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Casimir effect: The TGTG formula

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Abstract. The representation of the Casimir energy in terms of Lippmann-Schwinger operators in the TGTG formula is a convenient way to address both quantitative and qualitative aspects of the Casimir force. Since this form of the energy is already explicitly regularized it may be used in order to prove statements about the force. In addition, combined with approximations of the T operators involved, it allows for convergent asymptotic expressions for the force between bodies.

Keywords: *Casimir , Quantum Fluctuations , Van der Waals , Zero-Point Energy*

1. Introduction

In this talk we discuss various aspects of a representation of the Casimir energy in terms of T operators, which is adequate whenever one is interested in the interaction of two bodies.

The Casimir force [1] explores the interplay between a quantum field and external "classical" like objects such as boundary conditions, background dielectric bodies or space-time metric. While the classical objects modify the behavior of the field due to their presence, the field, in turn, acts on the objects by exerting forces. Much work has been devoted to understanding the effect, as it appears in varied branches of physics: from condensed matter (interaction between surfaces in fluids) to gravitation and cosmology.

The original method used by Casimir, that of mode summation, has led to a large body of work on the effect in simple geometries, where the modes may be computed exactly.

The scattering approach to Casimir physics has proved very useful in 1D. Indeed, many of the calculations of Casimir interaction between bodies are based on scattering theory, as the photon spectrum in an open geometry is continuous and its description requires scattering.

Here, we explore a scattering approach to the Casimir effect in higher dimensions. The approach is based on analysis of a determinant formula for Casimir interactions obtained in [2, 12], and may be viewed as a generalization of previous formulas, especially related to scattering, such as the Lifshitz formula [3], and the results of Balian and Duplantier [4]. Within this approach, the Casimir energy is encoded in the determinant of the operator $1-T_AG_0T_BG_0$ where T_A, T_B are Lippmann-Schwinger T operators associated with bodies A and B and G_0 is the photons Green's function; we shall therefore refer to the formula as the TGTG formula.

In [2] it was shown how general results regarding the direction of the force between bodies related by reflection can be obtained from the $TGTG$ formula. For example, the sign problem of interaction between two hemispheres was resolved. This result was subsequently extended to a large class of interacting fields possessing the "reflection positivity" property [5] (See also [6], where use is made of reflection positivity arguments to infer attraction between vortices and anti-vortices in a frustrated XY model). In [7] an alternative derivation of the formula was presented.

We start with a derivation of the determinant formula by showing how to derive the T operator of a pair of perturbations, once the T operator of each perturbation by itself is known. We then illustrate how one obtains the appropriate formula in the vector (electromagnetic) case. Next, we will consider the special case of a body placed next to a perfect mirror, and the dilute limit, dealing with very weak dielectrics by expanding round $\epsilon = 1$.

The mathematical validity of the present approach to calculations of Casimir forces is then addressed. In particular we show that the formula is given in terms of $\log\det(1 + A)$. Here the sum $\sum_i |\lambda_i| < \infty$ is absolutely convergent, where λ_i are the eigenvalues of A, and so $\log\det(1+A)$ is mathematically well defined, thereby explicitly showing that the usual infinities plaguing Casimir calculations have been completely accounted for.

A natural setting for computations using the TGTG formula is by writing it in a basis of partial wave expansions respecting symmetries of bodies as much as possible. In 1D where only two modes (left and right movers) exist at each ω the formula leads to a known closed form formula for the Casimir energy in terms of reflection coefficient (see, e.g. [8, 9]).

Spherical wave expansions are convenient when considering the Casimir interaction between spherical bodies. We demonstrate this by computing the force between two spheres at all distances, thereby generalizing the approach of [10] to spheres beyond Dirichlet boundary conditions, and going beyond the proximity force approximation.

2. The TGTG formula

Let us start by briefly recalling the Lippman-Schwinger operators from scattering theory. Most of the derivation is standard and may be skipped by readers interested only in new results. However, we point out that our approach where the T operators of combined scatterers are utilized seems new.

The T operator was introduced by Lippmann and Schwinger in their celebrated paper on scattering theory [11]. As an application of their formalism, they have computed the scattering properties of a neutron by a proton bound in an inert molecule. T appears in the Lippmann-Schwinger equation as follows: Assume that a solution ϕ_{in} of a free wave equation, without a potential $H_0(\omega)\phi_{in} = 0$ is known. Here H_0 corresponds to a free wave equation, for example:

$$H_0 = \nabla^2 + \omega^2$$

if we are dealing with a scalar field, or

$$H_0 = \nabla \times \nabla \times +\omega^2 \mathbf{I}$$

for the wave equation of the vector potential A in the radiation gauge $A_0 = 0$.

We are interested in the modes of the system in the presence of some local perturbation. Such a perturbation will be represented by a potential V (In the electromagnetic case, when the magnetic response is not considered V corresponds to $\omega^2(\epsilon(x) - 1)$), and so we are interested in the equation $(H_0 + V)\psi = 0$. Note that all the operators above depend on ω, which we omit from the notation to streamline the arguments.

A solution ψ of the eigenvalue equation $(H_0 + V)\psi = 0$ having ϕ_{in} as the incoming part is now constructed. Formally, this is done by looking for a solution of:

$$\psi = \phi - G_0 V \psi,$$

which is the Lippmann-Schwinger equation.

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It follows that

$$\psi = (I + G_0 V)^{-1}\phi = (I - G_0 V(I + G_0 V)^{-1})\phi = (I - G_0 T)\phi$$

where

$$T = V(I + G_0 V)^{-1}, \tag{1}$$

is called the transition matrix, or the Lippmann-Schwinger operator.

A typical example is choosing ϕ to be a plane wave solution of the scalar equation, and so the scattered wave-function is given by

$$\psi_k = e^{ikx} - \int \mathrm{d}k' G_0(k)\langle k|T|k'\rangle e^{ik'x}. \tag{2}$$

To find the interaction between two bodies, we consider the T operator of two perturbations V_A, V_B. We compute the joint transition matrix for both perturbations $T_{A\bigcup B}$, and show that the part independent of "self energy" is exactly (11).

Writing, formally, $V = G_0^{-1} - G^{-1}$ we have:

$$1 - G_0(V_1 + V_2) = 1 - G_0(2G_0^{-1} - G_1^{-1} - G_2^{-1}) = G_0 G_1^{-1} + G_0 G_2^{-1} - 1 \tag{3}$$

so that:

$$\begin{gathered}\tfrac{1}{1-G_0(V_1+V_2)} = \tfrac{1}{G_0G_1^{-1}+G_0G_2^{-1}-1} = \tfrac{1}{G_1^{-1}+G_2^{-1}-G_0^{-1}} G_0^{-1} = \\ G_1 \tfrac{1}{G_2+G_1-G_2G_0^{-1}G_1} G_2 G_0^{-1},\end{gathered} \tag{4}$$

where we used $A^{-1} = BB^{-1}A^{-1}C^{-1}C = B(CAB)^{-1}C$.

Using the identity:

$$G = G_0 - G_0 T G_0 \tag{5}$$

as $G_i = G_0 - G_0 T_i G_0$ (with $i = A, B$), together with the definition of T (1) we find:

$$\tfrac{1}{1+G_0(V_A+V_B)} = (1 - G_0 T_A)\tfrac{1}{1-G_0T_BG_0T_A}(1 - G_0 T_B) \tag{6}$$

and so the joint T operator of a pair of perturbations may be factored as:

$$\begin{gathered}T_{A\bigcup B} = (V_A + V_B)\tfrac{1}{1+G_0(V_A+V_B)} = \\ (V_A + V_B)(1 - G_0 T_A)\tfrac{1}{1-G_0T_BG_0T_A}(1 - G_0 T_B)\end{gathered} \tag{7}$$

This expression represents a *re-summation* of the Born series for T, given by:

$$T = (V_A + V_B) - (V_A + V_B)G_0(V_A + V_B) + (V_A + V_B)G_0 V G_0(V_A + V_B) + ... \tag{8}$$

in a way which identifies the terms which are responsible for directly mixing the A and B perturbations, namely $1 - G_0 T_B G_0 T_A$, as well as the terms which will be responsible for repeated scattering from the same perturbation, which are responsible for self energy: $(1 - G_0 T_A)$.

The expression for $T_{A\bigcup B}$ can be used to find the ground state energy of the system. It is well known that the change of the density of states in the system due to introduction of the perturbations is given by[1]:

$$\delta\rho(E) = \frac{1}{\pi}\mathrm{Im}\partial_E \,\mathrm{Tr}\log T(E) \tag{10}$$

We now may use this formula in two ways:

For the Casimir effect, we compute the energy by integrating over the density of states and summing the allowed modes of the electromagnetic field, giving the result

$$E_C(a) = \textstyle\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\log\det(1 - T_A G_0 T_B G_0). \tag{11}$$

From a more general point of view, we consider the change in the energy of the ground state of a many body system. In this case one needs to compute the many-body T operator, and try to identify how the pole (or cut) associated with the ground state in the density of states shifts.

Dirichlet boundary conditions

In many cases, and indeed in the original presentation by Casimir, one is interested in sharp boundary conditions, such as Dirichlet or Neumann. Sharp boundary conditions result in singular energy density at the surface, as field modes are required to vanish for all momentum scales. Typically, the local energy density diverges as the inverse fourth power of the distance from the boundary [13].

It is important to point out that the above considerations also describe the conducting case with minor changes. Following [14], assume conducting boundary conditions are given over a surface Σ, parameterized by internal coordinate u and by the embedding in $\mathbb{R}^3$ given by $\mathbf{x}(u)$. One may describe a simple metal by taking $\chi(i\omega) = \frac{\omega_p^2}{4\pi\omega^2}$ on Σ, and letting Σ have a thickness of a few skin depths l/ω_p, $l \sim \mathcal{O}(1)$, here ω_p is the plasma frequency (proportional to the effective electron density in the metal). In the limit of large ω_p one retains the same expression as (11), with the following substitutions:

$$E_C(a) = \\ -\tfrac{1}{4\pi}\textstyle\int_0^\infty \mathrm{d}\omega(\log\det(1 - \mathcal{M}_{BA}\frac{1}{1+\mathcal{M}_A}\mathcal{M}_{AB}\frac{1}{1+\mathcal{M}_B}) \tag{12}$$

where in the Dirichlet case $\mathcal{M}$ is given by:

$$M^{(D)}(u,u';\omega) = l\omega_p\sqrt{g(u)}G_0(\mathbf{x}(u),\mathbf{x}(u'))\sqrt{g(u')} \tag{13}$$

and acting on the surfaces Σ (See also [18]).

3. Analytical properties of the $T_A G_0 T_B G_0$-operator:

Casimir physics is largely involved with understanding and separating cutoff dependent self energies from universal energies. Often various regularization methods are employed in the calculation. The aim of this section is to show that the form (11) for the energy is already

[1] A related form is known as the Krein formula:

$$\delta\rho(E) = \frac{1}{\pi}\mathrm{Im}\partial_E \log\det S(E) \tag{9}$$

where S is the scattering matrix.

fully regularized. We do this in a rigorous fashion, showing that the object $\log\det(1 - T_A G_{0AB} T_B G_{0BA})$ is mathematically well defined and finite.

In practical terms this means that replacing the infinite dimensional matrix of $1 - T_A G_{0AB} T_B G_{0BA}$ by its upper-left $n \times n$ block with n large enough and calculating the resulting ordinary determinant, gives an arbitrarily good approximation to a (finite) quantity which we call $\det(1 - T_A G_{0AB} T_B G_{0BA})$. The main mathematical notions and theorems which we use here, are briefly reviewed in section 11.

As remarked already in the introduction it is well known that $\det(1 - M)$ is well defined whenever M is a *trace class* operator (definition 11.4, denoted *t.c.*). We would like to show that for a large class of situations (including a pair of disjoint finite bodies A, B, separated by a finite distance) the operator $T_A G_{0AB} T_B G_{0BA} : H_A \to H_A$ is trace class in the continuum limit , and so prove that indeed the expression (11) is finite and well defined.

Indeed by theorem 11.5 the mere fact that $G_0(x, y)$ is a smooth function for $x \neq y$ is sufficient to guarantee that for any pair of compact volumes $A, B \in \mathbb{R}^3$ at finite mutual distance the operator G_{0AB} is trace class. To deduce that $T_A G_{0AB} T_B G_{0BA}$ is trace class (and by similar argument also $1 - G_0 \mathcal{J} T_A$ appearing in eq (26)) it is then enough (proposition 11.6) to make sure $T_{A,B}(i\omega)$ are bounded (definition 11.2).

In the context of dielectric interaction it is particularly easy to show that $T(i\omega)$ is bounded. In physical systems at equilibrium, it follows from causality properties of the dielectric function [15], that $\chi(i\omega, x) \geq 0$. We then have the following

Lemma 3.1. *For $\chi(i\omega, x) > 0$, the T operators are positive and bounded.*

Proof: Since $G_0, \chi > 0$ (definition 11.3) one may write $T = \sqrt{\chi} \frac{\omega^2}{1+\omega^2 \sqrt{\chi} G_0 \sqrt{\chi}} \sqrt{\chi}$ from which it is seen that $T > 0$ and that in the operator norm $||T|| \leq \omega^2 ||\chi||$ $\Box$.
In fact, this holds also for nonlocal χ as long as $f(x) \to \int_A \chi(i\omega, x, x') f(x') \mathrm{d}x'$ is a bounded positive operator $H_A \to H_A$. In the context of a more general type of interactions which may not be positive, one needs to use some assumption on the stability of the system to guarantee that $T(i\omega)$ is bounded. We do not elaborate on this here.

An alternative approach to proving the trace class property of $T_A G_{0AB} T_B G_{0BA}$ is based on the notion of a Hilbert Schmidt operator (definition 11.7, also denoted *H.S.*). Here the frequently used strategy in operator analysis is to use the following fact: if $U \in H.S.$ and $V \in H.S.$, then $UV \in t.c.$. The advantage of this approach is that it is very easy to check if an operator is Hilbert-Schmidt. Since the Hilbert-Schmidt norm is $||A||^2_{H.S.} = \mathrm{Tr}(A^\dagger A)$, one may evaluate it directly, (e.g. by computing $\int |A(x, x')|^2$).

Theorem 3.2. *For any two bodies A, B such that $\int_{A\times B} \mathrm{d}x\mathrm{d}y |G_0(x, y)|^2 < \infty$, $T_A G_{0AB} T_B G_{0BA}$ is trace class.*

Proof: First we show that $T_A G_{0AB}$ and $T_B G_{0BA}$ are Hilbert Schmidt operators. This can be verified in the following way. We have just seen that T_A, T_B are bounded operators. Now note that G_{0AB} is Hilbert Schmidt, since,

$$||G_{0AB}||^2_{H.S} = \int_{A\times B} \mathrm{d}x \mathrm{d}y |G_{0AB}(x, y)|^2, \tag{14}$$

which is finite under the condition above. Now the inequality $||T_A G_{0AB}||_{H.S} \leq ||T_A|| ||G_{0AB}||_{H.S}$ implies that $T_A G_{0AB}$ is Hilbert-Schmidt. Finally using $U, V \in H.S. \Rightarrow UV \in t.c.$ we see that $T_A G_{0AB} T_B G_{0BA} \in t.c.$ $\Box$

Corollary 3.3. *For any finite bodies A, B, such that distance$(A, B) > 0$, and any Green's function which is finite away from the diagonal, $T_A G_0 T_B G_0 \in t.c.$*

Example 3.4. *For the scalar field discussed above,* $G_0(x,y) = \frac{e^{-\omega|x-y|}}{4\pi|x-y|}$*, the condition is satisfied. In the same way it is satisfied for the electromagnetic field (one has to take into account also matrix indices but these discrete indices do not change the finiteness of the integrals)*

Remark 3.5. *The* ω *integration in (11), is convergent. To see this note that* G_0 *decays exponentially with* ω*, therefore* $||G_0||_{H.S.}$ *decays exponentially, also the* $||T||$*'s do not grow more then quadratically in* ω*.*

In the EM case one may also worry due to the factor $\frac{1}{\omega^2}$ *appearing in* $\mathcal{D}_{0ij}(x,y) = (\delta_{i,j} - \frac{1}{\omega^2}\nabla_i^{(x)}\nabla_j^{(y)})G_0(x,y)$ *about convergence for* $\omega \sim 0$*. This factor, however, gets cancelled since* $||T|| \leq \omega^2||\chi||$ *as shown in lemma 3.1.*

One may also show that G_{0AB} are t.c. themselves by using H.S. properties. The bodies are assumed not to touch, thus we can choose a C_0^∞ (compactly supported and infinitely smooth) function f_A, such that $P_A f_A = P_A$, and $P_B f_A = 0$ where P_A, P_B are the projections on $L^2(A), L^2(B)$ (i.e. $f_A(x) = 1$ for $x \in A$, and it then smoothly goes to 0, before reaching body B) .

Writing:

$$\begin{gathered} G_{0AB} = L_1 L_2 \\ L_1 = P_A \tfrac{1}{(p^2+\omega^2)^\alpha} \;\; ; \;\; L_2 = (p^2+\omega^2)^\alpha f_A G_0 P_B, \end{gathered} \tag{15}$$

we see that if $2\alpha > d$,

$$\begin{gathered} ||L_1||^2_{H.S.} = \mathrm{Tr}(P_A \tfrac{1}{(p^2+\omega^2)^\alpha})(P_A \tfrac{1}{(p^2+\omega^2)^\alpha})^\dagger = \\ Vol(A) \int \mathrm{d}^d p |\tfrac{1}{(p^2+\omega^2)^{2\alpha}}| < \infty \end{gathered} \tag{16}$$

and so L_1 is Hilbert Schmidt. Next, we check that $L_2 \in H.S.$. To see this last point, note that

$$\begin{gathered} < x|L_2|x' > = < x|(p^2+\omega^2)^\alpha f_A G_0 P_B|x' > = \\ (-\triangle_x + \omega^2)^\alpha f_A(x) G_0(x-x') P_B(x') \end{gathered} \tag{17}$$

Since $G_0(x-x')$ is smooth away from $x = x'$, where the expression is anyway zero because $f_A P_B = 0$, and since $\langle x|L_2|x'\rangle$ has compact support we see that $||L_2||^2_{H.S.} = \int \mathrm{d}x\mathrm{d}x'|L_2|^2 < \infty$. Thus, G_{0AB} can be written as a product of two H.S. operators, and as such is trace class.

Finally, we have that

Theorem 3.6. *(***Eigenvalues of TGTG***) For* $\chi > 0$*, all eigenvalues* λ *of the (compact) operator* $T_A G_{0AB} T_B G_{0BA}$ *appearing in (11) satisfy* $1 > \lambda \geq 0$*.*

Proof: We will use repeatedly that for bounded operators X, Y the nonzero eigenvalues of XY and YX are the same. Note first that $G_0, \chi \geq 0$ (as operators) implies

$$\mathrm{spec}(\chi G_0) \setminus \{0\} = \mathrm{spec}(\sqrt{G_0}\chi\sqrt{G_0}) \setminus \{0\} \subset [0, \infty). \tag{18}$$

Writing $T_\alpha G_0 = 1 - \frac{1}{1+\omega^2\chi_\alpha G_0}$ as an operator on $L^2(\mathbb{R}^3)$ it is then clear that its spectrum lies in [0,1). The same conclusion then applies to the operator $\sqrt{G_0}T_\alpha\sqrt{G_0}$ but since it is hermitian one concludes also $||\sqrt{G_0}T_\alpha\sqrt{G_0}|| < 1$ from which it follows $||\sqrt{G_0}T_A G_0 T_B\sqrt{G_0}|| < 1$ and hence $\lambda < 1$. Similarly $\sqrt{G_0}T_\alpha\sqrt{G_0} \geq 0$ imply $\lambda \geq 0$ $\square$

4. The Electromagnetic Case

Here we follow the approach of [15]. The statistical properties of the electromagnetic field in a medium are described by the appropriate photonic Green's function. The electromagnetic fields are derived from the electromagnetic potentials A^α, $\alpha = 0,..,3$. (It is convenient to work in the gauge $A^0 = 0$.) The retarded Green's function $\mathcal{D}_{ik}$ is defined by:

$$\mathcal{D}^R_{ik}(X_1, X_2) = \begin{cases} \langle A_i(X_1)A_k(X_2) - A_k(X_2)A_i(X_1)\rangle & t_1 < t_2 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

where X_1, X_2 are 4-vectors $(X_1^0, ..X_1^3)$ and $k, i = 1,..3$. The angular brackets denote averaging with respect to the Gibbs distribution.

The interaction of the electromagnetic field with a classical current $\mathbf{J}$ put in the medium is given by:

$$V = -\frac{1}{c}\int \mathbf{J}\cdot\mathbf{A}.$$

Kubo's formula allows us to treat this interaction within linear response. By Kubo's formula the mean value $\overline{\mathbf{A_i}}$ in presence of a current $\mathbf{J}$ satisfies:

$$\overline{\mathbf{A_i}}(\mathbf{r})_\omega = -\frac{1}{\hbar c}\int \mathcal{D}^R_{ik}(\omega;\mathbf{r},\mathbf{r}')\mathbf{J}_k(\mathbf{r}')_\omega \mathrm{d}^3\mathbf{r}', \tag{20}$$

where

$$\mathcal{D}^R_{ik}(\omega;\mathbf{r},\mathbf{r}') = \int_0^\infty e^{i\omega t}\mathcal{D}^R_{ik}(t;\mathbf{r},\mathbf{r}')\mathrm{d}t \tag{21}$$

The function $\mathcal{D}$ is sometimes referred to as the generalized susceptibility of the system [15].

From Maxwell's equations it follows that in a medium with a given permittivity tensor ϵ_{ij}, permeability tensor μ_{ij}, and current $\mathbf{J}$, the vector potential A_i satisfies:

$$(\nabla\times(\mu^{-1}\nabla\times) - \frac{\omega^2}{c^2}\epsilon)\overline{\mathbf{A}} = \frac{4\pi}{c}\overrightarrow{J}_\omega \tag{22}$$

Substituting (20) in (22), we see that $\mathcal{D}$ is a Green's function for the equation:

$$\nabla\times\mu^{-1}\nabla\times\mathcal{D} - \frac{\omega^2}{c^2}\epsilon\mathcal{D} = -4\pi\hbar\mathrm{I}\delta(\mathbf{r}-\mathbf{r}') \tag{23}$$

where I is the 3-dimensional unit matrix.

To get the energy we now simply use the TGTG formula (11) with the scalar propagator G_0 replaced by the vector propagator $\mathcal{D}_0$ everywhere (including in the definition of the T operators) (for details see [12]). Here the explicit expression for $\mathcal{D}_0$ is

$$\mathcal{D}_{0ij}(k, i\omega) = -\frac{4\pi\hbar}{k^2+\omega^2c^{-2}}(\delta_{ij} + \frac{c^2k_ik_j}{\omega^2}) \tag{24}$$

5. Dielectric in front of a mirror

A somewhat simplified, but useful in practice, version of our formula is obtained in the case of a body placed close to a mirror. Consider the body A to the left of a Dirichlet mirror B located at $x_n = a/2$. It is well known (using the image method) that the effect of the Dirichlet mirror is to replace the free propagator G_0 by

$$G_B(\mathbf{x},\mathbf{x}') = G_0(\mathbf{x},\mathbf{x}') - G_0(\mathbf{x}, J(\mathbf{x}')) \tag{25}$$

where $J(\mathbf{x}_\parallel, x_\perp) = (\mathbf{x}_\parallel, a - x_\perp)$ denotes reflection through the mirror plane. This may be written as $G_B - G_0 = -G_0\mathcal{J}$ where $\mathcal{J}$ is the operator defined by $\mathcal{J}\psi(\mathbf{x}) = \psi(J(\mathbf{x}))$. Noting the standard relation (5) $G_B = G_0 - G_0 T_B G_0$ between the Green function in the presence of scatterer B to its T matrix one concludes $G_0 T_B G_0 = G_0 \mathcal{J}$ which when substituted in (11) gives

$$E_C(a) = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \log\det(1 - G_0 \mathcal{J} T_A) \tag{26}$$

An alternative (though closely related) approach is to note the energy it costs to place a body A near a mirror B is

$$E_C = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \log\det{}_\Lambda(1 + \omega^2 \chi_A(\mathbf{x}, i\omega) G_B(\mathbf{x}, \mathbf{x}')).$$

Subtracting the energy $E_C = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \log\det_\Lambda(1 + \omega^2\chi_A(\mathbf{x}, i\omega)G_0(\mathbf{x}, \mathbf{x}'))$ it costs to put A in vacuum then gives the Casimir interaction energy. Using the relation

$$\begin{gathered}(1 + G_B V_A)/(1 + G_0 V_A) = \\ 1 + (G_B - G_0) T_A = 1 - G_0 \mathcal{J} T_A\end{gathered} \tag{27}$$

leads again to (26).

Yet another way of obtaining the same result is by substituting $\chi_B = \lambda\delta(x_n - a/2)$ in the definition of T_B and doing the algebra. One then finds

$$\begin{gathered}G_0 T_B G_0 = \\ \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp(x-x')_\perp} \frac{\lambda\omega^2}{2q(\lambda\omega^2 + 2q)} e^{-q|x_n| - q|x_n'|}\bigg|_{q=\sqrt{\omega^2 + k_\perp^2}},\end{gathered} \tag{28}$$

which in the limit $\lambda \to \infty$ reduces, as expected, to the expression $G_0\mathcal{J}$ obtained through the image method.

We now address the case of a Neumann mirror. Note that the Green function in the presence of a Neumann mirror is $G = G_0 + G_0\mathcal{J}$. By repeating the arguments above we find that the Casimir interaction between an object A and a Neumann mirror is given by a similar formula to (26), involving the determinant $\det(1 + G_0\mathcal{J}T_A)$. We remark that while the Dirichlet mirror may be considered as the limit $\lambda \to \infty$ of a dielectric having e.g. $\chi_B = \lambda\delta(x_n - a/2)$ (or in more realistic model $\chi_B = \lambda\theta(x_n - a/2)$) it is hard to find a simple analog $\chi_B(x)$ that would lead in a similar limit to a Neumann mirror.

A similar treatment is applicable in the more physically relevant EM case. The boundary conditions $E_\parallel = 0$ may be enforced by requiring the vector potential to satisfy $\mathcal{J}A = -A$ where $\mathcal{J}$ is defined to act on vectors as $\mathcal{J}A(x) = (A_\parallel(J(x)), -A_\perp(J(x)))$ (Here $A_\parallel, A_\perp$ denote the components of A parallel and normal to the mirror surface. The temporal component is considered as a parallel component though in practice we usually choose a gauge where it vanishes.)

The EM Casimir interaction between a dielectric and a mirror is then given by a formula similar to (26) with $G_0, \mathcal{J}$ replaced by the EM propagator $\mathcal{D}_0$ and the vectorial $\mathcal{J}$ defined above.

It is interesting to consider also an ideal permeable mirror (having $\mu \to \infty, \epsilon = 1$). This corresponds to the boundary condition $B_\parallel = 0$ which may be enforced by requiring the vector potential to satisfy $\mathcal{J}A = +A$. Thus the Casimir interaction of body A with such a mirror will be given by an expression involving the determinant $\det(1 + T_A\mathcal{D}_0\mathcal{J})$.

6. Separable Potentials

Here we show a simple example where everything can be computed immediately in the TGTG approach. The example is based on so called "separable potentials". Such potentials arise in variety of situations in physics and in mathematics, and were first introduced for Casimir type problems by Jaffe and Williamson [21]. They correspond to "rank 1" perturbations, and can be written as $V = |f\rangle\langle f|$ for some function f, or in x space notation, the perturbation V has the kernel $V(x, x') = f(x)f(x')$.

For such potentials, one may readily calculate the T operators, which turn out to be also of rank 1, and so the interaction energy is easily calculated. We compute T:

$$T = \tfrac{1}{1+VG_0}V = \sqrt{V}\tfrac{1}{1+\sqrt{V}G_0\sqrt{V}}\sqrt{V} = \tfrac{1}{1+\langle f|G_0|f\rangle}|f\rangle\langle f|, \tag{29}$$

and so we have

$$\log\det(1 - T_A G_0 T_B G_0) = \log(1 - \tfrac{1}{1+\langle f_A|G_0|f_A\rangle}\tfrac{1}{1+\langle f_B|G_0|f_B\rangle}|\langle f_B|G_0|f_A\rangle|^2) \tag{30}$$

Since the (positive) terms $\frac{1}{1+\langle f_A|G_0|f_A\rangle}\frac{1}{1+\langle f_B|G_0|f_B\rangle}$ do not depend on the distance, to find the direction of the force it is enough to consider $|\langle f_B|G_0|f_A\rangle|$.

Let us show that the force can be repulsive when both V_A and V_B are positive perturbations. We take the following functions: $f_A = \alpha_1\delta(x + a) + \alpha_2\delta(x + a + 1)$ and the potential in B is defined by $f_B = \beta_1\delta(x - a) + \beta_2\delta(x - a - 1)$; here we assume that these points all lie on a particular line, which is the x direction.

We choose $\omega = 1$ for simplicity and compute:

$$\langle f_A|G_0|f_B\rangle = \alpha_1\beta_1\tfrac{e^{-2a}}{2a} + (\alpha_1\beta_2 + \alpha_2\beta_1)\tfrac{e^{-2a-1}}{2a+1} + \alpha_2\beta_2\tfrac{e^{-2a-2}}{2a+2}. \tag{31}$$

Upon choosing: $\alpha_1 = 1, \alpha_2 = -20, \beta_1 = 10, \beta_2 = -1$, and distance $a = 0.15$ we get

$$\partial_a|\langle f_A|G_0|f_B\rangle|^2 \sim 2200 > 0,$$

implying that (30) gives a repulsive contribution for the energy at this distance and frequency. It is possible to extend this example so that even integration over frequencies is still repulsive. This will be reported elsewhere.

7. Dilute limit

In the following sections we consider strategies of using the $TGTG$ formula in actual calculations. A particularly simple case is when χ is small, which is commonly referred to as the "dilute" case (and sometimes as "low contrast"). Here we briefly sketch how to best use the formula in this limit. As shown in Theorem 3.6, one always has $||TGTG|| < 1$, therefore we may expand the $\log\det(1 - ..)$ expression (11) in powers:

$$E_C = -\int \frac{\mathrm{d}\omega}{2\pi} \sum \frac{1}{m}\,\mathrm{Tr}(T_A G_0 T_B G_0)^m \tag{32}$$

In the dilute limit $\chi_\alpha < 1$, so one may also substitute the expansion

$$T_\alpha = -\sum_{n=0}^{\infty}(-\omega^2\chi_\alpha G_0)^n\omega^2\chi_\alpha \tag{33}$$

in (32) and compute the involved integrals to desired order. This expansion is the continuum equivalent to summation of two body forces, and is equal to the Born series appearing, for example, in [15].

8. Scattering Approach

As remarked above, the operator $T_A G_0 T_B G_0$ appearing in our formula is closely related to scattering data. The purpose of this section is to clarify this relation and make it more explicit. In order to keep better touch with conventions used in scattering theory, we usually avoid in this section using Wick rotation and thus we work in Lorentzian rather than Euclidean space, with a real rather then imaginary frequency and with the Feynman rather than the Euclidean propagator G_0.

As mentioned above, the arguments of G_0 in (11) never coincide, implying that when $G_0(x_a, x_b)$ is considered as a function of x_b alone it is a solution of the (homogeneous) free wave equation. Thus one may expand $G_0(x_a, x_b)$ in the form $\sum \mathcal{C}_{\alpha\beta}\phi^*_\alpha(x_a)\phi_\beta(x_b)$ where $\{\phi_\alpha(x_a)\}, \{\phi_\beta(x_b)\}$ are some sets of free wave solutions of energy ω. There is of course great freedom in choosing the sets $\{\phi_\alpha(x_a)\}, \{\phi_\beta(x_b)\}$. In practice one would choose these in a way that makes subsequent calculations easier. As mentioned earlier we consider $T_A G_0 T_B G_0$ as acting only on the volume of object A, therefore these considerations also apply to the propagator on the right of this expression.

The Lippmann-Schwinger operator $T(\omega)$ is related to the S-matrix by [2]

$$S = 1 - 2\pi i \delta(\omega^2 - \omega'^2) T(\omega). \tag{34}$$

Therefore $T(\omega)$ has the property that its matrix element $\langle \alpha | T | \beta \rangle$ between a pair of free states α, β having energy ω is equal to the corresponding matrix element of the transition matrix. Since the operator T_B in $T_A G_0 T_B G_0$ is sandwiched between a pair of free Feynman propagators corresponding to energy ω, we may identify it with the corresponding transition matrix. Due to the cyclicity of the determinant $\det(1 - T_A G_0 T_B G_0)$ the same is true of T_A.

Substituting the expansion $G_0(x_a, x_b) = \sum \mathcal{C}_{\alpha\beta}\phi^*_\alpha(x_a)\phi_\beta(x_b)$ we arrive at

$$T_A G_0 T_B G_0 = \sum_{\alpha\alpha'\beta\beta'} T_A |\alpha\rangle \mathcal{C}_{\alpha\beta} \langle \beta | T_B | \beta' \rangle \mathcal{C}_{\alpha'\beta'} \langle \alpha' |$$

The Casimir interaction will then be given explicitly by

$$E = \int_0^\infty \frac{d\omega}{2\pi} \log\det(1 - K(i\omega)). \tag{35}$$

Here $K_{\alpha''\alpha'} = (T_A)_{\alpha''\alpha} \mathcal{C}_{\alpha\beta} (T_B)_{\beta\beta'} \mathcal{C}_{\alpha'\beta'}$.

9. Partial waves expansion

In the following sections, we consider strategies of using the representation (35) by restricting the K matrix to a finite subspace which gives the dominant contribution to the force. Indeed, in many cases of interest only a few partial waves are significantly scattered, the best example for this is when objects are far apart, and from a large distance look point like. At this limit one expects significant contribution only from s-wave scattering. In the more general case, K may be approximated by a finite dimensional matrix corresponding to several partial waves. In order to see how this works in practice we consider below a few simple cases.

[2] Most standard textbooks discuss the non-relativistic case and therefore include a factor $\delta(\omega - \omega')$ instead of $\delta(\omega^2 - \omega'^2)$. Writing the delta function in terms of momentum the two cases reduce to the same expression: $\delta(k^2 - k'^2)$

One dimensional systems
A particularly simple case occurs when the system is one-dimensional. Consider, e.g., a scalar field in 1D. All states of energy ω are then spanned by two modes: left and right movers $|L\rangle, |R\rangle = \frac{1}{\sqrt{2\pi}} e^{\pm i\omega x}$. Hence in this case the determinant Eq (11) can easily be calculated exactly. To see how this is done we write the Feynman propagator explicitly as

$$G_0 = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{\omega^2 - k^2 + i0} = -\frac{i}{2\omega} e^{i\omega|x|} \tag{36}$$

We consider a pair of scatterers A, B such that A is to the left of B. This immediately implies that we have $x_a < x_b$ and therefore

$$G_{0BA}(x_b, x_a) = -\frac{i}{2\omega} e^{i\omega(x_b - x_a)} = \frac{-2i\pi}{2\omega} |R\rangle\langle R| \tag{37}$$

Similarly we also have $G_{0AB} = \frac{-2i\pi}{2\omega} |L\rangle\langle L|$. Using this we see that the operator K in (35) turns into the c-number

$$K = (\frac{-2i\pi}{2\omega})^2 \langle R|T_A|L\rangle\langle L|T_B|R\rangle = \tilde{r}_A(\omega) r_B(\omega). \tag{38}$$

Here r_B ($\tilde{r}_A$) is the reflection coefficient for a wave hitting scatterer B from the left (A from the right) to be reflected back. Note that the normalization of T implied by Eq (34) is responsible to the cancelling of the factor $\frac{-2i\pi}{2\omega}$. (Had we used relativistic normalization for $|L, R\rangle$ the factor 2ω would not have appeared.) We thus conclude

$$\det(1 - T_A G_0 T_B G_0) = 1 - \tilde{r}_A(\omega) r_B(\omega).$$

The tilde on r_A serves to remind us that it is the reflection coefficient from the right side of A.

We remark that $\tilde{r}_A(\omega) r_B(\omega)$ depends implicitly on the distance between A, B through the (phase) dependence of r_A, r_B on the scatterers locations. To make this explicit, note that moving a scatterer a distance a affects the reflection coefficients as $r \to e^{-2ia\omega} r, \tilde{r} \to e^{2ia\omega} \tilde{r}$.

Moving the scatterers a distance a apart therefore result in

$$\det(1 - T_A G_0 T_B G_0) \to (1 - e^{2ia\omega} \tilde{r}_A(\omega) r_B(\omega)).$$

Substituting in (35) we obtain the familiar formula for 1d Casimir interaction between scatterers [8, 9, 16].

Multi-component field in 1d
The considerations used above for a single scalar field in one dimension extend to a situation where $\phi = (\phi_1, \phi_2, ...\phi_n)$ is an n component field. In this case the reflection coefficients $r_{A,B}$ turn into $n \times n$ matrices and one finds $\det(1 - T_A G_0 T_B G_0) = \det(1 - \tilde{r}_A(\omega) r_B(\omega))$ where the determinant on the right is of a usual $n \times n$ matrix.

Plane wave expansion.
In physical three dimensional space there are many different possible ways to expand the propagator $G_0(x_a, x_b) = \sum \mathcal{C}_{\alpha\beta} \phi^*_\alpha(x_a) \phi_\beta(x_b)$ in terms of free wave solutions $\{\phi_\alpha(x_a)\}, \{\phi_\beta(x_b)\}$. In the next section we describe the expansion in spherical waves (which is probably the most useful expansion), and we demonstrate its use to calculating the Casimir force between compact object. However for the sake of simplicity we first describe here a plane wave expansion which is the immediate generalization of eq(37). A simple heuristic way to arrive at this generalization

is to formally think of the field ϕ in three dimensions as one dimensional field having infinitely many components labelled by its transverse momenta. Indeed, such point of view has been successfully used in describing transport in quasi 1D conductors in mesoscopic physics, whereby each transverse component corresponds to a scattering channel (see for example [17]). This suggests splitting $\vec{k}$ into its z-component k_z and its transverse components $k_\parallel = (k_x, k_y)$. The 3d propagator may then be written as:

$$G_0 = -\int \frac{d^2k_\parallel}{(2\pi)^2} \frac{ie^{i|z|k_z} e^{ik_\parallel x_\parallel}}{2k_z}\Big|_{k_z=\sqrt{\omega^2-k_\parallel^2+i0}}$$

Here $\sqrt{\omega^2 - k_\parallel^2 + i0}$ may be either real and positive (for $\omega^2 > k_\parallel^2$) or pure imaginary (for $\omega^2 < k_\parallel^2$) in which case the $i0$ prescription implies that it must be chosen on the positive imaginary axis. Assuming that A is located to the left of B along the z-axis it follows that

$$T_A G_0 T_B G_0 = \int \tfrac{dk_x dk_y dq_x dq_y}{(2\pi)^4} T_A |(q_x, q_y, -q_z)\rangle \times \\ \tfrac{1}{2q_z} \langle (q_x, q_y, -q_z)| T_B |(k_x, k_y, k_z)\rangle \tfrac{1}{2k_z} \langle (k_x, k_y, k_z)|, \tag{39}$$

where $q_z = \sqrt{\omega^2 - q_x^2 - q_y^2 + i0}$ and $k_z = \sqrt{\omega^2 - k_x^2 - k_y^2 + i0}$.

When considering only the terms satisfying $\omega^2 > q_x^2 + q_y^2, k_x^2 + k_y^2$, eq. (39) indeed looks like a straightforward generalization of the 1d result. However as this expression shows, to get the correct result one must also include the contribution of evanescent waves ($q_\parallel^2 > \omega^2$). Upon Wick rotation, however, the distinction between ordinary and evanescent waves disappears. It may also be noted that (since in general $q_z \neq k_z$) the variation of the $\langle (q_x, q_y, -q_z)|T_B|(k_x, k_y, k_z)\rangle$ matrix elements upon moving B along the z-axis is considerably more complicated then in the 1d case.

The above representation may be helpful in problems where the scatterers A, B have exact or approximate planar geometry (e.g. corrugated plates). Though the theorem guaranteeing finite trace does not apply for infinite plates one may show that dividing by the plate area leads to finite result. We remark that actual calculation of the determinant requires discretizing $k_\parallel$ which corresponds to assuming large but finite plates. Alternatively, one may use eq. (32) with continuous $k_\parallel$.

10. Spherical waves expansion

When describing interaction between two compact bodies, often it is convenient to represent the transition matrices T in a spherical wave basis. To do so, we choose two points P_A, P_B inside bodies A, B respectively. We parameterize the points of body A by the radius vector $\vec{r} = \vec{r_A}$ measured from the point P_A and the points of B by the radius vector $\vec{r'} = \vec{r_B}$ measured from the point P_B . The vector connecting P_A and P_B will be denoted by $\vec{a}$ (Fig.1). In the scalar case, the free spherical waves centered at P_A, P_B are given by

$$|(lm)_{A,B}\rangle = \sqrt{\frac{2\omega^2}{\pi}} j_l(\omega r_{A,B}) Y_{lm}(\hat{r}_{A,B}) \tag{40}$$

with the normalization $\langle \omega' l' m' | \omega l m\rangle = \delta_{ll'}\delta_{mm'}\delta(\omega - \omega')$.

To use (35), the scalar 3d Green function $G_0 = -\frac{e^{i\omega r}}{4\pi r}$, is expanded in terms of the spherical harmonic functions centered around P_A and those centered around P_B.

$$G_\omega = \sum_{lm;l'm'} |(lm)_B\rangle \mathcal{C}_{lm;l'm'} \langle (l'm')_A| \tag{41}$$

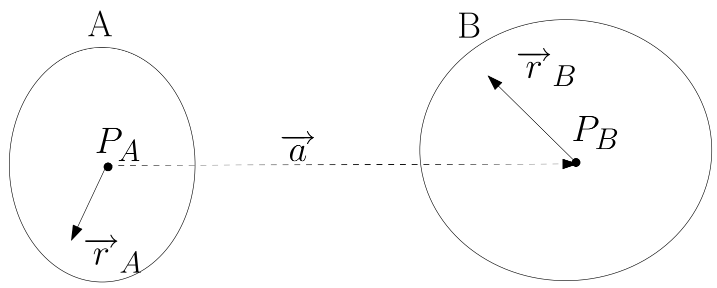

Figure 1. Coordinate system used for the partial wave approach

where

$$\mathcal{C}_{lm;l'm'}(\omega) = -\frac{i\pi}{2\omega}\sum_{l'',m''} C\left(\begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array}\right) i^{l''+l'-l} h^{(1)}_{l''}(\omega a) Y_{l''m''}(\hat{a}), \tag{42}$$

Y_{lm} are spherical harmonics, j_l, h_l are spherical Bessel and Hankel functions, and the coefficients $C\left(\begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array}\right)$ have known expressions in terms of the $3j$ symbol or as an integral of spherical functions:

$$C\left(\begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array}\right) = 4\pi \int d\Omega Y_{lm} Y^*_{l'm'} Y^*_{l''m''} = (-1)^m \sqrt{4\pi(2l+1)(2l'+1)(2l''+1)} \times \left(\begin{array}{ccc} l & l' & l'' \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} l & l' & l'' \\ m & -m' & -m'' \end{array}\right) \tag{43}$$

In actual computations it is often more convenient to use the Wick-rotated expression. This may be expressed as $\mathcal{C}_{lm;l'm'}(i\omega) = -\frac{\pi}{2\omega} i^{l'-l} g_{lm;l'm'}$. where the coefficients

$$g_{lm;l'm'} = \sum_{l'',m''} C\left(\begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array}\right) \sqrt{\frac{2}{\pi\omega a}} K_{l''+\frac{1}{2}}(\omega a) Y_{l''m''}(\hat{a}), \tag{44}$$

are real. Equations(42,44) may be somewhat simplified by choosing the z-axis along $\hat{a}$.

The above expansion of G_ω allows expressing $T_A G_0 T_B G_0$ in terms of matrix elements $\langle l'm'|T|lm\rangle$ of the transition matrices of the two scatterers. The Casimir interaction may then be written as in (35) where

$$K_{lm;l'm'} = (-1)^{l_1+l_2} (T_A)_{lm;l_1m_1} \mathcal{C}_{l_1m_1;l_2m_2} (T_B)_{l_2m_2;l_3m_3} \mathcal{C}_{l_3m_3;l'm'}. \tag{45}$$

Here $\mathcal{C}_{lm;l'm'}$ are given by (42) or (44), summation over $l_1, m_1, l_2, m_2, l_3, m_3$ is implied and we note that the extra sign resulted from $\mathcal{C}_{lm;l'm'}(-\hat{a}) \equiv (-1)^{l+l'} \mathcal{C}_{lm;l'm'}(\hat{a}) = \mathcal{C}_{l'm';lm}(\hat{a})$.

If we assume that only waves having $l \leq l_0$ are significantly scattered then K will turn into a finite $(l_0+1)^2 \times (l_0+1)^2$ matrix (since the dimension of the subspace $l \leq l_0$ is $\sum_{l=0}^{l_0}(2l+1) = (l_0+1)^2$). We stress that this argument *does not* require us to assume spherical symmetry of the scatterers.

When A, B are very far apart the interaction between them is governed by waves of very low frequency and therefore also low l. At this limit the leading contribution comes from the s-wave scattering transition matrix element $\langle l=0|T_{A,B}|l=0\rangle \simeq 2\omega^2\lambda_{A,B}/\pi$ where λ is the scattering length.

The matrix K then reduces to the scalar $K = -\omega^2\lambda_A\lambda_B\left(h_0^{(1)}(\omega a)\right)^2 = 4\pi\frac{\lambda_A\lambda_B}{a^2}e^{2ia\omega}$. Doing the integral (35) one arrives at

$$E_C = -\frac{\lambda_A\lambda_B}{a^3}.$$

This limit corresponds to the scalar version of the well known Casimir-Polder interaction. Our formalism however allows calculating corrections to it up to any desirable finite order in $\frac{1}{a}$. For example for two Dirichlet spheres of radii R_1, R_2 at distance a between their centers the expansion gives:

$$E = -\frac{R_1R_2}{4\pi a^3} - \frac{R_1R_2(R_1+R_2)}{8\pi a^4} - \frac{R_1R_2(34R_1^2+9R_1R_2+34R_2^2)}{48\pi a^5} - \frac{R_1R_2(R_1+R_2)(2R_1^2+21R_1R_2+2R_2^2)}{36\pi a^6} + \ldots \quad (46)$$

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11. Appendix: Some properties of (infinite dimensional) operators

Here we recall some mathematical notions that we have used in describing the trace class properties of (11).

Definition 11.1. *For an operator $B : H \to H$, the operator norm of $||B||$ is defined as* $||B|| = \sup_{\psi\in H,\psi\neq 0}\frac{|<\psi|B|\psi>|}{<\psi|\psi>}$

Definition 11.2. *An operator B is bounded if $||B|| < \infty$*

Definition 11.3. *An operator $A : H \to H$ is called a* **positive operator** *(denoted $A > 0$) iff $\langle\psi|A|\psi\rangle \geq 0$ for every $\psi \in H$.*

This implies that A is hermitian and its spectrum nonnegative. If $A : H \to H$ is a positive operator then there exist a unique positive operator $B : H \to H$ satisfying $A = B^2$. B is called the **square root** of A and denoted $\sqrt{A}$.

Definition 11.4. *An operator $A : H_1 \to H_2$ is called* **trace class** *(and denoted $A \in t.c.$ or $A \in \mathcal{J}_1$) iff $\sum ||A\psi_n|| < \infty$ where $\{\psi_n\}_{n=1}^{\infty}$ is some orthonormal basis of H_1. It can be shown that this condition does not depend on the choice of the orthonormal basis. (Note that the definition makes sense even when $H_1 \neq H_2$.)*

If $A : H \to H$ is trace class then for any orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$ of H the sum $\sum\langle\psi_n|A|\psi_n\rangle$ converges to the same (finite) value which is denoted $tr(A)$ and called the trace of A. One then also has $tr(A) = \sum\lambda_n$ where $\{\lambda_n\}$ are the eigenvalues of A (Lidski's theorem)

If $A : H \to H$ is trace class then the determinant $\det(1 + A)$ may also be rigorously defined and one has $\det(1 + A) = \prod(1 + \lambda_n)$.

The following theorem may be proved using the well known fact that the Fourier coefficients of a smooth $K(x, y)$ decay faster than any power. (Note that these coefficients also serve as the matrix elements with respect to Fourier basis of the operator defined by K.)

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Theorem 11.5. *Consider an operator $A : L^2(D_1) \to L^2(D_2)$ where D_1, D_2 are some domains in $\mathbb{R}^n$ which is given explicitly as an integral $A\psi(x) = \int_{D_1} K(x,y)\psi(y)dy$. A sufficient condition for A to be trace class is that D_1, D_2 are compact and $K(x,y)$ is smooth in a neighborhood of $D_1 \times D_2$.*

Proposition 11.6. *If A is trace class and B bounded then AB and BA are also trace class and $Tr(|AB|), Tr(|BA|) \leq ||B||Tr(|A|)$.*

Definition 11.7. *M is a Hilbert Schmidt operator (denoted $M \in H.S.$ or $M \in J_2$) if $||M||^2_{H.S.} \equiv \mathrm{Tr}\, M^\dagger M < \infty$*

In particular we mention that the product of two Hilbert Schmidt operators always gives a trace class operator.

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