# Simulation of number sets based on probability distributions 

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# Simulation of number sets based on probability distributions 

M S Tokmachev<br>Yaroslav-the-Wise Novgorod State University, ul. B. St. Petersburgskaya, 41 173003 Veliky Novgorod, Russia<br>E-mail: Mikhail.Tokmachov@novsu.ru


#### Abstract

A three-parameter probability distribution is presented, modeled on the basis of the negative binomial distribution. This distribution corresponds structurally to the author's distribution of the hyperbolic cosine type by the form of the characteristic function. The difference lies in the use of standard trigonometric cosine and sine for the characteristic function instead of the corresponding hyperbolic functions. The moment-forming polynomials of two arguments are found, derived from the recurrent differential relation to calculate the moments of distribution. A recurrent algebraic formula to produce the integer coefficients of these polynomials is introduced. The set of the coefficients depending on three arguments is ordered and geometrically interpreted as a numerical prism. Numerical prism sections are numerical triangles and numerical sequences. Among the sections of the prism there are well-known and new numerical sets, for example, Stirling numerical triangle, Bessel numerical triangle with alternating signs of elements, alternating tangential numbers, etc. A connection with the numerical prism derived from the hyperbolic cosine type distribution is indicated.


## 1. Introduction

The study is based on a three-parameter probability distribution of the hyperbolic cosine type with a characteristic function

$$
\begin{equation*}
f(t)=\left(\operatorname{ch} \frac{\beta}{m} t-\mathrm{i} \frac{\mu}{\beta} \operatorname{sh} \frac{\beta}{m} t\right)^{-m}, \quad \text { where } \mu, \beta, m \in \mathbf{R} ; m>0, \beta \neq 0 \tag{1}
\end{equation*}
$$

This distribution is obtained by the author in the problem of characterization of probability distributions by the condition of constancy of regression of quadratic statistics $Q$ on linear statistics $\Lambda$ [1, 2]:

$$
\begin{equation*}
E(Q \mid \Lambda)=E(Q), \quad \text { where } Q=\sum_{j=1}^{n} \sum_{k=1}^{n} a_{j k} X_{j} X_{k}, \quad \Lambda=\sum_{j=1}^{n} X_{j} \tag{2}
\end{equation*}
$$

$X_{1}, X_{2}, \ldots, X_{n}$ - independent, equally distributed random variables. With different ratios of the coefficients of the statistic $Q$ the condition (2) is also characteristic for a number of known distributions. A summarizing result of characterization for all possible cases is presented in [2].

The hyperbolic cosine type distribution is studied [3-5]. Being a generalization of the Meixner twoparameter probability distribution known in the literature [6], and also when $\mu=0, m=1$ of the classical one-parameter distribution of hyperbolic cosine (secant), this distribution is very interesting in terms of its properties. It is used in mathematical statistics. [4, 7, 8].

Finding the distribution moments of the hyperbolic cosine type forms a class of new recurrent polynomials based on features when differentiating the tangent, such as Derivative polynomials in work
[9]. Specified moment-forming polynomials $P_{n}(m ; b) \equiv P_{n}\left(m ; \frac{\mu}{\beta}\right)$ are expressed through distribution parameters [10].They are defined both in the form of recurrent differential relations, and with the help of a set of integer coefficients $\{U(n ; k, j)\}$. This ordered set of coefficients $\{U(n ; k, j)\}$, depending on three arguments, is geometrically interpreted as a numerical prism. Структурирование призмы позволяет выделить и систематизировать различные подмножества: числовые треугольники и числовые последовательности [11]. Among the sections of the prism there are not only new, but also fairly well-known from the literature, for example, a Stirling numerical triangle or a sequence of tangential numbers, secant numbers, etc. [12-13]. The underlying probability distribution and formulas for constructing $U\{n ; k, j\}$ lead to dependencies between subsets, as components of a prism. Many new numerical triangles and sequences, their interrelations with already known sets, as well as interesting relationships (for example, between tangential and secant numbers) are presented in [14-17].

The literature continues to show interest in numerical sequences and triangles, which can be seen for example in [13]. And their associations into more complex structures allow finding relations and relationships between mathematical objects. Particularly, [18] presents a numerical pyramid, among sections of which are well-known numerical sets: Euler, Stirling, Deleham, MacMahon, and also some numerical sequences.

Consideration of the hyperbolic cosine type distribution, as the basis for constructing a numerical prism, suggests using this technique for other probability distributions.

Distribution with a similar in structure characteristic function is used in the work for the simulation of a new numerical set:

$$
\begin{equation*}
f(t)=\left(\cos \frac{\alpha}{m} t-\mathrm{i} \frac{\mu}{\alpha} \sin \frac{\alpha}{m} t\right)^{-m}, \quad \text { where } \propto, \mu \in \mathbf{R} ;|\mu|>|\alpha| ; \alpha \neq 0 ; m \in \mathbf{N} \tag{3}
\end{equation*}
$$

The indicated probability distribution attracts attention by belonging to the group of classical distributions united by a common characterization method [2]: normal, gamma distribution, negative binomial, etc.

Formulas for calculating the moments were found, moment-forming polynomials were established, the corresponding numerical prism of the coefficients of the polynomials was constructed for the introduced distribution.

## 2. Main distribution

We consider a function (3), obtained from the characteristic function (1) by replacing the hyperbolic cosine and sine with the corresponding standard functions cos and $\sin$. Parameter $\beta$ is replaced with $\alpha$.

Theorem 1. Function (3)

$$
f(t)=\left(\cos \frac{\alpha}{m} t-\mathrm{i} \frac{\mu}{\alpha} \sin \frac{\alpha}{m} t\right)^{-m}, \text { at } \propto, \mu \in \mathbf{R} ;|\mu|>|\alpha| ; \alpha \neq 0 ; m \in \mathbf{N}
$$

is a characteristic function for the distribution of a random variable $Y=X_{1}+X_{2}+\cdots+X_{m}$, where $X_{1}, X_{2}, \ldots, X_{m}$ - independent identically distributed random variables, each $X_{j}$ - linear function of a random variable with a geometric distribution, $j=\overline{1, m}$.

Proof. 1. Analyze the function

$$
\begin{gathered}
f_{x}(t)=\left(\cos \frac{\alpha}{m} t-\mathrm{i} \frac{\mu}{\alpha} \sin \frac{\alpha}{m} t\right)^{-1}=\left(\frac{e^{\mathrm{i} \frac{\alpha}{m} t}+e^{-\mathrm{i} \frac{\alpha}{m} t}}{2}-\frac{\mu}{\alpha} \frac{e^{\mathrm{i} \frac{\alpha}{m} t}-e^{-\mathrm{i} \frac{\alpha}{m} t}}{2}\right)^{-1} \\
=\left(e^{\mathrm{i} \frac{\alpha}{m} t}\left(\frac{1}{2}-\frac{\mu}{2 \alpha}\right)+e^{-\mathrm{i} \frac{\alpha}{m} t}\left(\frac{1}{2}+\frac{\mu}{2 \alpha}\right)\right)^{-1}=\left(e^{\mathrm{i} \frac{\alpha}{m} t} \lambda_{1}+e^{-\mathrm{i} \frac{\alpha}{m} t} \lambda_{2}\right)^{-1}
\end{gathered}
$$

where, obviously, $\lambda_{1}+\lambda_{2}=1$, and one of the factors $\lambda_{1}, \lambda_{2}$ negative and the other one is more than 1 (because $|\mu|>|\alpha|$ ).

Suppose $\lambda_{1}>1$ (i.e. $\mu, \propto-$ of different signs). Write $f_{x}(t)$ as

$$
f_{x}(t)=\lambda_{1}^{-1} e^{-\mathrm{i} \frac{\alpha}{m} t}\left(1+\frac{\lambda_{2}}{\lambda_{1}} e^{-\mathrm{i} \frac{2 \alpha}{m} t}\right)^{-1}=e^{-\mathrm{i} \frac{\alpha}{m} t}\left(\frac{1 / \lambda_{1}}{1+\frac{\lambda_{2}}{\lambda_{1}} e^{-\mathrm{i} \frac{2 \alpha}{m} t}}\right)
$$

Let us set $\frac{1}{\lambda_{1}}=p$, then $\left(-\frac{\lambda_{2}}{\lambda_{1}}\right)=-\frac{1-\lambda_{1}}{\lambda_{1}}=1-\frac{1}{\lambda_{1}}=1-p=q$.
Consequently,

$$
f_{x}(t)=e^{-\mathrm{i} \frac{\alpha}{m} t}\left(\frac{p}{1-q e^{\mathrm{i} \frac{(-2 \alpha)}{m} t}}\right)
$$

Considering that the characteristic function of a random variable $X$, having a geometric distribution equals $\left(\frac{p}{1-q e^{i t}}\right)$, we get that $f_{x}(t)-$ is a characteristic function for a linear function $X$, namely: for a random variable $-2 \frac{\alpha}{m} X-\frac{\alpha}{m}$.

Suppose $\lambda_{2}>1$ (i.e. $\mu, \propto-$ of one sign). Then likewise

$$
f_{x}(t)=\lambda_{2}^{-1} e^{\mathrm{i} \frac{\alpha}{m} t}\left(1+\frac{\lambda_{1}}{\lambda_{2}} e^{\mathrm{i} \frac{2 \alpha}{m} t}\right)^{-1}=e^{\mathrm{i} \frac{\alpha}{m} t}\left(\frac{1 / \lambda_{2}}{1-\left(-\frac{\lambda_{1}}{\lambda_{2}}\right) e^{\mathrm{i} \frac{\alpha}{m} t}}\right)=e^{\mathrm{i} \frac{\alpha}{m} t}\left(\frac{p}{1-q e^{\mathrm{i} \frac{2 \alpha}{m} t}}\right)
$$

where $p=\frac{1}{\lambda_{2}}, q=1-p=\left(-\frac{\lambda_{1}}{\lambda_{2}}\right)$. Consequently $f_{x}(t)$-characteristic function of a random variable $2 \frac{\alpha}{m} X+\frac{\alpha}{m}$, where $X$ has a geometric distribution with the parameter $p$.

It should be noted that for. $\mu, \propto$ of one sign, we get the distribution $2 \frac{\alpha}{m} X+\frac{\alpha}{m}$, and for $\mu, \propto$ of different signs, we get the distribution $-2 \frac{\alpha}{m} X-\frac{\alpha}{m}$.

Let us express the parameters of the found distributions:
In the first case,

$$
p=\frac{1}{\lambda_{1}}=\frac{1}{\frac{1}{2}-\frac{\mu}{2 \alpha}}=\frac{2 \alpha}{\alpha-\mu} ; q=\frac{\mu+\alpha}{\mu-\alpha} .
$$

In the second case,

$$
p=\frac{1}{\lambda_{2}}=\frac{1}{\frac{1}{2}+\frac{\mu}{2 \alpha}}=\frac{2 \alpha}{\alpha+\mu} ; q=\frac{\mu-\alpha}{\mu+\alpha}
$$

2. For $m \in \mathbf{N}$ function $f(t)=\left(\cos \frac{\alpha}{m} t-\mathrm{i} \frac{\mu}{\alpha} \sin \frac{\alpha}{m} t\right)^{-m}$ is the product of the characteristic functions $f_{x}(t)$ with the corresponding ratios of coefficients. Therefore, $f(t)-$ is the characteristic function of the sum $Y=X_{1}+X_{2}+\cdots+X_{m}$, of independent, equally distributed random variables with the above distribution.

Theorem 1 is proven.
Let us note that for $|\mu|<|\alpha|$ function $f(t)$ of the form (3) is not characteristic, since it turns out to be inverse to the characteristic function

$$
\left(e^{\mathrm{i} \frac{\alpha}{m} t}\left(\frac{1}{2}-\frac{\mu}{2 \alpha}\right)+e^{-\mathrm{i} \frac{\alpha}{m} t}\left(\frac{1}{2}+\frac{\mu}{2 \alpha}\right)\right)^{m}=\left(e^{\mathrm{i} \frac{\alpha}{m} t} \lambda_{1}+e^{-\mathrm{i} \frac{\alpha}{m} t} \lambda_{2}\right)^{m}
$$

of the discrete distribution, where $\lambda_{1}, \lambda_{2}-$ are the $\operatorname{probabilities}\left(\lambda_{1}+\lambda_{2}=1\right)$. And for $|\mu|=|\alpha|$ there is only a degenerate distribution.

A random variable with a negative binomial distribution is the sum of independent random variables, each of which has a geometric distribution. The distribution of a linear function of random variables with a geometric distribution is called a geometric type distribution. Accordingly, the distribution of the sum of such independent random variables is called the negative binomial type distribution. This distribution, modeled on the basis of negative binomial distribution, has values on the entire real axis,
and not necessarily integers, depending on the parameters, and has a characteristic function of the form (3).

## 3. Calculation of moments. Moment generating polynomials

As is well known, [19], one can obtain recurrent moment ratios for a wide class of discrete distributions. In particular, the negative binomial distribution belongs to this class. A negative binomial distribution is considered a discrete analogue of Erlang distribution. A graph of the probability of the distribution of the negative binomial type corresponds to the graph of the density distribution of the hyperbolic cosine type. Next, we will confirm analytically the general recurrent basis for the construction of the numerical characteristics of these distributions.

Let us transform the characteristic function (3) by replacing

$$
\begin{align*}
\mathrm{i} \frac{\mu}{\alpha} & \equiv \tan a \equiv b:  \tag{4}\\
f(t)=\left(\cos \frac{\alpha}{m} t-\mathrm{i} \frac{\mu}{\alpha} \sin \frac{\alpha}{m} t\right)^{-m} & =\left(\frac{\cos a}{\cos \frac{\alpha}{m} t \cdot \cos a-\sin a \cdot \sin \frac{\alpha}{m} t}\right)^{m}=\left(\frac{\cos a}{\cos \left(a+\frac{\alpha}{m} t\right)}\right)^{m}
\end{align*}
$$

Then we differentiate the resulting function using the ratio
$\frac{1}{\cos ^{2} x}=1+\tan ^{2} x$.

$$
\begin{gathered}
f^{\prime}(t)=m\left(\frac{\alpha}{m}\right) f(t) \tan \left(a+\frac{\alpha}{m} t\right)=\left(\frac{\alpha}{m}\right) f(t)\left[m \tan \left(a+\frac{\alpha}{m} t\right)\right] \\
=\left(\frac{\alpha}{m}\right) f(t) P_{1}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right) ; \\
f^{\prime \prime}(t)=\left(\frac{\alpha}{m}\right)^{2} f(t)\left[\left(m^{2}+m\right) \tan ^{2}\left(a+\frac{\alpha}{m} t\right)+m\right]=\left(\frac{\alpha}{m}\right)^{2} f(t) P_{2}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right) ; \\
f^{\prime \prime \prime}(t)=\left(\frac{\alpha}{m}\right)^{3} f(t)\left[\left(m^{3}+3 m^{2}+2 m\right) \tan ^{3}\left(a+\frac{\alpha}{m} t\right)+\left(3 m^{2}+2 m\right) \tan \left(a+\frac{\alpha}{m} t\right)\right] \\
=\left(\frac{\alpha}{m}\right)^{3} f(t) P_{3}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right)
\end{gathered}
$$

In a general case, we obtain

$$
\begin{equation*}
f^{(n)}(t)=\left(\frac{\alpha}{m}\right)^{n} f(t) P_{n}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right) \tag{5}
\end{equation*}
$$

Where the function $P_{n}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right)$ - is polynomial of two arguments, $n$ - degree of polynomial for each of the arguments.

Supposing $t=0$, we obtain from (5)

$$
\begin{equation*}
f^{(n)}(0)=\left(\frac{\alpha}{m}\right)^{n} f(0) P_{n}(m ; \tan a)=\left(\frac{\alpha}{m}\right)^{n} P_{n}(m ; b) \tag{6}
\end{equation*}
$$

where, according to (4), the argument $b$ takes purely imaginary values. From the above derived characteristic function $f(t)$, it follows

$$
\begin{gathered}
P_{0}(m ; b)=1 ; P_{1}(m ; b)=m b ; \\
P_{2}(m ; b)=\left(m^{2}+m\right) b^{2}+m ; P_{3}(m ; b) \stackrel{\left(m^{3}+3 m^{2}+2 m\right) b^{3}+\left(3 m^{2}+2 m\right) b .}{ } .
\end{gathered}
$$

And also

$$
P_{4}(m ; b)=\left(m^{4}+6 m^{3}+11 m^{2}+6 m\right) b^{4}+\left(6 m^{3}+14 m^{2}+8 m\right) b^{2}+\left(3 m^{2}+2 m\right) .
$$

Since $b$ takes purely imaginary values, then the values of the polynomials $P_{n}(m ; b)$ for odd $n$ will be purely imaginary

Theorem 2. Polynomials $P_{n+1}(m ; b)$ satisfy the recurrent ratio.

$$
\begin{equation*}
P_{n+1}(m ; b)=m b P_{n}(m ; b)+\left(1+b^{2}\right) \frac{\partial P_{n}(m ; b)}{\partial b} \tag{7}
\end{equation*}
$$

Proof. To calculate $f^{(n+1)}(t)$ we differentiate the relation (5).
We obtain

$$
f^{(n+1)}(t)=\left(\frac{\alpha}{m}\right)^{n}\left[f^{\prime}(t) P_{n}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right)+f(t) P_{n}^{\prime}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right)\right]
$$

Instead of $f^{\prime}(t)$ we place $\left(\frac{\alpha}{m}\right) f(t)\left[m \tan \left(a+\frac{\alpha}{m} t\right)\right]$ and differentiate $P_{n}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right)$ as complex function. It follows

$$
\begin{aligned}
f^{(n+1)}(t)=\left(\frac{\alpha}{m}\right)^{n} & {\left[\left(\frac{\alpha}{m}\right) f(t)\left(m \tan \left(a+\frac{\alpha}{m} t\right)\right) P_{n}\left(m ; \tan \left(a+\frac{\propto}{m} t\right)\right)\right.} \\
& \left.+f(t) P_{n}^{\prime}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right)\right] \\
& =\left(\frac{\alpha}{m}\right)^{n+1} f(t)\left[\left(m \tan \left(a+\frac{\alpha}{m} t\right)\right) P_{n}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right)\right. \\
& \left.+\frac{\partial P_{n}\left(m ; \tan \left(a+\frac{\alpha}{m} t\right)\right)}{\partial \tan \left(a+\frac{\alpha}{m} t\right)}\left(1+\tan ^{2}\left(a+\frac{\propto}{m} t\right)\right)\right]
\end{aligned}
$$

Supposing $t=0$ and denoting $\tan a=b$, we come to ratios

$$
\begin{aligned}
& f^{(n)}(0)=\left(\frac{\alpha}{m}\right)^{n} P_{n}(m ; b) f^{(n+1)}(0)=\left(\frac{\alpha}{m}\right)^{n+1} P_{n+1}(m ; b), \\
& f^{(n+1)}(0)=\left(\frac{\alpha}{m}\right)^{n+1}\left[m b P_{n}(m ; b)+\left(1+b^{2}\right) \frac{\partial P_{n}(m ; b)}{\partial b}\right]
\end{aligned}
$$

Consequently, the recurrent formula takes place for polynomials (7)

$$
P_{n+1}(m ; b)=m b P_{n}(m ; b)+\left(1+b^{2}\right) \frac{\partial P_{n}(m ; b)}{\partial b}
$$

Theorem 2 is proven.
Let us express the moments of distribution, based on a known ratio $M\left(X^{n}\right)=\frac{1}{\mathrm{i}^{n}} f^{(n)}(0)$. According to (6), we obtain $M\left(X^{n}\right)=\frac{1}{\mathrm{i}^{n}}\left(\frac{\alpha}{m}\right)^{n} P_{n}(m ; b)$. Moreover, given that $b$ is a purely imaginary value, see (4), and moments $M\left(X^{n}\right)$ - are real numbers, we introduce moment generating polynomials in the form

$$
\begin{equation*}
Q_{n}(m ; b) \equiv \frac{1}{\mathrm{i}^{n}} P_{n}(m ; b) . \tag{8}
\end{equation*}
$$

Then $M\left(X^{n}\right)=\left(\frac{\alpha}{m}\right)^{n} Q_{n}(m ; b)$.
Considering that the relationship $b=\mathrm{i} \frac{\mu}{\alpha}$, we will write out the first few polynomials for clarity:

$$
\begin{gathered}
Q_{0}(m ; b)=1, \quad Q_{1}(m ; b)=m \frac{\mu}{\alpha} \\
Q_{2}(m ; b)=\left(m^{2}+m\right)\left(\frac{\mu}{\alpha}\right)^{2}-m ; Q_{3}(m ; b)=\left(m^{3}+3 m^{2}+2 m\right)\left(\frac{\mu}{\alpha}\right)^{3}-\left(3 m^{2}+2 m\right) \frac{\mu}{\alpha} \\
Q_{4}(m ; b)=\left(m^{4}+6 m^{3}+11 m^{2}+6 m\right)\left(\frac{\mu}{\alpha}\right)^{4}-\left(6 m^{3}+14 m^{2}+8 m\right)\left(\frac{\mu}{\alpha}\right)^{2}+\left(3 m^{2}+2 m\right)
\end{gathered}
$$

Relations (8) and (7) imply recurrence relations for polynomials and for moments of the distribution of the negative binomial type:

$$
\begin{gather*}
Q_{n+1}(m ; b)=\frac{1}{\mathrm{i}}\left(m b Q_{n}(m ; b)+\left(1+b^{2}\right) \frac{\partial Q_{n}(m ; b)}{\partial b}\right),  \tag{9}\\
M\left(X^{n+1}\right)=\left(\frac{\alpha}{m}\right)^{n+1} Q_{n+1}(m ; b)=\frac{1}{\mathrm{i}}\left(\frac{\alpha}{m}\right)^{n+1}\left(m b Q_{n}(m ; b)+\left(1+b^{2}\right) \frac{\partial Q_{n}(m ; b)}{\partial b}\right) . \tag{10}
\end{gather*}
$$

## 4. Moment generating polynomial coefficients

In [10], polynomials of the form $P_{n}(m ; b)$ are moment generating for the distribution of the hyperbolic cosine type and are represented as

$$
P_{n}(m ; b)=\sum_{k=1}^{n} \sum_{j=1}^{n} U(n ; k, j) m^{k} b^{j}
$$

where

$$
U(0 ; 0,0)=1 ; \quad U(0 ; k, j)=0 \text { at } k \neq 0 \text { or } j \neq 0
$$

$U(n+1 ; k, j)=U(n ; k-1, j-1)+(j-1) U(n ; k, j-1)+(j+1) U(n ; k, j+1)$ at $n=0,1,2, \ldots$
In our case, for the distribution of the negative binomial type, the polynomials $P_{n}(m ; b)$ have the same structure, but $b=\mathrm{i} \frac{\mu}{\alpha}$.

Accordingly, the polynomials $Q_{n}(m ; b)$ are represented as

$$
Q_{n}(m ; b) \equiv \frac{1}{\mathrm{i}^{n}} P_{n}(m ; b)=\frac{1}{\mathrm{i}^{n}} \sum_{k=1}^{n} \sum_{j=1}^{n} U(n ; k, j) m^{k}\left(\mathrm{i} \frac{\mu}{\alpha}\right)^{j}=\sum_{k=1}^{n} \sum_{j=1}^{n} U(n ; k, j) m^{k}\left(\frac{\mu}{\alpha}\right)^{j}{ }_{\mathrm{i}}{ }^{j-n}
$$

Introducing new ratios

$$
\begin{equation*}
W(n ; k, j)=U(n ; k, j) \mathrm{i}^{j-n} \tag{11}
\end{equation*}
$$

We obtain

$$
\begin{equation*}
Q_{n}(m ; b)=\sum_{k=1}^{n} \sum_{j=1}^{n} W(n ; k, j) m^{k}\left(\frac{\mu}{\alpha}\right)^{j} \tag{12}
\end{equation*}
$$

Polynomials $Q_{n}(m ; b)$ of the form (12) are moment generating for the distribution of the type of negative binomial, and (11) is the formula for connecting the coefficients of moment generating polynomials $Q_{n}(m ; b)$ and $P_{n}(m ; b)$.

Theorem 3. The coefficients of the moment generating polynomials $Q_{n}(m ; b)$ of the form (12) are related by the relations

$$
\begin{gather*}
W(0 ; 0,0)=1 ; W(0 ; k, j)=0 \text { at } k \neq 0 \text { or } j \neq 0 ; \\
W(n+1 ; k, j)=W(n ; k-1, j-1)+(j-1) W(n ; k, j-1)-(j+1) W(n ; k, j+1)  \tag{13}\\
\text { at } n=0,1,2, \ldots
\end{gather*}
$$

Proof. To obtain the formulas (13) of the theorem, we use the well-known numbers $\{U(n ; k, j)\}-$ the coefficients in the decomposition of the hyperbolic cosine-type moment generating polynomials and the connecting relation (11).

$$
\begin{gathered}
W(0 ; 0,0)=U(0 ; 0,0) \mathrm{i}^{0}=1 . \\
W(0 ; k, j)=U(0 ; k, j) \mathrm{i}^{j}=0 \text { at } k \neq 0 \text { or } j \neq 0 . \\
W(n+1 ; k, j)=U(n+1 ; k, j) \mathrm{i}^{j-n-1} \\
=\mathrm{i}^{j-n-1}[U(n ; k-1, j-1)+(j-1) U(n ; k, j-1)+(j+1) U(n ; k, j+1)] \text { at } n=0,1,2, \ldots \\
=\left[\mathrm{i}^{j-1-n} U(n ; k-1, j-1)+(j-1) \mathrm{i}^{j-1-n} U(n ; k, j-1)+(j+1) \mathrm{i}^{-2} \mathrm{i}^{j+1-n} U(n ; k, j+1)\right] \\
=W(n ; k-1, j-1)+(j-1) W(n ; k, j-1)-(j+1) W(n ; k, j+1) \text { at } n=0,1,2, \ldots
\end{gathered}
$$

Theorem 3 is proven.
Based on the formulas (13), you can find some numerical sequences associated with the moment generating polynomials $Q_{n}(m ; b)$, depending on the parameters $m, \mu, \alpha$.

## 5. Numerical prism

The set of coefficients $\{W(n ; k, j)\}$ can be ordered by its three arguments. We will call this set a numerical prism in geometrical terminology. A numerical prism based on a negative binomial type distribution is constructed using formulas (13) in the same way as a numerical prism [11, 15, 16] associated with the hyperbolic cosine type distribution.

Sections of a numerical prism are numerical sets obtained from the set $\{W(n ; k, j)\}$ upon fixing or interconnection of arguments. Prism sections are numeric triangles or numeric sequences. As examples of numerical triangles, we present the sets $\{W(n ; k, 1)\}$, see table 1 , and $\{W(2 n-j ; n, j)\}$, see table 2 .

Table 1. Values $W(n ; k, 1)$.

|  | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ | $k=7$ | $k=8$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n=1$ | 1 |  |  |  |  |  |  |  |
| $n=2$ |  |  |  |  |  |  |  |  |
| $n=3$ | -2 | -3 |  |  |  |  |  |  |
| $n=4$ |  |  |  |  |  |  |  |  |
| $n=5$ | 16 | 30 | 15 |  |  |  |  |  |
| $n=6$ |  |  |  |  |  |  |  |  |
| $n=7$ | -272 | -588 | -420 | -105 |  |  |  |  |
| $n=8$ |  |  |  |  |  |  |  |  |
| $n=9$ | 7936 | 18960 | 16380 | 6300 | 945 |  |  |  |
| $n=10$ |  |  |  |  |  |  |  |  |
| $n=11$ | -353792 | -911328 | -893640 | -429660 | -103950 | -10395 |  |  |
| $n=12$ |  |  |  |  |  |  |  |  |
| $n=13$ | 22368256 | 61152000 | 65825760 | 36636600 | 11351340 | 1891890 | 135135 |  |
| $n=14$ |  |  |  |  |  |  |  |  |
| $n=15$ | -1903757312 | -5464904448 | -6327135360 | -3918554640 | $-1427025600$ | -310269960 | 37837800 | -2027025 |

Table 2. Values $W(2 n-j ; k, j)$.

|  | $j=0$ | $j=1$ | $j=2$ | $j=3$ | $j=4$ | $j=5$ | $j=6$ | $j=7$ |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $n=0$ | 1 |  |  |  |  |  |  |  |
| $n=1$ | -1 | 1 |  |  |  |  |  |  |
| $n=2$ | 3 | -3 | 1 |  |  |  |  |  |
| $n=3$ | -15 | 15 | -6 | 1 |  |  |  |  |
| $n=4$ | 105 | -105 | 45 | -10 | 1 |  |  |  |
| $n=5$ | -945 | 945 | -420 | 105 | -15 | 1 |  |  |
| $n=6$ | 10395 | -10395 | 4725 | -1260 | 210 | -21 | 1 | 1 |
| $n=7$ | -135135 | 135135 | -62370 | 17325 | -3150 | 378 | -28 | 1 |
| $n=8$ | 2027025 | -2027025 | 945945 | -270270 | 51975 | -6930 | 630 | -36 |
| $n=9$ | -34459425 | 34459425 | -16216200 | 4729725 | -945945 | 135135 | -13860 | 990 |

The elements of the numeric triangle $\{W(n ; k, 1)\}$ in table 1 coincide modulo with the elements of the numeric triangle $\{U(n ; k, 1)\}$, given in $[15,16]$, but alternate in column. In particular, the sequence $\{W(2 n-1 ; 1,1)\}-$ is alternating tangential numbers.

The elements of the numeric triangle $\{W(2 n-j ; k, j)\}$, shown in table 2 , coincide modulo with the coefficients of the Bessel polynomials [20, 21]. The corresponding sections are given in [14, 15, 16]. The elements $\{U(2 n-j ; k, j)\}$-are coefficients in the function $f(x)$ of the form $\frac{P_{m}(x)}{\sqrt{(1-2 x)^{2 S+1}}}$, namely:
$\frac{1}{\sqrt{(1-2 x)}}, \frac{1+x}{\sqrt{(1-2 x)^{5}}}, \frac{1+3 x}{\sqrt{(1-2 x)^{7}}}, \frac{1+6 x+1.5 x^{2}}{\sqrt{(1-2 x)^{9}}}, \ldots$ [14]. Bessel numerical triangle [13] consists of the sequence of coefficients $\left\{f^{(k)}(0)\right\}$. The coefficients of Bessel polynomials of the ith order are written in the nth row of the Bessel numerical triangle.

The numeric triangle $\{W(2 n-j ; k, j)\}$ in table 2 represents the expansions of the corresponding functions

$$
f(-x): \frac{1}{\sqrt{(1+2 x)}}, \frac{1-x}{\sqrt{(1+2 x)^{5}}}, \frac{1-3 x}{\sqrt{(1+2 x)^{7}}}, \frac{1-6 x+1.5 x^{2}}{\sqrt{(1+2 x)^{9}}}, \ldots
$$

This entails alternating the sign in the sequences that make up the Bessel numeric triangle.
Other numerical triangles and sequences in the prism $\{W(n ; k, j)\}$ are also interesting. They may differ in the alternation of the sign of the elements from the corresponding sets in $\{U(n ; k, j)\}$.

## 6. Conclusion

The probability distribution modeled on the basis of the negative binomial type distribution turns out to be close to the hyperbolic cosine type distribution resulting from the characterization. In particular, the moments of these distributions have a general structure and principles of construction. The corresponding moment generating polynomials are constructed according to the recurrent formulas associated with the differentiation of the tangent. For polynomials and moments of the negative binomial type distribution, these recurrence relations are found, see (9), (10). Structurally identical polynomials are obtained by introducing a complex (purely imaginary) argument. The generated sets of coefficients of these polynomials differ only in the alternation of the sign of the elements. The numerical prism formed during the ordering of the coefficients contains positive and negative integer elements. At the same time, for example, the sections of the prism $\{W(n ; k, n)\}$ and $\{U(n ; k, n)\}$, which represent the famous Stirling numerical triangle (the Stirling numbers of the first kind) coincide.

Further study of the introduced numerical sets $\{W(n ; k, j)\}$ and $\{U(n ; k, j)\}$, which are geometrically interpreted as numerical prisms, seems very promising. We can note two aspects of the study: obtaining and systematization of various sequences and establishing new properties of relations and construction algorithms for known numerical sets.

The scheme of research "from probability distributions to numerical sets" can be applied to other distributions.

The results of the study can find applications in probability theory, number theory, cryptography and other mathematical and applied disciplines.

## References

[1] Tokmachev M S 1994 Characterization of the hyperbolic cosine type distribution by the property of regression constancy Dep. in VINITY 21.06.94. 1542-B94 11
[2] Tokmachev M S 1995 Constancy of regression of quadratic statistics on linear statistics Vestnik NovSU 1 139-141
[3] Tokmachev M S and Tokmachev A M 2001 Hyperbolic Cosine Type Distribution Vestnik NovSU 17 85-88
[4] Tokmachev M S 2005 The applied aspect of the generalized hyperbolic cosine distribution Vestnik NovSU 34 96-99
[5] Tokmachev M S 2017 Investigation of the probability distribution of the type of hyperbolic cosine Bulletin of South Ural State University. Series "Mathematics. Mechanics. Physics" Vol 93 18-26
[6] Lai C D 1982 Meixner classesand Meixner hypergeometric distributions Aust. J. Stat. 24 221233
[7] Medik V A and Tokmachev M S 2018 Mathematical statistics in medicine in 2 volumes: study guide for baccalaureate, specialty and magistracy. $2^{\text {nd }}$ ed (M.: Publishing house Urait) Vol 1 p 471
[8] Medik V A and Tokmachev M S 2018 Mathematical statistics in medicine in 2 volumes: study guide for baccalaureate, specialty and magistracy. $2^{\text {nd }}$ ed (M.: Publishing house Urait) Vol 2 p 347
[9] Hoffman M 1995 E Derivative polynomials for tangent and secant Amer. Math. Monthly 102 2330
[10] Tokmachev M S 2013 Calculation of cumulants and Meixner distribution points Vestnik NovSU 75 Vol 2 47-51
[11] Tokmachev M S 2015 On numerical sets and sequences in connection with the distribution of hyperbolic cosine type Vestnik NovSU. Ser.: Phys-math. Sciences 3 (86) 35-39
[12] Graham P, Knut D and Patashnik O 1998 Concrete Mathematics. Foundation of Informatics (Mir, M.)
[13] The On-Line Encyclopedia of Integer Sequences ${ }^{\mathrm{TM}}$ (OEIS ${ }^{\mathrm{TM}}$ ) http://oeis.org (assessed 04.03.2019)
[14] Tokmachev M S 2016 Numerical prism cross sections associated with Bessel polynomials Bulletin of South Ural State University. Series "Mathematics. Mechanics. Physics" ed Tokmachev M S Vol 83 64-71
[15] Tokmachev M S 2016 The set of subsets in the structure of a numerical prism (Monograph). Dep. in VINITY 09.06.2016 91 p 90
[16] Tokmachev M S 2018 Probability distributions: from characterization to number sets (Saarbrucken: LAP LAMBERT Academic Publ.) p 224
[17] Tokmachev M S 2019 Correlations Between Elements and Sequences in a Numerical Prism. Bulletin of South Ural State University. Series of "Mathematics. Mechanics. Physics" Vol 11 124-33 doi: $10.14529 / \mathrm{mmph} 190104$
[18] Franssens, Ghislain R 2006 On a Number Pyramid Related to the Binomial, Deleham, Eulerian, MacMahon and Stirling number triangles Journal of Integer Sequences Vol 9 Article 06.4.1
[19] Noack A A 1950 Class of variables with discrete distributions Ann. Math. Stat. 21 127-132
[20] Krall H L and Fink O A 1949 New Class of Orthogonal Polynomials: The Bessel Polynomials. Trans.Amer. Math. Soc. 65 100-115
[21] Carlitz L A 1957 Note on the Bessel Polynomials. Duke Math. J. 24 151-162

