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Inverse Problem for the Schrödinger Operator in an Unbounded Strip

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Abstract. We consider the operator \(H := i\partial_t + \nabla \cdot (c(x,y)\nabla q)\) in an unbounded strip \(\Omega = \mathbb{R} \times (-\frac{d}{2}, \frac{d}{2})\) of \(\mathbb{R}^2\) with a fixed width \(d\). We will consider the Schrödinger equation

\[
\begin{aligned}
Hq := i\partial_t q + \nabla \cdot (c(x,y)\nabla q) &= 0 \quad \text{in} \quad Q = \Omega \times (0,T), \\
q(x,y,t) &= b(x,y,t) \quad \text{on} \quad \Sigma = \partial\Omega \times (0,T), \\
q(x,y,0) &= q_0(x,y) \quad \text{on} \quad \Omega,
\end{aligned}
\]

where \(c(x,y) \in C^3(\Omega)\) and \(c(x,y) \geq c_{\text{min}} > 0\). Moreover, we assume that \(c\) and all its derivatives up to order three are bounded. If we assume that \(q_0\) belongs to \(H^1(\Omega)\) and \(b\) is sufficiently regular (e.g. \(b \in H^1(0,T, H^{\frac{5}{2}+\varepsilon}(\partial\Omega)) \cap H^2(0,T, H^{\frac{3}{2}+\varepsilon}(\partial\Omega))\) and some additional conditions), then (1) admits a solution in \(H^1(0,T, H^{\frac{3}{2}+\varepsilon}(\Omega))\). We will use this regularity result later. The aim of this paper is to give a stability and uniqueness result for the coefficient \(c(x,y)\) using global Carleman estimates and energy estimates. We denote by \(\nu\) the outward unit normal to \(\Omega\) on \(\Gamma = \partial\Omega\). We denote \(\Gamma = \Gamma^+ \cup \Gamma^-\), where \(\Gamma^+ = \{(x,y) \in \Gamma; \ y = \frac{d}{2}\}\) and \(\Gamma^- = \{(x,y) \in \Gamma; \ y = -\frac{d}{2}\}\). We use the following notations \(\nabla \cdot (c\nabla u) = \partial_x(c\partial_x u) + \partial_y(c\partial_y u), \ \nabla u \cdot \nabla v = \partial_x(u\partial_x v) + \partial_y(u\partial_y v), \ \partial_n u = \nabla u \cdot \nu\).

We shall use the following notations \(Q = \Omega \times (0,T), \ \tilde{Q} = \Omega \times (-T,T), \ \Sigma = \Gamma \times (-T,T), \ \Lambda(R_1) := \{\Phi \in L^\infty(\Omega), 0 < R_1 \leq \|
abla\Phi\|_{L^\infty(\Omega)}\}, \ \text{and} \ \Lambda(R_2) := \{\Phi \in L^\infty(\Omega), \|
abla\Phi\|_{L^\infty(\Omega)} \leq R_2\}, \ \text{where} \ R_1 \ \text{and} \ R_2 \ \text{are positive constants with} \ R_1 \leq R_2\).

Our problem can be stated as follows:

Is it possible to determine the coefficient \(c(x,y)\) from the measurement of \(\partial_\nu(\partial_t q)\) on \(\Gamma^+\)?

Let \(q\) (resp. \(\overline{q}\)) be a solution of (1) associated with \((c, b, q_0)\) (resp. \((\overline{c}, b, q_0)\)) satisfying some regularity properties:
• \( \partial_t \tilde{q}, \nabla (\partial_t \tilde{q}) \) and \( \Delta (\partial_t \tilde{q}) \) are in \( \Lambda(R_2) \),
• \( q_0 \) is a real valued function in \( C^3(\Omega) \),
• \( q_0 \) and all its derivatives up to order three are in \( \Lambda(R_2) \).

Our main result is

\[
|c - \tilde{c}|_{H^1(\Omega)}^2 \leq C|\partial_\nu (\partial_t q) - \partial_\nu (\partial_t \tilde{q})|_{L^2((0,T) \times \Gamma^+)}^2,
\]

where \( C \) is a positive constant which depends on \((\Omega, \Gamma, T, R_1, R_2)\) and where the above norms are weighted Sobolev norms.

The major novelty of this paper is to give an \( H^1 \) stability estimate for the diffusion coefficient with only one observation in an unbounded domain. We prove an adapted global Carleman estimate and an energy estimate for the operator \( H \) with a boundary term on \( \Gamma^+ \). Such energy estimate has been proved in [23] for the Schrödinger operator in a bounded domain in order to obtain a controllability result. Then using these estimates and following the method developed by Imanuvilov, Isakov and Yamamoto for the Lamé system in [16], [17], we give a stability and uniqueness result for the diffusion coefficient \( c(x,y) \). Note that this stability result corresponds to a stability result for three linked coefficients \( (c, \partial_x c \text{ and } \partial_y c) \) with only one observation. For independent coefficients, in our knowledge, there is no stability result with one observation. The method of Carleman estimates was introduced in the field of inverse problems in the works of Bukhgeim and Klibanov (see [1], [3], [19], [20]). The first stability result for a multidimensional inverse problem (for a hyperbolic equation) was obtained by Puel and Yamamoto [24] using a modification of the idea of [3]. For the non stationary Schrödinger equation, [2] gives a stability result for the potential in a bounded domain. For the stationary Schrödinger equation, we can cite recent results concerning uniqueness for the potential from partial Cauchy data (see for exemple [18] and the references herein). In unbounded domains Carleman estimate with an internal observation has been proved for the heat equation in [5]. A physical background could be the characterization of the diffusion coefficient for a strip in geophysics. Indeed if we look for time harmonic solutions of \((1)\), the problem can be written, after some changes of variables as the reconstruction of a non local potential \( P \) in a strip for the operator \(-\Delta + P\). Few results for inverse problems exist in a two-dimensional strip (see [9]). For the layer \( \mathbb{R}^n \times [0,b] \) with \( n \geq 2 \), several results exist for the stationary inverse problems (see [4], [10], [8], [13], [15], [25], ...).

On the other hand, we can link our problem to the determination of the curvature function for a curved quantum guide (see [12], [7], [11], ...).

We first prove a global Carleman estimate. Let \( c = c(x,y) \) be a bounded positive function in \( C^6(\Omega) \) such that

**Assumption 1** \( c(x,y) \in \Lambda(R_1), c \text{ and all its derivatives up to order three are in } \Lambda(R_2). \)

Let \( q = q(x,y,t) \) be a function equals to zero on \( \partial \Omega \times (-T,T) \) and solution of the Schrödinger equation

\[
i \partial_t q + \nabla \cdot (c(x,y)\nabla q) = f.
\]

We prove here a global Carleman-type estimate for \( q \) with a single observation acting on a part \( \Gamma^+ \) of the boundary \( \Gamma \) in the right-hand side of the estimate. Let \( \beta \) be a \( C^4(\Omega) \) positive function such that there exists positive constant \( C_{pc} \) which satisfies

**Assumption 2**

• \( |\nabla \tilde{\beta}| \in \Lambda(R_1) \)
• \( \partial_\nu \tilde{\beta} \leq 0 \) on \( \Gamma^- \)
• \( \tilde{\beta} \) and all its derivatives up to order four are in \( \Lambda(R_2) \).
• \( 2\Re(D^2 \tilde{\beta}(\zeta, \bar{\zeta})) - c \nabla c \cdot \nabla \tilde{\beta}|\zeta|^2 + 2c^2|\nabla \tilde{\beta} \cdot \zeta|^2 \geq C_{pc}|\zeta|^2, \text{ for all } \zeta \in \mathbb{C} \)
where
\[ D^2 \tilde{\beta} = \begin{pmatrix} c \partial_x (c \partial_x \tilde{\beta}) & c \partial_x (c \partial_y \tilde{\beta}) \\ c \partial_y (c \partial_x \tilde{\beta}) & c \partial_y (c \partial_y \tilde{\beta}) \end{pmatrix}. \]

Note that the last assertion of Assumption 2 expresses the pseudo-convexity condition for the function \( \tilde{\beta} \). This Assumption imposes restrictive conditions for the choice of the functions \( c(x, y) \) in connection with the function \( \tilde{\beta} \). Note that there exists functions satisfying such Assumptions; indeed, if we consider

\[ \tilde{\beta}(x, y) = \tilde{\beta}(y) \text{ is available (for example, } c(x, y) = (\frac{1}{1+y^2}+1) e^{-y} \text{ and } \tilde{\beta}(x, y) = e^y). \]

Similar restrictive conditions have been highlighted for the hyperbolic case in [21], [22] and for the Schrödinger operator in [14].

Then, we define \( \beta = \tilde{\beta} + K \) with \( K = m\|\tilde{\beta}\|_\infty \) and \( m > 1 \). For \( \lambda > 0 \) and \( t \in (-T, T) \), we define the following weight functions

\[ \varphi(x, y, t) = \frac{e^{\lambda \tilde{\beta}(x, y)}}{(T + t)(T - t)}, \quad \eta(x, y, t) = \frac{e^{2\lambda K - e^{\lambda \beta(x, y)}}}{(T + t)(T - t)}. \] (2)

Let \( H \) be the operator defined by

\[ Hq := i \partial_t q + \nabla \cdot (c(x, y)\nabla q) \text{ in } \tilde{Q} = \Omega \times (-T, T). \] (3)

We set \( \psi = e^{-sq}q, M\psi = e^{-sq}H(e^{sq}\psi) \) for \( s > 0 \) and we introduce the following operators

\[ M_1\psi := i \partial_t \psi + \nabla \cdot (c\nabla \psi) + s^2 c|\nabla \eta|^2 \psi, \] (4)

\[ M_2\psi := i s \partial_t \eta \psi + 2cs \nabla \eta \cdot \nabla \psi + s \nabla \cdot (c \nabla \eta) \psi. \] (5)

Then the following result holds.

**Theorem 3** Let \( H, M_1, M_2 \) be the operators defined respectively by (3), (4), (5). We assume that Assumptions 1 and 2 are satisfied. Then there exist \( \lambda_0 > 0, s_0 > 0 \) and a positive constant \( C = C(\Omega, \Gamma, T, C_p, R_1, R_2) \) such that, for any \( \lambda \geq \lambda_0 \) and any \( s \geq s_0 \), the next inequality holds:

\[ s^3 \lambda^4 \int_{-T}^{T} \int_{\Omega} e^{-2sq} \varphi^3 |q|^2 \, dx \, dy \, dt + s\lambda \int_{-T}^{T} \int_{\Omega} e^{-2sq} \varphi |\nabla q|^2 \, dx \, dy \, dt + \|M_1(e^{-sq}q)\|_{L^2(\tilde{Q})}^2 \] (6)

\[ + \|M_2(e^{-sq}q)\|_{L^2(\tilde{Q})}^2 \leq C \left[ s\lambda \int_{-T}^{T} \int_{\Gamma} e^{-2sq} |\partial_\omega \eta| \varphi |\partial_\nu \beta| \, d\sigma \, dt + \int_{-T}^{T} \int_{\Omega} e^{-2sq} |Hq|^2 \, dx \, dy \, dt \right], \]

for all \( q \) satisfying \( Hq \in L^2(\Omega \times (-T, T)) \), \( q \in L^2(-T, T; H^1_0(\Omega)) \), \( \partial_\nu q \in L^2(-T, T; H^2(\Gamma)) \).

For the proof see [6].

We now establish a stability result for the coefficient \( c \). The Carleman estimate (6) will be the key ingredient in the proof of such a stability estimate.

Let \( q \) be solution of

\[ \begin{cases} i \partial_t q + \nabla \cdot (c \nabla q) = 0 \quad \text{in } \Omega \times (0, T), \\
q(x, y, t) = b(x, y, t) \quad \text{on } \partial \Omega \times (0, T), \\
q(x, y, 0) = q_0(x, y) \quad \text{in } \Omega, \end{cases} \] (7)
and \( \tilde{q} \) be solution of
\[
\begin{aligned}
\begin{cases}
i\partial_t \tilde{q} + \nabla \cdot (c \nabla \tilde{q}) = 0 & \text{in } \Omega \times (0, T), \\
\tilde{q}(x, y, t) = b(x, y, t) & \text{on } \partial \Omega \times (0, T), \\
\tilde{q}(x, y, 0) = q_0(x, y) & \text{in } \Omega,
\end{cases}
\end{aligned}
\]
where \( c \) and \( \tilde{c} \) both satisfy Assumption 1. If we set \( u = q - \tilde{q}, \ v = \partial_t u \) and \( \gamma = \tilde{c} - c \), then \( u \) and \( v \) satisfy respectively
\[
\begin{aligned}
\begin{cases}
i\partial_t u + \nabla \cdot (c \nabla u) = \nabla \cdot (\gamma \nabla \tilde{q}) & \text{in } \Omega \times (0, T), \\
u(x, y, t) = 0 & \text{on } \partial \Omega \times (0, T), \\
u(x, y, 0) = 0 & \text{in } \Omega,
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
\begin{cases}
i\partial_t v + \nabla \cdot (c \nabla v) = \nabla \cdot (\gamma \nabla \partial_t \tilde{q}) = f & \text{in } \Omega \times (0, T), \\
v(x, y, t) = 0 & \text{on } \partial \Omega \times (0, T), \\
v(x, y, 0) = \frac{1}{\gamma} \nabla \cdot (\gamma \nabla q_0) & \text{in } \Omega.
\end{cases}
\end{aligned}
\]

**Assumption 4** \( q_0 \) is a real valued function in \( C^3(\Omega) \)

We extend the function \( v \) on \( \Omega \times (-T, T) \) by the formula \( v(x, y, t) = -\overline{v}(x, y, -t) \) for every \( (x, y, t) \in \Omega \times (-T, 0) \). Note that this extension is available if the initial data is a real valued function. For a pure imaginary initial data, the right extension is \( v(x, y, t) = \overline{v}(x, y, -t) \). Note that these extensions satisfy the previous Carleman estimate.

We prove the three following lemma.

We set \( \psi = e^{-s\eta} v \). With the operator
\[
M_1 \psi = i\partial_t \psi + \nabla \cdot (c \nabla \psi) + s^2 |\nabla \eta|^2 \psi,
\]
we introduce, following [2],
\[
I = 2\Im \left( \int_{-T}^T \int_{\Omega} M_1 \psi \overline{\psi} \, dx \, dy \, dt \right).
\]

**Assumption 5** \( \partial_t \tilde{q}, \nabla (\partial_t \tilde{q}), \Delta (\partial_t \tilde{q}) \) are in \( \Lambda(R_2) \).

We have the following estimate

**Lemma 6** We assume that Assumption 5 is satisfied. Then there exists a positive constant \( C = C(\Omega, \Gamma, T, R_1, R_2) \) such that for any \( \lambda \geq \lambda_0 \) and \( s \geq s_0 \), we have
\[
I = \int_{\Omega} e^{-2s\eta(x,y,0)} |\partial_t u(x, y, 0)|^2 \, dx \, dy
\]
and
\[
|I| \leq C s^{-3/2} \lambda^{-2} \int_{\Omega} e^{-2s\eta(x,y,0)} (|\gamma|^2 + |\nabla \gamma|^2) \, dx \, dy
\]
\[
+ C s^{-1/2} \lambda^{-1} \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_\nu \beta \, |\partial_\nu u|^2 \, d\sigma \, dt.
\]

We then give an estimate of \( E(0) = \int_{\Omega} \varphi^{-1}(x, y, 0) e^{-2s\eta(x,y,0)} |\partial_t \nabla u(x, y, 0)|^2 \, dx \, dy \).

**Lemma 7** Let \( v \) be solution of (10) in the following class
\[
v \in C([0, T], H^1(\Omega)), \ \partial_\nu v \in L^2(0, T, L^2(\Gamma)).
\]
We assume that Assumptions 1 and 2 are checked. Then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1, R_2) > 0$ such that

$$E(0) \leq C \left( s^2 \lambda^2 \int_{\Gamma^-}^{T} \phi_0 e^{-2s\eta} |\nabla u|^2 \right)\left( dx dy \right) \leq C \left( \int_{\Omega} \phi_0^{-1} \int_{\Omega} e^{-2s\eta} |\nabla u(x,y,0)|^2 \right) \left( dx dy \right),$$

(13)

for $s$ and $\lambda$ sufficiently large.

We adapt the proof of lemma 3.2 of [16] to an unbounded domain.

**Assumption 8**
- $q_0$ and all its derivatives up to order three are in $\Lambda(R_2)$
- $|\nabla \beta \cdot \nabla q_0| \in \Lambda(R_1)$

**Lemma 9** Let $u$ be solution of (10). We assume that Assumptions 2, 4 and 8 are satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1, R_2)$ such that for $s$ and $\lambda$ sufficiently large, the following estimates hold true

$$\int_{\Omega} \phi_0 e^{-2s\eta} |c - \tilde{c}|^2 + |\nabla(c - \tilde{c})|^2 \left( dx dy \right) \leq C \left( s^2 \lambda^2 \right) \left( \int_{\Gamma^-}^{T} \phi_0^{-1} \int_{\Omega} e^{-2s\eta} |\nabla u(x,y,0)|^2 \right) \left( dx dy \right),$$

(14)

for $\gamma \in H_0^2(\Omega)$.

For the proofs see [6].

**Theorem 10** Let $q$ and $\tilde{q}$ be solutions of (7) and (8) such that $c - \tilde{c} \in H_0^2(\Omega)$. We assume that Assumptions 1, 2, 5, 4 and 8 are satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1, R_2)$ such that for $s$ and $\lambda$ sufficiently large,

$$s^2 \lambda^2 \int_{\Omega} \phi_0 e^{-2s\eta} \left( |\nabla u|^2 + |\nabla u(x,y,0)|^2 \right) \left( dx dy \right) \leq C \left( s^2 \lambda^2 \right) \left( \int_{\Omega} \phi_0^{-1} e^{-2s\eta} \left( |\nabla u(x,y,0)|^2 + |\nabla u(x,y,0)|^2 \right) \right) \left( dx dy \right)$$

$$\leq C(\mathcal{I} + E(0))$$

$$\leq C s^{-3/2} \lambda^{-2} \int_{\Omega} e^{-2s\eta} \left( |\nabla u|^2 + |\nabla u(x,y,0)|^2 \right) \left( dx dy \right) + C s^{-1/2} \lambda^{-1} \int_{\Gamma^-}^{T} \phi_0 e^{-2s\eta} \varphi \partial_{\alpha} \beta \left( \partial_{\nu} v \right)^2 \left( ds dt \right)$$

$$+ C s^2 \lambda^2 \int_{\Gamma^-}^{T} \phi_0 e^{-2s\eta} \varphi \partial_{\alpha} \beta \left( \partial_{\nu} v \right)^2 \left( ds dt \right) + C s \lambda \int_{Q} e^{-2s\eta} \left( \int_{\Omega} \phi_0 e^{-2s\eta} \right) \left( dx dy \right).$$

So we get

$$s^2 \lambda^2 \int_{\Omega} \phi_0 e^{-2s\eta} \left( |\nabla u|^2 + |\nabla u(x,y,0)|^2 \right) \left( dx dy \right) \leq C s^2 \lambda^2 \int_{\Gamma^-}^{T} \phi_0 e^{-2s\eta} \varphi \partial_{\alpha} \beta \left( \partial_{\nu} v \right)^2 \left( ds dt \right)$$

$$+ C s \lambda \int_{Q} e^{-2s\eta} \left( \nabla \cdot (\nabla \partial_{\nu} q)^2 \right) \left( dx dy \right).$$
Remark: This result is also available for the heat equation in bounded or unbounded domains. Then, for $s$ if we adapt the regularity properties of the initial and boundary conditions.