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Stability analysis of the lattice Boltzmann schemes with body force action

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Abstract. Stability analysis of lattice Boltzmann equations for the case of body force action is realized. The analysis is performed by von Neumann method and is based on the linearized equations. The stability is estimated by the values of the areas of the stability domains in parametric space. As the results of the analysis, the recommendation on the use of the models in practical simulations may be performed. The proposed approach to the analysis may be extended to the investigation of the other cases of body force actions, especially in the cases of multiphase and multicomponent flows.

1. Introduction

Lattice Boltzmann method (LBM) in the last several decades is considered as a powerful tool for computational modelling of complex flows with multiple phases and components [1]. These types of flows may be characterized by the existence of the body force, which model interphase or intercomponent interactions. Other types of flows with the body force action are realized in the cases of flows with vector fields of external body forces, such as gravitational or electromagnetic forces. So for the practical simulations it is necessary to correctly represent the model of the body force action into computational schemes used in the LBM.

The first models of body force action were proposed in the early years of the development of the LBM. In [2, 3] models based on the perturbation of the equilibrium flow regime are proposed for the simulation of multiphase and multicomponent flows. In [4, 5] the models for the representation of the body force term in the discrete Boltzmann equations are proposed. Ladd et al [6] and Guo et al [7] proposed the models for terms with body forces, which are based on the expansions on the generalized Hermite polynomials. The models are based on the theory of the discretization of kinetic equations in the velocity space of kinetic equations [8]. In [9] the so-called exact difference method for incorporating of body force term is proposed. The model is based on the representation of body force term by a difference of the values of equilibrium distribution functions on perturbed and unperturbed flows. Comprehensive reviews of the models used for body force action modeling are presented at [1, 10].

Despite the widely development of body force action models used in the LBM, investigation of their stability properties and comparison of this property are still not realized. So the presented paper is dedicated to the stability analysis of LBE's with body force action, proposed by different authors.



The structure of the paper is the following. In Section 2 discrete Boltzmann equations and LBE's are discussed. In Section 3 the models of body force action widely used in the LBM are presented. In Section 4 the problem of the stability investigation is stated and some numerical results of the analysis are discussed. Some concluding remarks are made in Section 5.

2. Lattice Boltzmann equations

Let us consider the Boltzmann kinetic equation with the relaxation collision term (Bhatnagar — Gross — Krook equation):

$$\frac{\partial f}{\partial t} + \mathbf{V}\nabla f + \mathbf{a}\nabla_{\mathbf{V}}f = -\frac{f - f^{(eq)}}{\lambda}, \quad (1)$$

where $f = f(t, \mathbf{r}, \mathbf{V})$ is a particle distribution function, t is a time, \mathbf{r} is a vector of space variables, \mathbf{V} is a vector of particle velocity, $\mathbf{a} = \mathbf{F}/m$, \mathbf{F} is a body force, m is a particle mass, $\lambda > 0$ is a relaxation time, $f^{(eq)} = f^{(eq)}(\rho, \mathbf{U})$ is a Maxwellian distribution function, ρ is a gas density, \mathbf{U} is a gas velocity. These macrovariables may be calculated as a moments of the distribution function [8]:

$$\rho(t, \mathbf{r}) = \int f(t, \mathbf{r}, \mathbf{V})d\mathbf{V}, \quad \rho(t, \mathbf{r})\mathbf{U}(t, \mathbf{r}) = \int \mathbf{V}f(t, \mathbf{r}, \mathbf{V})d\mathbf{V}.$$

After the discretization of (1) on the velocities the following grid in the velocity space is considered: $\mathbf{V}_i = V\mathbf{e}_i$, $i = \overline{1, n}$, where $V = l/\delta t$ is a typical particle velocity, l is a mean free path, δt is a mean free time, \mathbf{e}_i are the dimensionless vectors. After the computation of f at nodes \mathbf{V}_i the following system on $f_i(t, \mathbf{r}) = f(t, \mathbf{r}, \mathbf{V}_i)$ is obtained:

$$\frac{\partial f_i}{\partial t} + \mathbf{V}_i\nabla f_i + (\mathbf{a}\nabla_{\mathbf{V}}f)_i = -\frac{f_i - f_i^{(eq)}}{\lambda}, \quad i = \overline{1, n}. \quad (2)$$

In cases of fluids and dense gases, number n is not so high for the exact approximation of $(\mathbf{a}\nabla_{\mathbf{V}}f)_i$, so for the approximation of this terms the model representations are used. In this paper the so-called D2Q9 lattice is considered. This lattice is defined by the following set of \mathbf{e}_i : $\mathbf{e}_1 = (0, 0)$, $\mathbf{e}_2 = (1, 0)$, $\mathbf{e}_3 = (0, 1)$, $\mathbf{e}_4 = (-1, 0)$, $\mathbf{e}_5 = (0, -1)$, $\mathbf{e}_6 = (1, 1)$, $\mathbf{e}_7 = (-1, 1)$, $\mathbf{e}_8 = (-1, -1)$, $\mathbf{e}_9 = (1, -1)$.

After the approximation of (2) on time and space grids, the following LBE's are obtained:

$$f_i(t_j + \delta t, \mathbf{r}_{kl} + \mathbf{V}_i\delta t) = f_i(t_j, \mathbf{r}_{kl}) + \Delta f_i - \frac{1}{\tau} \left(f_i(t_j, \mathbf{r}_{kl}) - f_i^{(eq)}(\rho, \mathbf{U}) \right), \quad (3)$$

where t_j is a node of the time grid, \mathbf{r}_{kl} is a node of the grid on space variables, $\tau = \lambda/\delta t$ is a dimensionless relaxation time and terms Δf_i model the influence of the body force.

In the case of the absence of body forces, (3) is rewritten as:

$$f_i(t_j + \delta t, \mathbf{r}_{kl} + \mathbf{V}_i\delta t) = f_i(t_j, \mathbf{r}_{kl}) - \frac{1}{\tau} \left(f_i(t_j, \mathbf{r}_{kl}) - f_i^{(eq)}(\rho, \mathbf{U}) \right). \quad (4)$$

3. Body force action models

At the time interval $[t_j, t_j + \delta t]$ the body force changes the momentum at the node \mathbf{r}_{kl} on the value $\Delta(\rho\mathbf{U}) = \mathbf{F}\delta t$. If the case of weakly-compressible flows is considered, so density at the node \mathbf{r}_{kl} may be approximated by a constant, so the velocity change in this node is equal to $\Delta\mathbf{U} = \mathbf{F}\delta t/\rho$.

In this paper the following widely-used models of body force action are considered:

1) Model of Shan and Chen. This model is proposed in [2, 3] and is based on the following assumption: the body force action at one time step leads to the equilibrium state of the system

of particles with velocities \mathbf{V}_i . So at this approach $\delta t = \lambda$ and the velocity value after the body force action is computed as:

$$\mathbf{U}^{(eq)} = \mathbf{U} + \frac{\mathbf{F}\lambda}{\rho}.$$

To the next time moment system is evolved from the equilibrium state, which corresponds to the velocity $\mathbf{U}^{(eq)}$. So, according to this approach the body force terms Δf_i are not introduced and the model is based on the discrete equations (4) in the case, when in right part of (4) $\mathbf{U} = \mathbf{U}^{(eq)}$. The actual velocity is calculated by the correction based on the following expression [3, 10]:

$$\mathbf{U}^* = \mathbf{U} + \frac{\mathbf{F}\delta t}{2\rho}. \quad (5)$$

2) Model of L.S. Luo. The model may be realized in the case of the complete kinetic equation (1) and is based on the following assumption:

$$\nabla_{\mathbf{V}} f \approx \nabla_{\mathbf{V}} f^{(eq)}, \quad (6)$$

which is based on the asymptotic representation of f , where the main part of f is defined by its equilibrium part $f^{(eq)}$. According to (6), the following representation of Δf_i is obtained:

$$\Delta f_i = \rho W_i \left(\frac{(\mathbf{V}_i - \mathbf{U})}{\theta} + \frac{(\mathbf{V}_i \cdot \mathbf{U})}{\theta^2} \mathbf{V}_i \right) \Delta \mathbf{U},$$

where $\theta = kT/m$, k is a Boltzmann constant, W_i are the parameters of the equilibrium distribution function and T is a temperature. From this model the actual velocity is obtained as: $\mathbf{U}^* = \mathbf{U}$.

3) Model of Guo et al. The model, proposed in [7], is based on the representation of body force term by expansion on Hermite polynomials. According to this approach, after the discretization Δf_i are represented by:

$$\Delta f_i = \left(1 - \frac{1}{2\tau} \right) \rho W_i \left(\frac{(\mathbf{V}_i - \mathbf{U}^*)}{\theta} + \frac{(\mathbf{V}_i \cdot \mathbf{U}^*)}{\theta^2} \mathbf{V}_i \right) \Delta \mathbf{U}.$$

4) Model of Kupershtokh et al. In [9] the following representation of $\nabla_{\mathbf{V}} f^{(eq)}$ is obtained: $\nabla_{\mathbf{V}} f^{(eq)} = -\nabla_{\mathbf{U}} f^{(eq)}$. After the changing of velocity on one time interval of length δt on $\Delta \mathbf{U}$, the following expression of the body force term is obtained:

$$\mathbf{a} \nabla_{\mathbf{V}} f \delta t = -(f_i^{(eq)}(\rho, \mathbf{U} + \Delta \mathbf{U}) - f_i^{(eq)}(\rho, \mathbf{U})),$$

and the following approximation is presented in [9]:

$$\Delta f_i = \rho W_i \left(\frac{(\mathbf{V}_i - \mathbf{U})}{\theta} + \frac{(\mathbf{V}_i \cdot \mathbf{U})}{\theta^2} \mathbf{V}_i \right) \Delta \mathbf{U} + \rho W_i \left(\frac{(\mathbf{V}_i \cdot \Delta \mathbf{U})^2}{2\theta^2} - \frac{(\Delta \mathbf{U})^2}{2\theta} \right).$$

The actual velocity is computed by using of (5).

4. Stability analysis

4.1. Statement of the problem

Let us consider dimensionless forms of all the proposed schemes. The stability is investigated in the cases of spatially homogeneous stationary flows in unbounded domain, corresponds to following values of macrocharacteristics: $\rho = 1$, $U_x = U$, $U_y = 0$, $U \in [0, 1]$. The case of

the constant body force $\mathbf{F} = (F_x, F_y)$ is considered. Components of \mathbf{F} are represented as: $F_x = F \cos(\varphi)$, $F_y = F \sin(\varphi)$, where $\varphi \in [0, 2\pi)$ is the angle between vectors \mathbf{F} and \mathbf{U} . The case of $F \in [0, 1]$ is considered. So for the stability investigation stability domains in the parameter space (U, F, φ) are considered.

Solutions of the LBE's are presented in following form:

$$f_i(t_j, \mathbf{r}_{kl}) = \bar{f}_i + \delta f_i(t_j, \mathbf{r}_{kl}), \quad (7)$$

where δf_i are the perturbations of the unperturbed solutions \bar{f}_i , which are calculated as: $\bar{f}_i = f_i^{(eq)}(\rho, \mathbf{U}) = const$, where \mathbf{U} corresponds to the stationary regime discussed above. After the substitution of (7) into LBE's and its linearization the system for δf_i may be obtained. The solution of this system may be presented in following form [11]:

$$\delta f_i(t_j, \mathbf{r}_{kl}) = F_i(t_j) \exp(\mathbf{i}\Theta \mathbf{r}_{kl}), \quad (8)$$

where $\mathbf{i}^2 = -1$, $\Theta = (\theta_x, \theta_y)$, $\theta_\alpha \in [-\pi, \pi]$, $\alpha = x, y$.

After the substitution of (8) into linearized LBE's, the following system of linear difference equations for F_i is obtained:

$$F_i(t_{j+1}) = \sum_{s=1}^9 G_{is} F_s(t_j), \quad (9)$$

where the components G_{is} for the case of eqs. (4) are presented as:

$$G_{is} = \begin{cases} \left(\left(1 - \frac{1}{\tau}\right) + \frac{1}{\tau} \frac{\partial f_i^{(eq)}}{\partial f_s}(\bar{\mathbf{f}}) \right) \exp(-\mathbf{i}\Theta \mathbf{e}_i), & i = s, \\ \left(\frac{1}{\tau} \frac{\partial f_i^{(eq)}}{\partial f_s}(\bar{\mathbf{f}}) \right) \exp(-\mathbf{i}\Theta \mathbf{e}_i), & i \neq s, \end{cases}$$

and for the case of eqs. (3) the following expressions are written:

$$G_{is} = \begin{cases} \left(\left(1 - \frac{1}{\tau}\right) + \frac{1}{\tau} \frac{\partial f_i^{(eq)}}{\partial f_s}(\bar{\mathbf{f}}) + \frac{\partial(\Delta f_i)}{\partial f_s}(\bar{\mathbf{f}}) \right) \exp(-\mathbf{i}\Theta \mathbf{e}_i), & i = s, \\ \left(\frac{1}{\tau} \frac{\partial f_i^{(eq)}}{\partial f_s}(\bar{\mathbf{f}}) + \frac{\partial(\Delta f_i)}{\partial f_s}(\bar{\mathbf{f}}) \right) \exp(-\mathbf{i}\Theta \mathbf{e}_i), & i \neq s. \end{cases}$$

So the problem of the stability of unperturbed solutions of systems (3) or (4) is reduced to the problem of the stability of the null solution of a system (9). This solution is stable, when the following inequality on the absolute values of the eigenvalues of matrix $\{G_{is}\}$ is realized [11]:

$$|\lambda(\tau, U, F, \varphi, \theta_x, \theta_y)| \leq 1. \quad (10)$$

For the finding of the eigenvalues of $\{G_{is}\}$ the QR-algorithm realized in EISPACK package is used.

4.2. Results of computations

At the first stage we try to compare all of the proposed models at the case of $\tau = 1$ (because the model of Shan and Chen is valid only in this case). For the construction of the stability domains the following uniform grid $M \times N$ nodes is constructed: $\Omega = \{(F_p, \varphi_r) | F_p = p h_F, \varphi_r = r h_\varphi, h_F = (F_M - F_1)/(M - 1), h_\varphi = (\varphi_N - \varphi_1)/(N - 1), p = \overline{1, M}, r = \overline{1, N}\}$. For every node of Ω the eigenvalues of matrix $\{G_{is}\}$ are obtained for the grids, constructed in the domain, where parameter U and vector Θ are changed. If inequality (10) in node (F, φ) is realized for all the

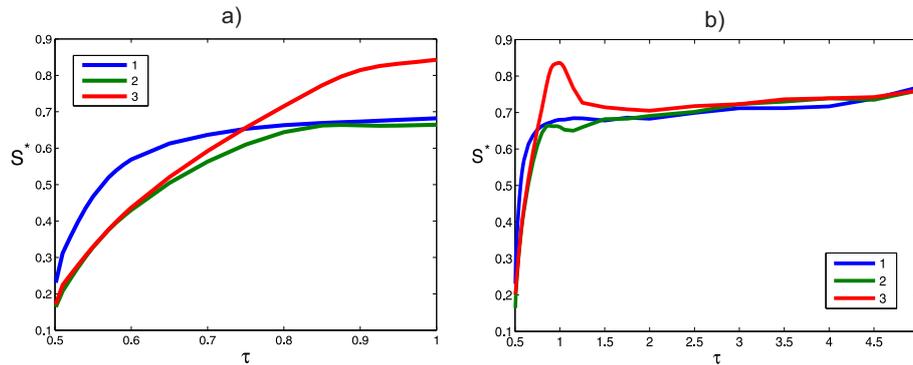


Figure 1. Plots of the maximal values $S^*(\tau)$ for considered models in the cases of $\tau \in [1/2, 1]$ (a) and $\tau \in [1/2, 5]$: 1 — model proposed by Luo; 2 — model proposed by Guo et al; 3 — model proposed by Kupershtokh et al

nodes in the domains, where U and Θ are changed, so this node is included to the stability domain. The area of the stability domain $S(F, \varphi)$ is calculated by rectangular quadrature rule.

All calculations were performed at the grids with 100×100 nodes. At table 1 the maximal values of S and its arguments $(F^*, \varphi^*) = \arg \max S(F, \varphi)$ are presented.

Table 1. The results of the numerical stability analysis: the maximal values of the areas of the stability domains $S^* = \max_{(F, \varphi)} S(F, \varphi)$ and its arguments for the case of $\tau = 1$.

Model	S^*	F^*	φ^*
Shan and Chen model	0,8480	1	3,1733
Luo model	0,6735	0,3434	3,1733
Guo et al model	0,6621	0,6465	3,1733
Kupershtokh et al model	0,8480	1	3,1733

As it can be seen, the largest values of S are achieved for the case of $\varphi \approx \pi$, that corresponds to the case of the opposite directions of vectors \mathbf{F} and \mathbf{U} . The values of S^* and F^* are the same at the cases of Shan and Chen model and Kupershtokh et al model. The smallest value of S takes place in the case of Guo et al model.

At the second stage the models, which may be realized in the case, when τ may be varied, are investigated. For these cases we compute the values of $S^*(\tau) = \max_{(F, \varphi)} S(F, \varphi, \tau)$ and $F^*(\tau)$ corresponds to this maximal value. We consider the cases of $\tau \in [1/2, 1]$, which is widely used in practical calculations and case of $\tau \in [1/2, 5]$ for investigation of the role of τ on the stability on LBE's with the body force action. At fig. 1 the plots of $S^*(\tau)$ are presented in both cases, corresponding plots of $F^*(\tau)$ are presented in fig. 2. As it can be seen, at the values of τ close to $1/2$, the maximal area of the stability domain is realized for Luo's model, but in this case the values of $F^*(\tau)$ are smaller, when in the case of Kupershtokh et al model. When $\tau \in [1/2, 1]$, maximal values of $F^*(\tau)$ takes place in the case of Guo et al model, but in a bounded interval. For a wide range of the τ values the maximal values of S^* and F^* are realized in the case of Kupershtokh et al model. In the cases of large values of τ all the models have the same behavior. So due to the realized analysis at values of $\tau \approx 1/2$ the models proposed in [5, 9] may be recommended. In the cases of the highest values of τ model proposed in [9] may be recommended to practical simulations due to its stability properties.

The typical examples of the practical problems, where τ is a very close to $1/2$, are the problems of solid-liquid phase transitions, such as melting or freezing [12, 13]. In these problems models

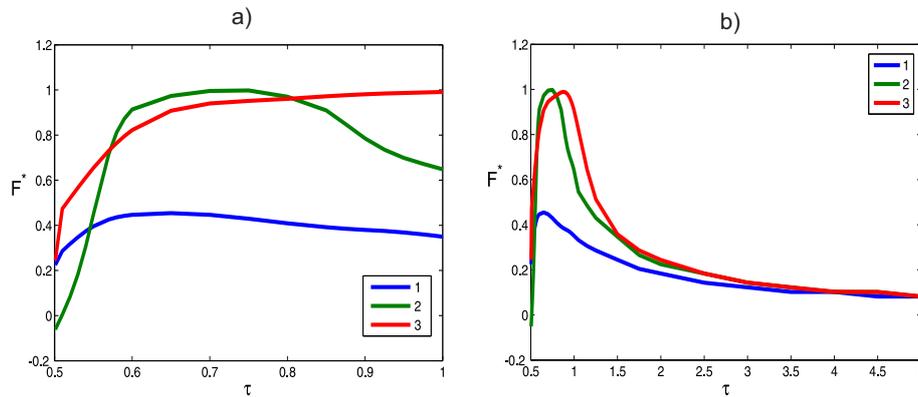


Figure 2. Plots of the maximal values $F^*(\tau)$ for considered models in the cases of $\tau \in [1/2, 1]$ (a) and $\tau \in [1/2, 5]$: 1 — model proposed by Luo; 2 — model proposed by Guo et al; 3 — model proposed by Kupershtokh et al

proposed in [5, 9] may be used. An example of the model with a wide range of τ (from 1/2 to 2) is the problem of interaction of multicomponent fluids [14, 15, 16]. For example, in [14] the case of $\tau = 1$ is considered for the simulation. So, for this kind of problem, the models, proposed in [2, 3] may be used. For the general case of multicomponent systems, the model proposed in [9] may be used for simulation due to the better stability properties in a wide range of τ .

5. Conclusion

In the presented paper stability analysis of LBE's for the case of body force action is realized. The analysis is performed by von Neumann method and is based on the linearisation of LBE's. The stability is estimated by the values of the areas of the stability domains in parametric space. as the results of the analysis, the recommendation on the use of the models in practical simulations may be performed. The proposed approach to the analysis may be extended to the investigation of the other cases of body force actions, especially for the cases of multiphase and multicomponent flows.

References

- [1] Li Q, Luo K H, Kang Q J, He Y L, Chen Q and Liu Q 2016 *Progr. in Energy and Combust. Sc.* **52** 62
- [2] Shan X and Chen H 1993 *Phys. Rev. E* **47** 1815
- [3] Shan X and Chen H 1994 *Phys. Rev. E* **49** 2941
- [4] He X, Chen X and Doolen G D 1998 *Phys. Rev. E* **57** R13
- [5] Luo L S 1998 *Phys. Rev. Lett.* **81** 1618
- [6] Ladd A J C and Verberg R 2001 *J. Stat. Phys.* **104** 1191
- [7] Guo Z, Zheng C and Shi B 2002 *Phys. Rev. E* **65** 046308
- [8] Shan X, Yuan X-F and Chan H 2006 *J. Fluid. Mech.* **550** 413
- [9] Kupershtokh A L, Medvedev D A and Karpov D I 2009 *Comp. Math. Appl.* **58** 965
- [10] Huang H, Krafczyk M and Lu X 2011 *Phys. Rev. E* **84** 046710
- [11] Richtmyer R D and Morton K W 1994 *Difference Methods for Initial-Value Problems* (New York: John Wiley and Sons)
- [12] Li D, Ren Q, Tong Z X and He Y L 2017 *Int. J. Heat and Mass Transf.* **112** 795
- [13] Huo Y and Rao Z 2017 *Int. J. Heat and Mass Transf.* **122** 292
- [14] Joshi A S, Peracchio A A, Grew K N and Chiu W K S 2007 *J. Phys. D* **40** 2961
- [15] Xu H and Dang Z 2017 *Numer. Heat Transf. B* **72** 250
- [16] Succi S 2001 *The Lattice Boltzmann Equation for Fluid Dynamics and Beyond* (Oxford: Clarendon Press)