generator of degree three of first-integrals for $\Gamma^{(2)}$, as corresponding to a master symmetry of the dynamics [1]. More explicitly, we have

$$\Gamma^{(2)}T_1 = T_2 = \frac{k\,x}{v + k\,x^2}\,,$$

$$\Gamma^{(2)}(\Gamma^{(2)}T_1) = \Gamma^{(2)}T_2 = k\,,$$

$$\Gamma^{(2)}(\Gamma^{(2)}(\Gamma^{(2)}T_1)) = 0\,.$$

Hence T_1 and T_2 are generators of t-dependent constants of motion for the Lagrangian. The two functions

$$J^{(2)}_{t1} = T_2 - k\,t\,, \quad J^{(2)}_{t2} = T_1 - t\,T_2 + \frac{1}{2}\,k\,t^2\,,$$

are the corresponding constants of the motion obtained from T_1 and T_2.

4. A geometric approach to third-order Riccati equation

For the case of the third-order Riccati equation in the Riccati sequence one can proceed in a similar way but of course we cannot have a (regular) Lagrangian formulation.

First, when applying the vector field (12) to $\widetilde{R}^{(2)}(x,v,a)$ we see that

$$(\Gamma^{(3)}\widetilde{R}^{(2)})(x,v,a) = -kx\,\widetilde{R}^{(2)}(x,v,a)\,,$$

and therefore the polynomial function $\mathcal{D}_1 = \widetilde{R}^{(2)}$ is a Darboux polynomial with cofactor equal to $-k\,x$. Consequently, we can consider the corresponding generalisation of the function L_1 given by (13):

$$L_2(x,v,a) = \frac{1}{a + 3k\,x\,v + k^2\,x^3} = (\widetilde{R}^{(2)}(x,v,a))^{-1}\,, \tag{21}$$

which is not a Lagrangian for (5) but note that the function L_2 is such that

$$\Gamma^{(3)}(L_2) = k\,x\,L_2\,. \tag{22}$$

This means that the function L_2 is not a constant of motion for the dynamics $\Gamma^{(3)}$ and the foliation of level surfaces of L_2 is not invariant under the vector field (12), but its zero level set is invariant. Furthermore, as $\operatorname{div}\Gamma^{(3)} = -4\,k\,x$, we can find a Jacobi multiplier as a power $(\mathcal{D}_1)^{-4}$ of the Darboux polynomial $\mathcal{D}_1$, i.e. we find a solution of (19) with only one term.

Note also that

$$dL_2 = -\frac{3k(kx^2+v)dx + 3kx\,dv + da}{(a + 3k\,x\,v + k^2\,x^3)^2}\,,$$

and therefore we can define a 2-form ω_{L_2} in $T^2\mathbb{R}$ in full similarity with the second relation in (15) by

$$\omega_{L_2} = dL_2^2 \wedge dx, \tag{23}$$

whose coordinate expression turns out to be given by

$$\omega_{L_2} = 2\frac{3kx\,dx\wedge dv + dx\wedge da}{(a + 3k\,x\,v + k^2\,x^3)^3}\,.$$

Such 2-form ω_{L_2} is exact and therefore $d\omega_{L_2} = 0$, but as $T^2\mathbb{R}$ is three-dimensional, ω_{L_2} is degenerate: the rank of ω_L is two and therefore its kernel is one-dimensional. More explicitly, one can see that such kernel is generated by the vector field in $T^2\mathbb{R}$

$$K = \frac{\partial}{\partial v} - 3kx\frac{\partial}{\partial a}\,,$$

because such vector field is such that $K(L_2) = K(x) = 0$.

The presymplectic form ω_{L_2} gives rise to a foliation determined by $\ker\omega_{L_2}$, i.e. spanned by the vector field K, and induces a symplectic form in the quotient, the set of leaves. Moreover, both functions L_2 and x are K-invariant and therefore projectable functions. This points out the convenience of using instead of coordinates (x,v,a) three different coordinates (x,f,L_2) with f being a function such that $Kf = 1$. In such coordinates the projected 2-form looks like in (23).

The function L_2^2 is such that there exist vector fields whose contraction with ω_{L_2} is dL_2^2. Such vector fields are those of the form

$$Y = -\frac{\partial}{\partial x} + 3k(v + kx^2)\frac{\partial}{\partial a} + b(x, v, a)K\,.$$

In particular for $b(x, v, a) = kx$ we obtain the vector field $X_1^{(2)}$ given by

$$X_1^{(2)} = -\frac{\partial}{\partial x} + kx\,\frac{\partial}{\partial v} + 3kv\,\frac{\partial}{\partial a}\,.$$

In fact, note that as L_2 is given by (13), we have

$$X_1^{(2)}(x) = -1\,, \qquad X_1^{(2)}(L_2) = -\frac{X_1^{(2)}(a + 3k\,x\,v + k^2\,x^3)}{(a + 3k\,x\,v + k^2\,x^3)^2} = 0\,,$$

and consequently, from the definition (23) one easily check that

$$i(X_1^{(2)})\omega_{L_2} = (X_1^{(2)}L_2^2)\;dx - (X_1^{(2)}x)\;dL_2^2 = dL_2^2\,.$$

Let us explore the existence of t-dependent constants of motion. The function $T_1 = L_2$ is such that

$$\Gamma^{(3)}(T_1) = T_2 = \frac{kx}{a + 3k\,x\,v + k^2\,x^3}\,,$$

$$\Gamma^{(3)}(T_2) = T_3 = \frac{k(v + kx^2)}{a + 3k\,x\,v + k^2\,x^3}\,,$$

$$\Gamma^{(3)}(T_3) = T_4 = k\,.$$

Hence T_1, T_2 and T_3 are generators of integrals of motion for the vector field $\Gamma^{(3)}$ and the t-dependent functions

$$J_{t1}^{(3)} = T_3 - t\,, \qquad J_{t2}^{(3)} = T_2 - t\,T_3 + \frac{1}{2}\,kt^2\,, \qquad J_{t3}^{(3)} = T_1 - t\,T_2 + \frac{1}{2}\,t^2 T_3 - \frac{1}{6}\,kt^3\,,$$

are constants of the motion for the dynamical vector field $\Gamma^{(3)}$.

5. Darboux function, closed two form and fourth-order Riccati equation

Let us now consider the fourth-order case of Riccati equation $R^{(4)}(x, \dot x, \ddot x, \dddot x, \ddddot x) = 0$, i.e.

$$\ddddot x + 5kx\,\dddot x + 10k\,\dot x\ddot x + 15k^2\,x\,\dot x^2 + 10k^2x^2\ddot x + 10k^3x^3\dot x + k^4\,x^5 = 0\,, \tag{24}$$

with the following associated system of first-order differential equations:

$$\left\{\begin{array}{rcl} \dfrac{dx}{dt} &=& v \\ \dfrac{dv}{dt} &=& a \\ \dfrac{da}{dt} &=& w \\ \dfrac{dw}{dt} &=& -(5\,kxw + 10k\,v\,a + 15k^2\,x\,v^2 + 10k^2x^2\,a + 10k^3x^3\,v + k^4\,x^5) \end{array}\right. \tag{25}$$

whose solutions are the integral curves of the vector field

$$\Gamma^{(4)} = v\frac{\partial}{\partial x} + a\frac{\partial}{\partial v} + w\frac{\partial}{\partial a} - (5kxw + 10k\,v\,a + 15k^2\,x\,v^2 + 10k^2x^2\,a + 10k^3x^3\,v + k^4\,x^5)\frac{\partial}{\partial w}\,. \quad (26)$$

Note that now in full similarity with the preceding cases

$$(\Gamma^{(4)}\widetilde{R}^{(3)})(x, v, a, w) = -kx\,\widetilde{R}^{(3)}(x, v, a, w)\,,$$

i.e. $\widetilde{R}^{(3)}(x, v, a, w)$ is a Darboux polynomial for $\Gamma^{(4)}$ with respective cofactor $-k\,x$.

Moreover, as $\operatorname{div}\Gamma^{(4)} = -5\,k\,x$, we can find a Jacobi multiplier as a power $R = (\mathcal{D}_1)^{-5}$ of the Darboux polynomial $\mathcal{D}_1$, as a solution of (19) with only one term.

By analogy with the preceding cases we can define the function L_3 given by

$$L_3(x, v, a, w) = \frac{1}{w + 4\,k\,x\,a + 6k^2\,x^2\,v + 3k\,v^2 + k^3\,x^4} = (\widetilde{R}^{(3)}(x, v, a, w))^{-1} \quad (27)$$

which satisfies the condition analogous to (22) with the new vector field $\Gamma^{(4)}$ and the function L_3. Define the 2-form ω_{L_3} as in the preceding example by an expression analogous to (15) and (23) but with L_3 instead of L_1 and L_2 respectively, which is written in this case as

$$\omega_{L_3} = -2\,\frac{6k(v + kx^2)\,dv \wedge dx + 4k\,x\;da \wedge dx + dw \wedge dx}{(w + 4\,k\,x\,a + 6k^2\,x^2\,v + 3k\,v^2 + k^3\,x^4)^3}\,.$$

Then the 2-form ω_{L_3} is presymplectic, its kernel defining a foliation generated by the two commuting vector fields

$$K_1 = \frac{\partial}{\partial v} - 6k(v + kx^2)\frac{\partial}{\partial w}\,, \qquad K_2 = \frac{\partial}{\partial a} - 4kx\frac{\partial}{\partial w}\,,$$

and therefore the functions x and L_3 are projectable because

$$K_1(x) = K_1(L_3) = 0\,, \qquad K_2(x) = K_2(L_3) = 0\,.$$

Consequently, the symplectic form induced in the quotient space is given by the same expression $dL_3^2 \wedge dx$.

This fourth-order Riccati equation also admits a set of t-dependent constants of the motion. Starting with the function $T_1 = L_3$ it is easy to check that

$$\Gamma^{(4)}(T_1) = T_2 = \frac{kx}{w + 4kxa + 3kv^2 + 6k^2x^2v + k^3x^4}$$

$$\Gamma^{(4)}(T_2) = T_3 = \frac{k(v + kx^2)}{w + 4kxa + 3kv^2 + 6k^2x^2v + k^3x^4}\,,$$

$$\Gamma^{(4)}(T_3) = T_4 = \frac{k(a + 3kvx + k^2x^3)}{w + 4kxa + 3kv^2 + 6k^2x^2v + k^3x^4}\,,$$

$$\Gamma^{(4)}(T_4) = T_5 = k\,,$$

and the functions $J^{(4)}_{t1}$, $J^{(4)}_{t2}$, $J^{(4)}_{t3}$ and $J^{(4)}_{t4}$ given by

$$\begin{array}{rcl} J^{(4)}_{t1} &=& T_4 - kt \\ J^{(4)}_{t2} &=& T_3 - t\,T_4 + \frac{1}{2}\,kt^2 \\ J^{(4)}_{t3} &=& T_2 - t\,T_3 + \frac{1}{2}\,t^2T_4 - \frac{1}{6}\,kt^3 \\ J^{(4)}_{t4} &=& T_1 - t\,T_2 + \frac{1}{2}\,t^2T_3 - \frac{1}{6}\,t^3T_4 + \frac{1}{24}\,kt^4 \end{array}$$

are t-dependent constants of the motion.

6. The n-th-order Riccati equation of the sequence

The theory can be extended to the n-th-order Riccati equation of the above Riccati sequence for a generic value of n, which is obtained from an iterated action of the operator $\mathbb{D}_k$,

$$R^{(n)}(x, \dot{x}, \ldots, x^{(n)}) = \mathbb{D}^n_k x = 0\,,$$

we can write the n-th-order Riccati equation of the Riccati sequence both as a section for $\tau_{n,n-1} : T^n\mathbb{R} \to T^{n-1}\mathbb{R}$ defined by the zeros of the map $\widetilde{R}^{(n)}(x, v^{(1)}, \ldots, v^{(n)})$ or as vector field $\Gamma^{(n)}(x, v^{(1)}, \ldots, v^{(n-1)}) \in \mathfrak{X}(T^{n-1}\mathbb{R})$. We remark that $\Gamma^{(n+k)}$ acting on (the pull-back of) a function $\widetilde{F}(x, v^{(1)}, \ldots, v^{(n)})$ on $T^n\mathbb{R}$ obtained from the function $F(x, \dot{x}, \ldots, x^{(n)})$ when replacing $\dot{x}$ by $v^{(1)}$, $\ddot{x}$ by $v^{(2)}$, and so on, coincides with the function obtained from the total time derivative dF/dt with the same replacements. Therefore $(\Gamma^{(n+1)} + kx)\widetilde{R}^{(n)} = \widetilde{R}^{(n+1)}$, as a consequence of the definition of the Riccati sequence, and therefore we obtain

$$(\Gamma^{(n+1)} + kx)\widetilde{R}^{(n)} = 0\,,$$

because $T^n\mathbb{R}$ is identified as a submanifold of $T^{n+1}\mathbb{R}$ given by the vanishing of the function $\widetilde{R}^{(n+1)}$. Consequently, $\mathcal{D} = \widetilde{R}^{(n)}$ is a Darboux polynomial for $\Gamma^{(n+1)}$ with a cofactor $-k\,x$.

On the other side, one can see form the explicit expression (8) that

$$\operatorname{div}\Gamma^{(n)} = -(n+1)\,k\,x$$

and therefore we find a Jacobi multiplier for $\Gamma^{(n)}$ as a power of $\mathcal{D}$,

$$R = (\mathcal{D})^{-(n+1)}\,.$$

Note that if a function $\mathcal{F}$ is such that $X\mathcal{F} = f\,\mathcal{F}$ for a vector field X and a polynomial function f, then

$$X^2\mathcal{F} = X(f\,\mathcal{F}) = (Xf + f^2)\,\mathcal{F} = ((X+f)f)\mathcal{F}\,,$$

and iterating the process

$$X^{n+1}\mathcal{F} = ((X+f)^n f)\ \mathcal{F}\,.$$

In our case $X = \Gamma^{(n)}$, and $\mathcal{F} = L_1 = (\widetilde{R}^{(n-1)})^{-1}$, and as we started with the cofactor $f = kx$ we find from the relation

$$(\Gamma^{(n)} + kx)^n x = \widetilde{R}^{(n)}(x) = 0\,,$$

that if $L_{n-1} = (\widetilde{R}^{(n-1)})^{-1}$

$$(\Gamma^{(n)})^{n+1}L_1 = ((\Gamma^{(n)} + kx)^n kx)L_1 = 0\,,$$

which shows that $T_1 = L_1$ is a generator of degree $n+1$ of first-integrals for $\Gamma^{(n)}$ and there will be n t-dependent constants of motion.

7. Conclusions

It has been proved that the use of geometrical techniques to deal with the elements of the Riccati equation is very efficient to unveil some previously hidden aspects of such equations. So this communication has raised many interesting problems, viz., the use of Darboux factors for finding Jacobi multipliers for higher order ODEs via Prelle-Singer method, a deeper understanding t-dependent constants of the motion and their relation with master symmetries and the Lagrangian formalism in the second order case and many other problems. The application of this differential geometric Prelle-Singer method to multicomponent systems and their integrability is a very interesting subject to be studied. In fact there are several interesting issues connected to this paper – we have only hit the tip of the iceberg. For instance the method here developed can also be used to study higher-order Abel equations [29]. We hope to answer some of these questions in forthcoming papers.

Acknowledgements.

JFC and MFR acknowledge support from research projects MTM-2006-10531, E24/1 (DGA), and MTM-2005-09183. PG thanks the Departamento de Física Teórica de la Universidad de Zaragoza for its hospitality and acknowledges support from Max Planck Institute for Mathematics in the Sciences, Leipzig.

References

[1] J.F. Cariñena, M.F. Rañada and M. Santander, "Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability", J. Math. Phys. **46**, no. 6, 062703 (2005).
[2] A.M. Grundland and D. Levi, "On higher-order Riccati equations as Bäcklund transformations", J. Phys. A: Math. Gen. **32**, 3931–3937 (1999).
[3] P. Guha and D. Mayer, "Riccati Chain, Ramanujan's Differential Equations For Eisenstein Series and Chazy Flows", to appear in: *Recent development in Nonlinear Dynamics*, edited by M. Daniel *et al*, Narosa (2009).
[4] K. Andriopoulos, P.G.L. Leach and A. Maharaj, "On Differential Sequences", arXiv:0704.3243v1 (2007).
[5] M. Euler, N. Euler and P.G.L. Leach, "The Riccati and Ermakov-Pinney hierarchies", J. Nonlinear Math. Phys. **14**, 290–310 (2007).
[6] F. Dumortier, J.S. Llibre and J.C. Artes, *Qualitative Theory of Planar Differential Systems*, Springer Verlag, Berlin 2006.
[7] A. Goriely, *Integrability and nonintegrability of dynamical systems*, Advanced Series in Nonlinear Dynamics, vol. 19. World Scientific, River Edge, NJ, 2001.
[8] G. Darboux, "Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré", Bull. Sci. Math. (2) **2**, 60–96, 123–144, 151–200 (1878).
[9] M.J. Prelle and M.F. Singer, "Elementary first integrals of differential equations", Trans. Amer. Math. Soc. **279**, 215–229 (1983).
[10] L.G.S. Duarte, S.E.S. Duarte, L.A.C. da Mota and J.E.F. Skea, "Solving second-order ordinary differential equations by extending the Prelle-Singer method", J. Phys. A: Math. Gen. **34**, 3015–3024 (2001).
[11] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, "Extended Prelle-Singer method and integrability/solvability of a class of nonlinear nth-order ordinary differential equations", J. Nonlinear Math. Phys. **12**, suppl. 1, 184–201 (2005).
[12] J.F. Cariñena, G. Marmo and J. Nasarre, "The non-linear superposition principle and the Wei–Norman method", Int. J. Mod. Phys. A **13**, 3601–3627 (1998).
[13] W.P. Johnson, "The curious history of Faà di Bruno formula", Amer. Math. Month. **109**, 217–234 (2002).
[14] S. Roman, "The formula of Faà di Bruno", Amer. Math. Month. **87**, 805–809 (1980).
[15] J.F. Cariñena, C. López and E. Martínez, "A new approach to the converse of Noether's theorem", J. Phys. A: Math. Gen. **22**, 4777–4787 (1989).

[16] J.F. Cariñena, C. López and E. Martínez, "Sections along a map applied to higher-order Lagrangian Mechanics. Noether's theorem", Acta Appl. Math. **25**, 127–151 (1991).

[17] E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge U.P., 1999.

[18] M.C. Nucci, "Jacobi Last Multiplier and Lie Symmetries: A Novel Application of an Old Relationship", J. Nonl. Math. Phys. **12**, 284–304 (2005).

[19] M.C. Nucci and P.G.L. Leach, "The Jacobi Last Multiplier and its applications in mechanics", Phys. Scr. **78**, 065011 (2008).

[20] W. Sarlet, "The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics", J. Phys. A **15**, 1503–1517 (1982).

[21] M. Crampin, G. Prince and G. Thompson, "A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics", J. Phys. A **17**, 1437–1447 (1984).

[22] J. Lopuszanski, *The inverse variational problem in classical mechanics* (World Scientific Publishing, 1999).

[23] J.F. Cariñena and M.F. Rañada, "Helmholtz conditions and alternative Lagrangians: Study of an integrable Henon-Heiles system", Intern. J. of Theor. Phys. **38**, 2049–2061 (1999).

[24] D.G. Currie, E.J. Saletan, "q-Equivalent Particle Hamiltonians I. The Classical one-dimensional case", J. Math. Phys. **7**, 967–74 (1966).

[25] W. Sarlet and F. Cantrijn, "Higher-order Noether symmetries and constants of motion", J. Phys. A: Math. Gen. **14**, 479–492 (1981).

[26] P.A. Damianou, "Symmetries of Toda equations", J. Phys. A: Math. Gen. **26**, 3791–3796 (1993).

[27] R.L. Fernandes, "On the master symmetries and bi-Hamiltonian structure of the Toda Lattice", J. Phys. A: Math. Gen. **26**, 3797–3803 (1993).

[28] M.F. Rañada, "Superintegrability of the Calogero-Moser system: constants of motion, master symmetries, and time-dependent symmetries", J. Math. Phys. **40**, 236–247 (1999).

[29] J.F. Cariñena, P. Guha and M.F. Rañada, "Higher-order Abel equations: Lagrangian formalism, Darboux polynomials and constants of the motion", Preprint, Univ. Zaragoza (2009).