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String-charge duality in integrable lattice models

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String-charge duality in integrable lattice models

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Abstract. We present an identification of the spectra of local conserved operators of integrable quantum lattice models and the density distributions of their thermodynamic particle content. This is derived explicitly for the Heisenberg XXZ spin chain. As an application we discuss a quantum quench scenario, in both the gapped and critical regimes. We outline an exact technique which allows for an efficient implementation on periodic matrix product states. In addition, for certain simple product states we obtain closed-form expressions for the density distributions in terms of solutions to Hirota difference equations. Remarkably, no reference to a maximal entropy principle is invoked.

Keywords: algebraic structures of integrable models, quantum integrability (Bethe Ansatz), quantum quenches, symmetries of integrable models

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1. Introduction

Over recent years, one of the main challenges in the theory of low-dimensional quantum many-body systems has been to understand the relaxation and equilibration of local observables. In particular, a central question has been to explain the emergence of statistical ensembles in isolated systems. For systems with generic interactions, one can resort to the argument of typicality, which is customarily formulated in the view of the Eigenstate Thermalization Hypothesis [1, 2], stating that eigenfunctions with similar energy densities cannot be distinguished on the level of local correlators, and that the behaviour in the long-time limit should agree with the predictions of the canonical Gibbs ensemble, with temperature fixed by the initial energy density. Exceptions to this paradigm are non-interacting models and integrable interacting quantum many-body problems, for which this picture breaks down due to the existence of a macroscopic number of charges, i.e. conserved local operators.

A recent study on the anisotropic Heisenberg spin chain has shown that there exist sufficient local conservation laws to completely determine the equilibrium steady state after a quantum quench [3]. This followed an observation [4] that although these conservation laws are comprised of a tower of charges which are functionally related, they are nevertheless linearly independent, and represent an adequate vector space for description of equilibrium ensembles.

In this paper we clarify this relationship and present a direct identification between the steady state of an integrable system and its charges, see equation (3.5), expressed as a discrete wave equation. The conceptual consequence of this result is that the conventional perspective [3, 5–9], based on generalized free energy functionals and a maximal entropy principle, is not a necessary framework for the characterization of equilibrium states in quantum integrable lattice models.

More specifically, we examine quantum quenches in the full range of the anisotropic Heisenberg spin chain, restricting to the properties of the steady state. We cast our discussion in the language of fusion hierarchies, which is the universal language of integrability, and renders manifest the locality of the charges. Special emphasis is devoted to the case of the quantum critical (gapless) regime where several interesting exceptional features arise. Moreover, we re-derive exact results for the steady state for quenches for both the Néel and dimer states [10, 11], as closed-form solutions to the quantum Hirota difference equation [12, 13].

The paper is organised as follows. In section 1 we introduce the technical tools at our disposal: Lax and transfer operators, Hirota equation and Y -system, Baxter’s

Q -operator, and string densities. The charges are defined in section 3, and their relationship to the string densities is derived. In section 4 we discuss the quantum quench scenario, describe the evaluation of the charges on initial states, and present exact results for quenches from both the dimer and Néel states. We summarize our presentation in section 5, and discuss open ends. We focus our discussion on the gapped regime, and present the modifications necessary for the critical regime in sections and appendices.

2. Anisotropic Heisenberg spin-1/2 chain

In this paper we focus on the Heisenberg XXZ Hamiltonian, which on a periodic chain of N sites reads

$$H = \sum_{n=1}^N \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta (\sigma_n^z \sigma_{n+1}^z - 1), \quad (2.1)$$

written in the basis of Pauli matrices σ^α for $\alpha \in \{x, y, z\}$. The Hamiltonian density has a $U(1)$ symmetry associated with a local conserved operator $S^z = \sum_n s_n^z$. Anisotropy of the interaction is controlled by parameter $\Delta \in \mathbb{R}$ conventionally parametrized by q -deformation parameter, $\Delta = \frac{1}{2}(q + q^{-1})$. There are two regimes which are to be distinguished, meeting at the isotropic point $\Delta = 1$: (i) *gapped regime* with $q = e^\eta$, $\eta > 0$, and (ii) *gapless regime* corresponding to the interval $|\Delta| \leq 1$, where q -parameter takes values on the unit circle, $q = \exp(i\gamma)$ for $\gamma \in [0, \pi)$. In the following we focus our discussion on the gapped regime. Necessary modifications required to treat the gapless regime are discussed in section 3.2.

2.1. Lax and transfer operators

The Heisenberg XXZ Hamiltonian belongs to an infinite family of commuting local operators. The underlying integrable structure is encoded in the $\mathcal{U}_q(\mathfrak{su}(2))$ -invariant Lax matrix. We employ a general unitary spin- $\frac{j}{2}$ representation over auxiliary spaces \mathcal{V}_j of dimension $j + 1$, and define a family of Lax operators on $\mathcal{V}_1 \otimes \mathcal{V}_j$

$$L_j(\mu) = \frac{1}{\sinh(\eta)} \begin{pmatrix} \sin(\mu + i\eta s^z) & i \sinh(\eta) s^- \\ i \sinh(\eta) s^+ & \sin(\mu - i\eta s^z) \end{pmatrix}, \quad (2.2)$$

which act as 2×2 matrices over the local spin space \mathcal{V}_1 with \mathcal{V}_j -valued components, and μ is a complex-valued parameter called the spectral parameter. Spaces \mathcal{V}_j are spanned by a basis $|n\rangle$, with indices $n = 0, 1, \dots, j$, and where $|0\rangle$ is the highest-weight vector. Spin operators s^α in equation (2.2), with $\alpha = +, -, z$, fulfil the q -deformed⁴ $\mathfrak{su}(2)$ commutation relations $[s^+, s^-] = [2s^z]_q$, $q^{\pm s^z} s^\pm = q^{\pm 1} s^\pm q^{\pm s^z}$, and are prescribed as

$$s^z |n\rangle = \left(\frac{j}{2} - n \right) |n\rangle, \quad (2.3)$$

⁴ We use the following notation for q -numbers, $[x]_q \equiv \sinh(\eta x)/\sinh(\eta)$.

$$s^-|n\rangle = \sqrt{[j-n]_q[n+1]_q} |n+1\rangle, \quad (2.4)$$

$$s^+|n+1\rangle = \sqrt{[j-n]_q[n+1]_q} |n\rangle. \quad (2.5)$$

Lax operators serve as local building units for construction of a commuting set of *higher-spin* quantum transfer operators $T_j(\mu)$, for $j \in \mathbb{Z}_{\geq 0}$, using the ‘traces over monodromies’ construction,

$$T_j(\mu) = \text{Tr}_{\mathcal{V}} L_j^{(1)}(\mu) L_j^{(2)}(\mu) \cdots L_j^{(N)}(\mu), \quad (2.6)$$

where the superscript indices pertain to embeddings of operators equation (2.2) into the spin-chain Hilbert space $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes N}$. Here the trivial (i.e. $j = 0$) representation is the scalar $T_0(\mu) = (\sin(\mu)/\sinh(\eta))^N$. Commutation $[T_j(\mu), T_{j'}(\mu')] = 0$ holds for all $j, j' \in \mathbb{Z}_{\geq 0}$ and $\mu, \mu' \in \mathbb{C}$ by virtue of Yang–Baxter relation [14, 15].

2.2. Hirota equation and Y-system

The Hirota difference equation, also known as the T -system [12, 16–18], is a discrete system of bilinear relations of the form⁵

$$T_j^+ T_j^- = \phi^{[j]} \bar{\phi}^{[-j]} + T_{j-1} T_{j+1}, \quad j \geq 0, \quad (2.7)$$

with the boundary condition $T_{-1} \equiv 0$ and the bar denoting complex conjugation. Here and subsequently we employ a compact notation for representing k -unit imaginary shifts,

$$f^{\pm} = f\left(\mu \pm \frac{i\eta}{2}\right), \quad f^{[\pm k]} = f\left(\mu \pm k \frac{i\eta}{2}\right). \quad (2.8)$$

Physically, equation (2.7) encodes the fusion rules of the symmetry algebra of a quantum integrable lattice model. The sequence of higher-spin transfer operators $T_j(\mu)$ defined in equation (2.6) represents a solution to the Hirota equation with potentials $\phi = T_0^+$ and $\bar{\phi} = T_0^-$. We shall refer to it as the *canonical* solution [19].

The T -system functional hierarchy governing the T -operators exhibits a gauge symmetry

$$T_j \mapsto g^{[j]} \bar{g}^{[-j]} T_j, \quad \phi \mapsto g^+ g^- \phi. \quad (2.9)$$

The T -operators can be combined in a gauge-invariant way as Y -operators, related through a non-linear transformation

$$Y_j = \frac{T_{j-1} T_{j+1}}{\phi^{[j]} \bar{\phi}^{[-j]}} = \frac{T_j^+ T_j^-}{\phi^{[j]} \bar{\phi}^{[-j]}} - 1. \quad (2.10)$$

Y -operators obey the Y -system functional hierarchy

$$Y_j^+ Y_j^- = (1 + Y_{j-1})(1 + Y_{j+1}), \quad (2.11)$$

⁵ We note that the form provided here is not the general one. For quantum symmetries which belong to higher-rank algebras an additional discrete representation label is required, see e.g. [17].

with the boundary condition $Y_0 \equiv 0$. A convenient way of expressing equation (2.10) is

$$\square \log T_j = \log \left(1 + \frac{1}{Y_j} \right), \quad (2.12)$$

where the operator \square is a *discrete d'Alembertian* [13, 20, 21] which acts on a discrete family of functions f_j as

$$\square f_j \equiv f_j^+ + f_j^- - f_{j-1} - f_{j+1}. \quad (2.13)$$

In what follows we will consider such f_j that are analytic inside the *physical region*

$$\mathcal{P}_\eta = \left\{ x \in \mathbb{C} \mid \operatorname{Re}(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \operatorname{Im}(x) < \frac{\eta}{2} \right\}. \quad (2.14)$$

2.3. Baxter's Q -operator and Bethe equations

The fundamental transfer matrix T_1 admits a useful decomposition in terms of the so-called Baxter's Q -operator [18, 22, 23]

$$T_1 Q = T_0^+ \bar{Q}^{[-2]} + T_0^- Q^{[+2]}. \quad (2.15)$$

The commutativity $[T_j(\mu), Q(\mu')] = 0$ for all values $\mu, \mu' \in \mathbb{C}$ and $j \in \mathbb{Z}_{\geq 0}$ enables us to operate on the level of operator spectra^{6,7}.

The Bethe equations can be derived in an algebraic fashion from the TQ -equation (2.15), as the condition for the cancellation of superficial poles in T_1 (see e.g. [23]). Physically the Bethe equations specify the quantization conditions for momenta of excitations, and for a periodic chain of length N they take the form

$$e^{ip(\lambda)N} \prod_{k=1}^M S_{1,1}(\lambda - \lambda_k) = -1, \quad \lambda = \lambda_1, \dots, \lambda_M, \quad (2.16)$$

where the two-particle scattering matrix $S_{1,1}(\lambda)$ is introduced in appendix A. The integer M counts the number of flipped spins with respect to the totally polarized eigenstate, and $p(\lambda_j)$ is the momentum of a single magnon excitation which is related to the rapidity variable as

$$e^{ip(\lambda)} = \frac{\sin\left(\lambda + \frac{i\eta}{2}\right)}{\sin\left(\lambda - \frac{i\eta}{2}\right)}. \quad (2.17)$$

⁶ Although the Q -operator associated with the transfer matrix is hermitian for $\lambda \in \mathbb{R}$, we keep the distinction between Q and \bar{Q} in equation (2.15) because in section 4 we will examine certain solutions of the TQ -equation for which the Q -functions are not real-valued.

⁷ For periodic boundary conditions, the Q -operator can exhibit singular behaviour, which can nevertheless be regularised, e.g. by a boundary twist [23]. We stress however that in the following we do not construct the Q -operator explicitly [24], but instead only deal with well-behaved ratios of its eigenvalues.

In the non-deformed theory (i.e. for $q = 1$), the Baxter Q -function (eigenvalue of the Q -operator on a Bethe state) is a degree N polynomial whose roots are the Bethe roots λ_j . Under the quantum deformation, the Q -function becomes a product of trigonometric factors,

$$Q(\mu) = \prod_{j=1}^M \sin(\mu - \lambda_j). \quad (2.18)$$

The Hirota equation (2.7) permits a solution in terms of the Q -operator. The linearity of equation (2.15) allows one to express the higher-spin T -operators explicitly in terms of combinations of Q -operators [12, 13, 19]

$$\frac{T_j^+}{T_0^{[j+1]}} = Q^{[j+2]} \bar{Q}^{[-j]} \sum_{k=0}^j \frac{\zeta_{j,k}^N}{Q^{[2k-j]} Q^{[2k-j+2]}}, \quad (2.19)$$

where the scalar functions

$$\zeta_{j,k}(\mu) = \frac{\sin^{[2k-j+1]}(\mu)}{\sin^{[j+1]}(\mu)}, \quad (2.20)$$

are defined as $\zeta_{j,k}^N = T_0^{[2k-j+1]} / T_0^{[j+1]}$.

2.4. String densities

In the thermodynamic limit, solutions of the Bethe equations organize into regular patterns in the complex plane referred to as (Bethe) strings. These are classified according to the *string hypothesis* [25–28] and represent the thermodynamic particle content of the model. In the large- N limit many solutions of the Bethe equations become in fact indistinguishable and are characterized by the density distributions ρ_j of the string centers. These densities satisfy the set of integral equations known as string Bethe equations

$$\rho_j + \bar{\rho}_j = a_j - a_{j,k} \star \rho_k, \quad (2.21)$$

adopting Einstein summation convention. Here $\bar{\rho}_j$ are the densities of *holes*, solutions of equation (2.16) which are *not* Bethe roots, and the kernels a_j and $a_{j,k}$, along with further details, can be found in appendix A. As observed already in Bethe's seminal work [29], the strings are inherently local objects, and we will show in the next section that their density distributions are completely fixed by the local symmetries of the model.

3. Thermodynamic spectra of local charges

In this section we introduce the local conserved charges and derive their thermodynamic spectra. This leads to a direct relationship to the string densities. Again we focus our discussion on the $|\Delta| \geq 1$ regime, and refer the reader to section 3.2 for subtleties related to the critical $|\Delta| < 1$ regime.

3.1. String-charge identification

We define a *continuous* family of conserved operators $X_j(\mu)$ [4]

$$X_j(\mu) = \frac{1}{N} \frac{1}{2\pi i} \partial_\mu \log \frac{T_j^+(\mu)}{T_0^{[j+1]}(\mu)}, \quad j = 1, 2, \dots, \quad (3.1)$$

where the scalar $T_0^{[j+1]}(\mu)$ provides a convenient normalization, see equation (2.19). We will refer to these objects as the *charges*, and the prefactor N^{-1} ensures that their eigenvalues remain finite in the thermodynamic limit. They become *local* when μ is restricted to the physical region \mathcal{P}_η , and are hermitian for $\mu \in \mathbb{R}$.

To extract the spectra of X_j we employ the Hirota equation and its solution in terms of the eigenvalues of the Q -operator. By inspecting the large- N behaviour of equation (2.19), we note that for $k = j$ we have $\zeta_{j,j}(\mu) = 1$, whereas $|\zeta_{j,k}(\mu)| < 1$ for $k < j$, implying that only a single term survives the large- N limit

$$X_j(\mu) = \frac{1}{N} \frac{1}{2\pi i} \partial_\mu \log \frac{Q^{[-j]}(\mu)}{Q^{[+j]}(\mu)}, \quad \mu \in \mathcal{P}_\eta. \quad (3.2)$$

Using the explicit form of the Q -function given by equation (2.18) and the string hypothesis, expression equation (3.2) can be expanded over Bethe root densities

$$X_j(\mu) = \int_{-\pi/2}^{\pi/2} d\lambda G_{j,k}(\mu - \lambda) \rho_k(\lambda) \equiv G_{j,k} \star \rho_k, \quad (3.3)$$

where the kernels $G_{j,k}$ can be neatly expressed in terms of the fused scattering matrices

$$G_{j,k}(\lambda) = \sum_{i=1}^k \frac{1}{2\pi i} \partial_\lambda \log S_j \left(\lambda + (k+1-2i) \frac{i\eta}{2} \right) = \sum_{m=1}^{\min(j,k)} a_{|j-k|-1+2m}(\lambda). \quad (3.4)$$

It is instructive to note that in equation (3.4) the index j designates the size of the auxiliary spin, while the index k pertains to the string type. We note that G is the fundamental solution ($\square G = \delta$, see appendix B) of the operator \square , the discrete d'Alembert operator introduced earlier in equation (2.13), and so can be interpreted as a *discrete Green's function* of the problem. Moreover, equation (3.3) shows that the X_j manifestly comply with the *additivity principle*, nicely exposing the particle nature of thermodynamic excitations.

The relation given by equation (3.3) is readily inverted upon application of the operator \square , as described in appendix A, yielding the remarkable identity

$$\rho_j = \square X_j. \quad (3.5)$$

Furthermore, with help of the string Bethe equation (2.21), the hole distributions $\bar{\rho}_j$ are also related to the charges in a local way

$$\bar{\rho}_j = a_j - X_j^+ - X_j^-, \quad (3.6)$$

recovering the previously known result from [3].

Let us stress that though X_j are defined up to the gauge transformation of equation (2.9), the string densities are independent of this. Indeed, in the large- N limit

the string densities can be lifted to the operator level and expressed in a local gauge-invariant way in terms of the Y -operators from equation (2.10) as

$$\rho_j = \frac{1}{N} \frac{1}{4\pi i} \partial_\mu \log \frac{1 + 1/Y_j^+}{1 + 1/Y_j^-}, \quad \bar{\rho}_j = \frac{1}{N} \frac{1}{4\pi i} \partial_\mu \log \frac{1 + Y_j^-}{1 + Y_j^+}. \quad (3.7)$$

Equation (3.5) will play a central role in the later discussion of quantum quenches. It connects the charges, which can be explicitly evaluated on an initial state, to the density distributions of the strings in the steady state.

Remark 1. Expanding the charge $X_1(\mu)$ about the origin recovers the well-known *ultra-local* charges [14, 15],

$$X_1(\mu) = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} X_1^{(k)}. \quad (3.8)$$

In particular, the Heisenberg Hamiltonian equation (2.1) is given by $X_1(0)$, up to an overall rescaling and shift of the spectrum. We want to stress nevertheless that, in principle, any charge taken from the families X_j can be legitimately considered as a Hamiltonian that is consistent with the same two-particle scattering rule. For characterization of states which encode local observables at equilibrium, the entire two-parametric family X_j has to be accounted for on equal footing. While X_1 permits one to define Hamiltonians with ultra-local Hamiltonian densities, generally the Hamiltonians contained in X_j possess interactions which are of long range, but with exponentially decaying amplitudes (see [4]). This weaker form of locality impacts the physics of local observables at the same level as the ultra-local charges. Moreover, from the above consideration it is evident that it is preferable to operate with the continuous representation rather than dealing with a countable discrete basis of charges of say equation (3.8). We wish to emphasize that such a discrete basis has no obvious physical significance.

Remark 2. Let us mention a direct connection between locality as advocated in [3, 4] and the large-volume behaviour of the T -system. Considering the large- N limit of

$$\frac{T_j^-(\mu)}{T_0^{[-j-1]}(\mu)} \frac{T_j^+(\mu)}{T_0^{[j+1]}(\mu)} = 1 + Y_j(\mu), \quad j \in \mathbb{Z}_{\geq 0}, \quad \mu \in \mathcal{P}_\eta, \quad (3.9)$$

by taking into account equation (2.19), we observe that $1 + Y_j(\mu)$ converges toward the *identity* operator as $N \rightarrow \infty$. This provides an *inversion relation* [4] which allows one to evaluate the logarithmic derivative in the definition of the charges equation (3.1) as

$$X_j(\mu) \simeq \frac{1}{N} \frac{1}{2\pi i} \frac{T_j^-(\mu)}{T_0^{[-j-1]}(\mu)} \partial_\mu \frac{T_j^+(\mu)}{T_0^{[j+1]}(\mu)}, \quad (3.10)$$

up to a correction which is subleading in system size N . This result renders locality of X_j manifest. Moreover, locality of [4] can be understood as a corollary of the fusion rules among transfer matrices.

Remark 3. Recent studies of non-ergodic aspects of the Heisenberg spin chain uncovered a macroscopic family of ‘hidden’ local conservation laws, referred to as the

‘quasi-local’ charges [4, 30–33]. Curiously, quasi-locality in a weaker version appeared already in studies of non-ergodic Floquet dynamics of a quantum many-body problem outside of conventional integrability [34, 35]. Definition of quasi-locality does not make any direct use of Bethe Ansatz related concepts but instead resorts to requiring *extensive* $\sim N$ scaling of the Hilbert–Schmidt operator norm (a recent and more general formulation is presented in [36]). In the present work we evaded dealing with quasi-locality. The upshot of our analysis is that quasi-locality is a manifest property of the large-volume scaling of thermodynamic spectra of fused transfer operators. In addition, by providing a link to the thermodynamic particle content of the spectrum which allows for a genuine local interpretation, the omission of the prefix ‘quasi’ is readily justified. It remains an interesting open issue how the quasi-local conservation laws employed in [30–33] which lie outside of the ‘particle sector’ (i.e. objects pertaining to *compact* representations of the symmetry algebra) can be connected to the framework presented in this article.

Remark 4. Finally, let us not forget the remaining local conserved operator S^z , allowing one to distinguish states from symmetry multiplets which are characterized by same set of string densities.

3.2. Gapless regime

For the gapless regime, modifications of the above formulation arise because the derivation of the string content is much more involved compared to its gapped counterpart. This is a consequence of the exceptional spectral degeneracy which occurs when deformation parameter q becomes a root of unity [27, 37, 38]. The string hypothesis is briefly summarized in appendix A. Two novel features to keep in mind are that (i) string configurations acquire an additional quantum number, the so-called string *parity*, and (ii) that the number of distinct string types becomes *finite*. Here we describe how these properties manifest on the level of the charges.

We proceed by retaining the structural form of the conservation laws from the gapped regime, i.e. make use of logarithmic derivatives of traces for the higher-spin monodromy operators. Referring to the fact that strings are local objects, the number of (linearly) independent charges has to be in agreement with the number of distinct string types. In parallel to the stability condition for the strings (see equation (A.13)), our task is now to derive an analogous condition for the gapless regime. With this in mind, we first propose a three-parametric family of conserved operators $X_{(j,u)}(\mu)$ of the form

$$X_{(j,u)}(\mu) = \frac{1}{N} \frac{1}{2\pi i} \partial_\mu \log \frac{T_{(j,u)}^+(\mu)}{T_{(0,u)}^{[j+1]}(\mu)}, \quad j = 1, 2, \dots, \quad u = \pm 1, \quad (3.11)$$

writing shortly $T_{(j,u)}^{[\pm k]} = T_j(\mu \pm k \frac{i\gamma}{2} + (1-u) \frac{i\pi}{4})$ and restricting parameter μ to the physical strip

$$\mathcal{P}_\gamma \equiv \left\{ \mu \in \mathbb{C} \mid \text{Im}(\mu) \in \left(-\frac{i\gamma}{2}, \frac{i\gamma}{2} \right) \right\}. \quad (3.12)$$

It has to be emphasized that even for $u = 1$ conservation laws from equation (3.11) *do not* (in the $N \rightarrow \infty$ limit) automatically inherit locality from their gapped analogues

under substitution $\eta \rightarrow i\gamma$ and $\mu \rightarrow i\mu$. By keeping anisotropy parameter γ fixed, only a subset of operators $X_{(j,u)}$, which will be later on referred to as the ‘*charge content*’⁸, is compatible with an extensive large-volume scaling as we shall readily demonstrate. By restricting the discussion to the roots of unity [37, 38], $\gamma/\pi = \ell_1/\ell_2$ (see appendix A for details) the total number of charges is always finite.

Specializing to the generic root of unity case, the number of string types is given by m_l (see appendix A for definitions and details). Remarkably however, the number of (linearly) independent charges is only $m_l - 1$. As we shall learn from the subsequent discussion, the reason is actually quite subtle and has to do with the ‘truncation effect’ of the Y -system hierarchy. Its physical consequence is that the so-called boundary pair of strings do not carry independent dynamical information.

Once the local conserved charges $X_{(j,u)} \in \mathcal{C}$ have been identified, we can proceed along the lines of the gapped scenario and rewrite their thermodynamic spectra $X_{(j,u)}$ in terms of Bethe roots distributions ρ_k

$$X_{(j,u)}(\mu) = \int_{-\infty}^{\infty} d\lambda \, G_{(j,u),k}(\mu - \lambda) \rho_k(\lambda) \equiv G_{(j,u),k} \star \rho_k, \quad \mu \in \mathcal{P}_\gamma, \quad (3.13)$$

where k th string which carries a pair of labels (n_k, v_k) is now drawn from the string content for a particular value of γ . The matrix Green function is determined by kernels $G_{(j,u),(n,v)}$ which are provided in terms of scattering phases⁹

$$G_{(j,u),(n,v)}(\mu) = \sum_{i=1}^n \frac{1}{2\pi i} \partial_\mu \log S_j \left(\mu + (n+1-2i) \frac{i\gamma}{2} + (1-uv) \frac{i\pi}{4} \right) \quad (3.14)$$

$$= \sum_{m=1}^{\min(j,n)} a_{(|n-j|-1+2m, uv)}(\mu). \quad (3.15)$$

To find out which charge labels give rise to *local* conservation laws we derive a stability condition based on the explicit solution of Hirota equation in terms of Q -functions. Replacing $\lambda \rightarrow i\lambda$ and $\eta \rightarrow i\gamma$ in the scalars $\zeta_{j,k}(\mu)$ from equation (2.20), and incorporating the parity number by an appropriate $i\pi/2$ shift, the analysis reduces to study the moduli of

$$\zeta_{(j,u),k}(\mu) = \frac{\sinh^{[2k-j+1]} \left(\mu + (1-u) \frac{i\pi}{4} \right)}{\sinh^{[j+1]} \left(\mu + (1-u) \frac{i\pi}{4} \right)}, \quad (3.16)$$

and requiring the *stability condition*

$$\left| \zeta_{(j,u),k}(\mu) \right| < 1, \quad (3.17)$$

for all $k < j$, whereas by construction we have $\zeta_{(j,u),j}(\lambda) = 1$. Fulfilling condition (3.17) implies that conserved quantities with spectra given by expression (3.11) attain additivity in the large- N limit. By inspection we find that condition (3.17) is fulfilled (for

⁸ Bethe root configurations in the thermodynamic limit for a given value of γ will be referred to as the ‘*string content*’.

⁹ Note that kernels $G_{(j,u),(n,v)}$ are well-defined objects even when a pair of labels (n, v) is not taken from the string content.

fixed γ) precisely for $m_l - 1$ pairs of labels (j, u) which in effect determine $X_{(j,u)}$, i.e. the charge content of the theory.

3.2.1. Inverting matrix kernel G The problem of inverting relation (3.13) is much more involved in comparison to its gapped counterpart. Here we find ourselves forced to restrict the consideration to the zero magnetization sector, and defer comments on polarized states to the conclusions (see section 5). This is due to a mismatch between the number of strings m_l and the number of charges $m_l - 1$ which arises due to a redundancy linked to the boundary string pair (associated with indices $m_l - 1$ and m_l in the standard ordering of [37]). In particular, since these two special string types scatter identically with respect to all other string types, and ‘inversely’ among themselves [39, 40], i.e. $S_{m_l-1}S_{m_l} = 1$ and hence $G_{m_l-1,k} = -G_{m_l,k}$, the boundary strings are ascribed only a single charge. To proceed, we restrict attention to states for which the particle and hole distributions of the boundary string pair are related as

$$\rho_{m_l} = \bar{\rho}_{m_l-1}, \quad \bar{\rho}_{m_l} = \rho_{m_l-1}. \quad (3.18)$$

In appendix D we show that this identification implies that such states have zero magnetization¹⁰.

The redundancy arising from equation (3.18) can be incorporated by reducing the basis of m_l densities ρ_j to $m_l - 1$ densities by modifying the last distribution at node $m_l - 1$ according to the prescription

$$\rho_{m_l-1} \mapsto \tilde{\rho}_{m_l-1} = \rho_{m_l-1} - \rho_{m_l}. \quad (3.19)$$

This allows us to define the matrix kernel G as a $(m_l - 1)$ -dimensional matrix of non-degenerate rank. By inverting equation (3.13) we again arrive at an (almost) d’Alembert difference relation of the form

$$\rho_j = \square X_j. \quad (3.20)$$

We remark that operator \square now acquires an explicit dependence on the value of parameter γ (which is for clarity suppressed), which apart from determining its dimensions also introduces certain ‘non-local’ modifications. Before we state the explicit structure of \square some extra clarifications are first in order. In the gapless regime the auxiliary spin quantum numbers which label the charges no longer directly correspond to the ordering index j in X_j entering through equation (3.20), namely they should not be confused with sizes of auxiliary spin labels. It thus becomes crucial to decide about the ordering of charges explicitly. Here we assume the ordering of charges by *increasing* spin label j . This conveniently renders \square of an almost-tridiagonal form in the linear space of densities ρ_j .

3.2.2. Principal roots of unity The situation with simple roots of unity, given by $\ell_1 = 1$ and $\ell_2 \equiv \ell$, becomes equivalent to relation (2.13) in the gapped case,

$$\square f_j = (s^{-1} - I\delta)_{j,k} \star f_k, \quad (3.21)$$

¹⁰ In principle there may exist states with zero magnetization for which equation (3.18) is not true.

the only difference being that operators \square and I are of finite dimension $\ell - 1$. Equation (3.21) is a direct consequence of discrete d'Alembert equation (3.9) and is determined by a single s -kernel reading $\mathcal{F}[s^{-1}](\omega) = 2 \cosh(\frac{\pi}{2} p_0^{-1} \omega)$.

3.2.3. Generic roots of unity For *non-principal* roots of unity, $\gamma/\pi = (\nu_1, \nu_2, \dots, \nu_l)$ for $l \geq 2$ (see appendix A and equation (A.11) for details), the γ -modified discrete d'Alembertian \square component-wise reads

$$\rho_j = (-1)^i (s_{i+1}^{-1} \star X_j - X_{j-1} - X_{j+1}), \quad m_i < j \leq m_{i+1} - 1, \quad j \neq l - 1, \quad (3.22)$$

$$\rho_{m_i} = (-1)^i (s_{i+1}^{-1} \star X_{m_i} - X_{m_{i-1}-1} - X_{m_i+1}), \quad i = 1, \dots, l - 1, \quad (3.23)$$

$$\tilde{\rho}_{m_l-1} = (-1)^{l-1} (s_l^{-1} \star X_{m_l-1} - X_{m_l-2} - X_{m_{l-1}-1}), \quad (3.24)$$

for $i = 0, 1, \dots, l - 1$ and adopting the boundary condition $X_0 \equiv 0$.

We wish to draw the reader's attention to three features which qualitatively differ from those of the gapped regime:

- A modification of the 'backward coupling' occurring at irregular indices pertaining to l -many exceptional nodes at positions m_i , for $i = 1, 2, \dots, l - 1$, and finally $m_l - 1$. These nodes can be interpreted as 'band edges'.
- Nodes in the range $m_i \leq j \leq m_{i+1} - 1$ are assigned convolution kernels s_{i+1} which can be viewed as various intrinsic length-scales present in the spectrum of the model.
- Alternating overall signs $(-1)^i$ for distinct bands $m_i < j \leq m_{i+1} - 1$.

A straightforward approach to prove equation (3.24) is to resort to Fourier representation of kernels a_j which transforms the main relation $\rho = \square X$ into a set of *algebraic* equations, and subsequently employ identities among scattering phase shifts which can be found in appendix A.

Remark. This procedure is very reminiscent to the procedure used in the pioneering work on the string hypothesis [37], however it seems to us that the set of known kernel identities presented in [27, 37] do not suffice to complete the proof due to presence of functions $a_{(n,v)}$ in a typical expression for kernels $G_{j,k}$ which actually do not directly involve the scattering data for 'physical particles'. Nonetheless, one can easily express the action of $s_i^{-1} \star$ on an arbitrary function $a_{(n,v)}$ in Fourier space, i.e. evaluate $\mathcal{F}[s_i^{-1}] \cdot \mathcal{F}[a_{(n,v)}]$, and observe cancellation of terms by due to trigonometric addition formula $2 \cosh(\theta) \sinh(\psi) = \sinh(\psi + \theta) + \sinh(\psi - \theta)$.

Some concrete examples for the non-principal roots of unity can be found in appendix C.

4. Evaluation of charge densities

Having established the identification (3.5) we are in position to immediately obtain the string densities ρ_j by evaluating the whole set of charges X_j on a state $|\Psi\rangle$. In the scope of a quantum quench, $|\Psi\rangle$ would be the initial state. The charges can be conveniently contracted with respect to $|\Psi\rangle$ by resorting to the form of equation (3.10)

$$X_j^\Psi(\mu) = \lim_{N \rightarrow \infty} \frac{1}{N} \frac{1}{2\pi i} \langle \Psi | \frac{T_j^-(\mu)}{T_0^{[-j-1]}(\mu)} \partial_\mu \frac{T_j^+(\mu)}{T_0^{[j+1]}(\mu)} | \Psi \rangle. \quad (4.1)$$

Previously in the literature these objects have been customarily referred to as the ‘generating functions’ for the expectation values of the charges [3, 41–43] with respect to state $|\Psi\rangle$ ¹¹. Thanks to the Lax structure, an efficient computation is readily available with respect a large class of states, e.g. periodic matrix-product states [42]. Hence for purely practical reasons we concentrate on states of the form

$$|\Psi\rangle = |\psi\rangle^{\otimes (N/N_p)}, \quad (4.2)$$

where $N_p \in \mathbb{N}$ pertains to the periodicity of the state.

By taking advantage of this product structure we employ the computational scheme which has been developed previously in [3, 41, 42]. The main trick is to replace the logarithmic derivative of equation (3.1) with the product form given by equation (4.1), where crucially the μ -derivative is taken at the end of calculation, i.e. after contracting physical indices with respect to $|\Psi\rangle$. Performing this however requires a small displacement of the spectral parameter, here denoted by x . A local part of expression (4.1) is given by composite two-channel Lax operators acting over $\mathbb{C}^2 \otimes \mathcal{V}_j \otimes \mathcal{V}_j$

$$\mathbb{L}_j(\mu, x) = \frac{L_j^-(\mu)}{L_0^{[-j-1]}(\mu)} \frac{L_j^+(\mu + x)}{L_0^{[j+1]}(\mu + x)}. \quad (4.3)$$

Computation of quantities X_j^Ψ can be most elegantly achieved by means of the standard transfer matrix technique via *boundary partition functions*

$$Z_j^\Psi(\mu, x) = \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}_{\mathcal{V}_j \otimes \mathcal{V}_j} \mathbb{T}_j^\Psi(\mu, x)^{N/N_p}, \quad (4.4)$$

with

$$\mathbb{T}_j^\Psi(\mu, x) = \langle \psi | \mathbb{L}_j^{(1)}(\mu, x) \cdots \mathbb{L}_j^{(N_p)}(\mu, x) | \psi \rangle, \quad (4.5)$$

from where the charge distributions can be ultimately determined on taking the x -derivative at $x = 0$,

$$X_j^\Psi(\mu) = -i \partial_x Z_j^\Psi(\mu, x)|_{x=0}. \quad (4.6)$$

¹¹ Here we deal with a continuum of charges and hence the interpretation as the generating function does not make much sense.

The large- N behaviour of $Z_j^\Psi(\mu, x)$ is contained in the largest-modulus eigenvalue $\Lambda_j^\Psi(\mu, x)$, which in the unperturbed limit obeys $\Lambda_j^\Psi(\mu, 0) = 1$, by virtue of inversion relation (3.9).

4.1. Charge densities in the gapless regime

The procedure outlined above is equally valid in the gapless regime, the only difference being that the charge content becomes finite. All results can be computed along the lines of the gapped regime by making use of substitutions $\lambda \rightarrow i\lambda$ and $\eta \rightarrow i\gamma$, and taking into account for the presence of negative parity.

4.2. Explicit evaluation

For a typical initial matrix-product state there exists an efficient numerical procedure to compute the charges, introduced in [41] and extended in [42]. This is achieved by either directly referring to equation (4.6) or by using Jacobi's formula to rewrite equation (4.6) as

$$X_j^\Psi(\mu) = \frac{-i}{N_p} \frac{\text{Tr}(\text{Adj}(\mathbb{T}_j^\Psi(\mu, 0) - 1)\mathbb{D}_j^\Psi(\mu))}{\text{Tr}(\text{Adj}(\mathbb{T}_j^\Psi(\mu, 0) - 1))}, \quad (4.7)$$

where $\mathbb{D}_j^\Psi(\mu) = \partial_x \mathbb{T}_j^\Psi(\mu, x)|_{x=0}$ and the matrix coadjoint is given through $A \cdot \text{Adj}(A) = \det(A)$.

4.3. Analytic approach

In practice, explicitly evaluating equation (4.7) becomes a computational challenge already for relatively small values of j . In this section we show specific cases where this difficulty can be overcome. This is achieved by observing that for certain simple equilibrium states the set of functions $\eta_j = \bar{\rho}_j/\rho_j$, which can be obtained through the string-charge duality relations equations (3.5) and (3.6), satisfy the Y -system functional hierarchy equation (2.11), which allows for a simple recursive solution to the problem. Moreover, by switching to the corresponding T -system we obtain a closed-form solution for the η_j through the TQ -equation. The density distributions ρ_j then follow from the linear string Bethe equations (2.21). Let us stress however that it is an open question whether this procedure can be generalised to treat arbitrary initial states.

Specifically we consider two equilibrium states which have particularly simple 'representative'¹² states,

- *Néel state* $|N\rangle = |\uparrow\downarrow\rangle^{\otimes N/2}$, an eigenstate of the Ising limit $\Delta \rightarrow \infty$,
- *dimer state* $|D\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)^{\otimes N/2}$, a ground state of Majumdar–Ghosh Hamiltonian [44],

¹² Let us stress that these are not eigenstates, but rather (from a quantum quench perspective) are in the basin of attraction of the equilibrium state.

on which the charges are straightforwardly evaluated. We remark that the Néel and dimer states have recently been found to permit explicit evaluation of overlap coefficients with Bethe eigenstates, allowing for an exact implementation of the Quench Action method [11, 43, 45–48]. This time however we resort to a different technique.

We owe to stress that the solutions of Hirota equation (2.7) we shall now discuss are *distinct* from the canonical one of section 2.2. We relax the condition that $T_0 = \phi^- = \bar{\phi}^+$, and consider a general auxiliary linear problem for equation (2.7),

$$T_{j+1} Q^{[j]} - T_j^- Q^{[j+2]} = \phi^{[j]} \bar{Q}^{[-j-2]}, \quad (4.8)$$

$$T_{j-1} \bar{Q}^{[-j-2]} - T_j^- \bar{Q}^{[-j]} = -\bar{\phi}^{[-j]} Q^{[j]}, \quad (4.9)$$

enabling to express its solution in the explicit form

$$T_k = T_0^{[-k]} \frac{Q^{[k+1]}}{Q^{[-k+1]}} + Q^{[k+1]} \bar{Q}^{[-k-1]} \sum_{j=1}^k \frac{\phi^{[2j-k-1]}}{Q^{[2j-k-1]} Q^{[2j-k+1]}}. \quad (4.10)$$

We wish to emphasize that the function Q here should not be confused with Baxter's Q -operator given earlier by equation (2.18) whose spectrum is a deformed polynomial which stores the positions of the Bethe roots. On the contrary, here the analytic properties of Q encode the local physics of an equilibrium state.

As the TQ -equation is a second order difference equation, there generally exist two independent solutions to it. For our purpose it is nonetheless sufficient to find only one solution. The second solution can be in principle derived (modulo the addition of the first solution) by explicitly solving the quantum Wronskian condition [12].

We now present the explicit closed form solutions for the two states under consideration.

4.3.1. Dimer state

Isotropic point. At the isotropic point the auxiliary transfer operator

$$\mathbb{T}_j^D(\lambda, x) = \langle D | \mathbb{L}_j^{(1)}(\lambda, x) \mathbb{L}_j^{(2)}(\lambda, x) | D \rangle, \quad (4.11)$$

yields neat compact expressions for the first few charges,

$$X_1^D(\lambda) = \frac{1}{2\pi} \frac{2\lambda^2 + 5}{4(\lambda^2 + 1)^2}, \quad X_2^D(\lambda) = \frac{1}{2\pi} \frac{4(4\lambda^2 + 17)}{(4\lambda^2 + 9)^2}, \quad X_3^D(\lambda) = \frac{1}{2\pi} \frac{3(2\lambda^2 + 13)}{4(\lambda^2 + 4)^2}, \quad (4.12)$$

while expressions pertaining to the charges of higher order shall be omitted here. The corresponding expressions for η -functions are

$$\eta_1^D(\lambda) = \frac{3\lambda^2}{1 + \lambda^2}, \quad \eta_2^D(\lambda) = \frac{32\lambda^2}{9 + 4\lambda^2}, \quad \text{etc}, \quad (4.13)$$

and can be encoded in T -functions (modulo gauge freedom) simply as

$$T_j^D(\lambda) = (j + 1)\lambda. \quad (4.14)$$

The Q -function for this particular case reads $Q^D(\lambda) = \lambda^2$. Moreover, here we were able to find the other independent solution to equation (2.15), $Q^D(\lambda) = -i/2$, which allows for a determinant representation of T_k^D [12, 49, 50].

Gapped case. Repeating the procedure in the gapped regime for an arbitrary value of η one can extract the expressions for the values of $X_1^D(\lambda)$. Their analytic form is quite cumbersome and so we omit them here. The corresponding η -functions are somewhat simpler expressions and read e.g.

$$1 + \eta_1^D(\lambda) = \frac{\cos(4\lambda) - \cosh(2\eta)}{\cos(\lambda)^2(\cos(2\lambda) - \cosh(2\eta))}, \quad (4.15)$$

$$\eta_2^D(\lambda) = -\frac{4 \sin(2\lambda)^2(2 \cos(2\lambda) + \cosh(\eta) + \cosh(3\eta))}{(\cos(2\lambda) + \cosh(\eta))^2(\cos(2\lambda) - \cosh(3\eta))}. \quad (4.16)$$

Choosing $T_1^D(\lambda) = \sin(2\lambda)$ and setting the potential to $\phi^D(\lambda) = \sin(\lambda + \frac{i\eta}{2}) \cos(\lambda - \frac{i\eta}{2})$, we obtain a set of η -functions obeying the Y -system functional hierarchy compatible with the following set of T -functions,

$$T_0^D(\lambda) = \frac{1}{2} \sin(2\lambda), \quad (4.17)$$

$$T_1^D(\lambda) = \sin(2\lambda), \quad (4.18)$$

$$T_2^D(\lambda) = \frac{1}{2} \tan(\lambda)(3 \cos(2\lambda) + \cosh(2\eta) + 2), \quad (4.19)$$

$$T_3^D(\lambda) = \frac{\sin(2\lambda)(2 \cos(2\lambda) + \cosh(\eta) + \cosh(3\eta))}{\cos(2\lambda) + \cosh(\eta)}, \quad \text{etc.} \quad (4.20)$$

The Q -function is given by

$$Q^D(\lambda) = \cos\left(\lambda - \frac{i\eta}{2}\right). \quad (4.21)$$

4.3.2. Néel state

Isotropic point. Here

$$X_1^N(\lambda) = \frac{1}{2\pi} \frac{1}{2\lambda^2 + 1}, \quad X_2^N(\lambda) = \frac{1}{2\pi} \frac{12}{12\lambda^2 + 19}, \quad X_3^N(\lambda) = \frac{1}{2\pi} \frac{3\lambda^2 + 1}{2\lambda^4 + 7\lambda^2 + 2}, \quad (4.22)$$

and so forth, whence we readily calculate the first two η -functions

$$\eta_1^N(\lambda) = \frac{\lambda^2(19 + 12\lambda^2)}{(1 + \lambda^2)(1 + 4\lambda^2)}, \quad \eta_2^N(\lambda) = \frac{8(2\lambda^2 + 1)(2\lambda^4 + 7\lambda^2 + 2)}{\lambda^2(\lambda^2 + 1)(4\lambda^2 + 9)}. \quad (4.23)$$

The solution of the Hirota equation takes the form

$$T_0^N(\lambda) = \lambda, \quad T_1^N(\lambda) = 2\lambda + \frac{1}{\lambda}, \quad Q^N(\lambda) = 2\lambda. \quad (4.24)$$

Gapped case. The first three initial charges take the form

$$X_1^N(\lambda) = \frac{1}{2\pi} \frac{\sinh(2\eta)}{1 - 2\cos(2\lambda) + \cosh(2\eta)}, \quad (4.25)$$

$$X_2^N(\lambda) = \frac{1}{2\pi} \frac{2\sinh(3\eta)}{2\cosh(3\eta) + \cosh(\eta) - 3\cos(2\lambda)}, \quad (4.26)$$

$$X_3^N(\lambda) = \frac{(2\pi)^{-1} \sinh(4\eta)(3\cos(2\lambda) - \cosh(2\eta) - 2)}{\cos(2\lambda)(3\cosh(4\eta) + 2\cosh(2\eta) + 3) - 2\cosh(2\eta)^2(\cosh(2\eta) + 2) - 2\cos(4\lambda)}. \quad (4.27)$$

This gives

$$\eta_1^N(\lambda) = \frac{2\sin(2\lambda)^2(2\cosh(3\eta) + \cosh(\eta) - 3\cos(2\lambda))}{(\cos(2\lambda) - \cosh(\eta))(\cos(4\lambda) - \cosh(4\eta))}, \quad (4.28)$$

while expressions for higher η -functions are suppressed here. The solution of the Hirota equation is now cast in the form

$$T_0^N(\lambda) = \frac{1}{2} \sin(2\lambda), \quad (4.29)$$

$$T_1^N(\lambda) = \frac{1}{2} \cot(\lambda)(1 - 2\cos(2\lambda) + \cosh(2\eta)), \quad (4.30)$$

$$Q^N(\lambda) = 2\sin(\lambda). \quad (4.31)$$

Remark. The exact analytic form of the steady-state solution for the Néel quench problem has been presented before in [10, 11] using the Quench Action approach [45]. The authors of [10] already observed that the solution can be cast in the Y -system form, and in addition obtained an auxiliary function $\mathfrak{a}(\lambda)$ of the Quantum Transfer Matrix method [51–53]. In this section we have shown that the latter is still a composite object, reducible in terms of auxiliary functions Q and T_0 as

$$\mathfrak{a} = \frac{T_0^- Q^{[+2]}}{T_0^+ \bar{Q}^{[-2]}}. \quad (4.32)$$

The observation that functions Q^N and T_0^N reproduce \mathfrak{a}^N for the Néel state found in [10] (see also [54]) represents a non-trivial compatibility check of the two approaches.

5. Conclusions

In this paper we have developed an explicit and transparent framework to describe the relationship between local symmetries and equilibrium states, recently discovered in [3]. The charges in fact contain all information relevant to the steady-state expectation values of local observables, obviating the need to invoke statistical ensembles and a maximal entropy principle. Any set of initial states, which are indistinguishable with respect to the charges, relax, via dephasing, to the same equilibrium state in the late-time limit. Presence of entropy merely reflects the fact that there are many microstates corresponding to the same macrostate, distinguishable only non-locally.

We have identified the charges (which are expressed explicitly over the local spin basis) with the string densities in an appealing form as a discrete wave equation (3.5). We exemplified this connection explicitly on a prototypical integrable quantum lattice model—the anisotropic Heisenberg spin-1/2 chain—including both the gapped and the critical phases. While in the gapped regime the discrete d’Alembertian takes a purely local form, in the critical regime it reduces to a finite-dimensional object and undergoes non-local modifications.

We presented the formalism in the language of fusion hierarchies. This provides a unified perspective, and readily permits extensions to integrable lattice models based on higher-rank symmetry (super)algebras [20], where each independent node on the Y -system lattice is assigned an individual charge.

A key application of the identification equation (3.5) is to address the situation of quantum quenches [55–65]. We have presented a programme to access the steady state, which can be summarized with the following three-stage sequence

$$|\Psi\rangle \xrightarrow{(A)} X_j^\Psi(\lambda) \xrightarrow{(B)} \rho_j^\Psi(\lambda) \xrightarrow{(C)} \langle \mathcal{O}_{\text{loc}} \rangle. \quad (5.1)$$

While steps (A) and (B) were a part of considerations in the present work, step (C) requires invoking some extra tools and thus remains to be addressed in the future. Presently, the mapping (C) can be readily implemented for the gapped and isotropic regimes (for applications see [3, 10, 11, 43, 47]) by employing formulae provided in [54], building on previous works [53, 66, 67].

Further in this direction, the question of relaxation towards equilibrium is of central importance. A powerful tool to investigate this is the Quench Action method, which allows one to track the finite-time evolution of local correlators [45]. A recent study on the integrable Bose gas revealed a power-law decay in the late-time dynamics [68]. It would be interesting to address this issue in the framework of this article.

A distinguished property of the gapless regime is the appearance of an exceptional spectral degeneracy, which renders the particle content finite. A subtle artefact of this reduction is the presence of a pair of boundary strings, which cannot be distinguished on the level of charges. Presently, we have only succeeded in resolving this boundary effect by restricting to a subset of states with zero magnetization, while the general case of a polarized state remains a topic of future study. Along the lines of the discussion in [39, 40], for a polarized state the corresponding η -functions of the boundary pair are expected to be related as $\eta_{m_l-1}\eta_{m_l} = e^{-\chi}$, for some χ . Such a constraint is however

non-linear, and it remains an open question the matrix kernel G can be amended in such a way to account for it.

We succeeded in casting two equilibrium states as closed-form solutions to quantum Hirota equation. These states are however atypical, and general states lack such a compact description. It would be interesting to realize a modified hierarchy of functional relations which would encompass all equilibrium states.

Finally, we wish to stress that S^z and the family of charges X_j do not in fact exhaust *all* local symmetries of the model at hand. In the gapless regime there exist additional local conserved operators associated with non-compact highest-weight representations of $\mathcal{U}_q(\mathfrak{sl}(2))$ [30–33, 69]. These charges are related to current-carrying non-equilibrium ensembles and are responsible for anomalous transport properties, i.e. the diverging DC conductivity [30, 70–73]. Including them in a general classification of steady states remains a prominent and challenging task and will be pursued in the ongoing research.

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Appendix A. String hypothesis and Bethe equations for strings

A.1. Two-particle scattering matrices

Here we systematically introduce the key concepts and tools needed for implementation of our programme. We begin by considering a basic object of a quantum integrable theory, the two-particle scattering matrix. Integrability of the model reflects the fact that any particle scattering events can be factorized as a sequence of two-particle events.

The form of the scattering matrix is in fact related to the quantum R -matrix via Bethe Ansatz equations. In the Heisenberg model, the 2-particle scattering matrix admits the form

$$S(\lambda, \mu) = \frac{\sin(\lambda - \mu - \frac{i\eta}{2})}{\sin(\lambda - \mu + \frac{i\eta}{2})}. \quad (\text{A.1})$$

Notice that the class of so-called fundamental integrable models possess S -matrices which depend only on the *difference* of the particle's spectral parameters, i.e. $S(\lambda, \mu) = S(\lambda - \mu)$. The analogue of equation (A.1) at the isotropic point $\Delta = 1$ follows after taking the scaling limit $\lambda \rightarrow \lambda\eta$, and subsequently sending $\eta \rightarrow 0$. This will bring the scattering matrix into the rational form.

Gapless regime. The elementary 2-particle scattering matrix is readily obtained by taking the gapped counterpart equation (A.1) and applying the substitution $\lambda \rightarrow i\lambda$ and $\eta \rightarrow i\gamma$, in effect replacing factors $\sin(\lambda)$ with $\sinh(\lambda)$.

A.2. String hypothesis

A.2.1. String hypothesis in the gapped regime. The string hypothesis asserts that the Bethe roots align into vertical patterns in the complex rapidity plane [25–27, 37]. These formations are referred to as *strings*. Physically, they describe *bound states* in the spectrum of the model and represent thermodynamic particle excitations of a generic integrable lattice model.

Analysing the string content in the gapped regime of the Heisenberg chain shows that an infinite tower of string types emerge. An n -string is composed of n rapidities located on the real line at position λ_α^n ,

$$\{\lambda_\alpha^{n,i}\} \equiv \left\{ \lambda_\alpha^n + (n+1-2i)\frac{i\eta}{2} \mid i = 1, 2, \dots, n \right\}. \quad (\text{A.2})$$

Here index α enumerates different strings of length n , while the internal index i runs over all Bethe roots inside an individual string.

A scattering event with the string of length j is given by a *fused* scattering matrix

$$S_j(\lambda - \mu) = \frac{\sin(\lambda - \mu - j\frac{i\eta}{2})}{\sin(\lambda - \mu + j\frac{i\eta}{2})}, \quad (\text{A.3})$$

where we identify $S(\lambda) \equiv S_1(\lambda)$. For convenience we additionally define the trivial scattering matrix $S_0 = 1$.

A product of subsequent scatterings in Bethe equations involving M roots can be split as

$$\prod_{j=1}^M \longrightarrow \prod_{k=1}^{\infty} \prod_{\beta=1}^{M_k} \prod_{\alpha \in \{\lambda_\alpha^{n,i}\}} , \quad M = \sum_{k=1}^{\infty} k M_k. \quad (\text{A.4})$$

Then the *string Bethe equations* among different string types in a finite system of length N in the logarithmic form read

$$N \log S_j(\lambda_\alpha^j) = 2\pi I_\alpha^j + \log \prod_{k=1}^{\infty} \prod_{\beta=1}^{M_k} S_{j,k}(\lambda_\alpha^j - \lambda_\beta^k), \quad (\text{A.5})$$

where integer quantum numbers I_α^j are determined by fixing a branch of the logarithm. A central property of scattering matrices $S_j(\lambda)$ is a functional identity

$$\frac{S_j(\lambda + \frac{i\eta}{2}) S_j(\lambda - \frac{i\eta}{2})}{S_{j-1}(\lambda) S_{j+1}(\lambda)} = 1, \quad (\text{A.6})$$

which in the logarithmic form becomes a *discrete d'Alembert equation*.

In order to be able to describe any scattering event in the theory, we define a set of kernels $a_j(\lambda)$,

$$a_j(\lambda) = \frac{1}{2\pi i} \partial_\lambda \log S_j(\lambda), \quad j = 1, 2, \dots \quad (\text{A.7})$$

representing derivatives of the scattering phases which belong to individual strings. Subsequently we prefer to leave the dependence on the spectral parameter λ implicit. Furthermore, we introduce the kernel $s(\lambda)$ as a solution to the equation [27]

$$a_j - s \star (a_{j-1} + a_{j+1}) = 0, \quad j > 1, \quad (\text{A.8})$$

whereas at $j = 1$ we have $a_1 = s + s \star a_2$.

Unit parameter shifts $\pm \frac{i\eta}{2}$ play an instrumental role. The inverse of convolving with respect to s can be understood as the deconvolution operator $s^{-1} \star$. In particular, its action represents a symmetrized combination of spectral parameter shifts

$$(s^{-1} \star f)(\lambda) = \lim_{\epsilon \rightarrow 0} \left(f\left(\lambda + \frac{i\eta}{2} - i\epsilon\right) + f\left(\lambda - \frac{i\eta}{2} + i\epsilon\right) \right), \quad (\text{A.9})$$

for functions free of singularities inside the physical region \mathcal{P}_η . For instance, applying $s^{-1} \star$ to equation (A.8) and using d'Alembertian yields $\square a_j = 0$ for $j > 1$, while $\square a_1 = \delta$.

A.2.2. String hypothesis in the gapless regime. The above reasoning can be repeated in the critical regime $\Delta = \cos(\gamma)$. The main (however quite a profound) distinction with respect to the gapped phase is that certain string types become prohibited [37]. Even worse, which strings are allowed now depends on the anisotropy parameter γ is a quite dramatic way.

To analyse the so-called string content we follow the standard route presented in [27, 37] (see also [38]), while restricting our considerations to *roots of unity*,

$$\frac{\gamma}{\pi} = \frac{\ell_1}{\ell_2}, \quad (\text{A.10})$$

where $\ell_1 < \ell_2$ are two co-prime integer numbers and $\ell_2 > 2$. Without making any further restrictions on ℓ_1 and ℓ_2 , a set of points parametrized by equation (A.10) essentially densely covers the entire critical interval.

Following [27, 37] we adopt the terminated continued fraction representation of the anisotropy parameter,

$$\frac{\gamma}{\pi} \equiv p_0^{-1} = \frac{1}{\nu_1 + \frac{1}{\nu_2 + \frac{1}{\nu_3 + \dots}}}, \quad (\text{A.11})$$

which may be compactly written as $\gamma/\pi = (\nu_1, \nu_2, \dots, \nu_l)$. Parameter $l \in \mathbb{N}$ can be regarded as the degree of the root of unity, representing the numbers of distinct ‘bands’ in which the string particles can be arrange to. We shall moreover borrow a sequence of m -numbers defined as

$$m_0 = 0, \quad m_i = \sum_{k=1}^i \nu_k, \quad i = 1, \dots, l, \quad (\text{A.12})$$

which helps in determining the string content for given γ . A short summary of the main ingredients of the string hypothesis in the gapless regime comprises:

- Apart from the string length, an extra quantum number $v \in \{\pm 1\}$, called the string parity, arises. Centers of negative-parity strings are displaced by $i p_0$ away from the real axis.
- The allowed strings are determined by the *stability condition*

$$v \sin(\gamma(n-j)) \sin(\gamma j) > 0, \quad j = 1, 2, \dots, n-1. \quad (\text{A.13})$$
- The total number of distinct string types is m_l . Hence, the number of ‘degrees of freedom’ at roots of unity values of γ in the thermodynamic limit is always *finite*.

The allowed string lengths and parities are explicitly computable with aid of a sequence of auxiliary numbers y_i [27, 37],

$$y_{-1} = 0, \quad y_0 = 1, \quad y_1 = \nu_1, \quad y_i = y_{i-2} + \nu_i y_{i-1}, \quad (\text{A.14})$$

which allows to express

$$n_j = y_{i-1} + (j - m_i) y_i, \quad v_j = (-1)^{\lfloor (n_j-1)/p_0 \rfloor}. \quad (\text{A.15})$$

Another distinction of the gapless phase is that the set of s -kernels gets larger. Strings in the range $m_{i-1} \leq j \leq m_i - 1$ are associated with the kernel $s_i(\lambda)$. For this purpose another sequence of auxiliary numbers p_i is used,

$$p_0 = \frac{\pi}{\gamma}, \quad p_1 = 1, \quad p_i = p_{i-2} - p_{i-1} \nu_{i-1}. \quad (\text{A.16})$$

Scattering matrices are of the form

$$S_j(\lambda, \mu) \equiv S_{(n_j, v_j)}(\lambda, \mu) = \frac{\sinh(\lambda - \mu - n_j \frac{i\gamma}{2} + (1 - v_j) \frac{i\pi}{4})}{\sinh(\lambda - \mu + n_j \frac{i\gamma}{2} + (1 - v_j) \frac{i\pi}{4})}, \quad (\text{A.17})$$

while the scattering between j th and k th type of strings is described by

$$S_{(n_j, v_j), (n_k, v_k)} = S_{(|n_j - n_k|, v_j v_k)} S_{(n_j + n_k, v_j v_k)} \prod_{m=1}^{\min(n_j, n_k) - 1} S_{(|n_j - n_k| + 2m, v_j v_k)}^2. \quad (\text{A.18})$$

The corresponding kernels are given by

$$a_j(\lambda) = -\frac{1}{2\pi i} \partial_\lambda \log S_j(\lambda) = \frac{1}{2\pi} \frac{2 \sin(\gamma q_j)}{\cosh(2\lambda) + \cos(\gamma q_j)}, \quad j = 1, \dots, m_l, \quad (\text{A.19})$$

where q -numbers q_i (with $i = 0, 1, \dots, l-1$) can be provided recursively

$$q_j = (-1)^i (p_i - (j - m_i) p_{i+1}), \quad m_i \leq j \leq m_{i+1} - 1, \quad i = 0, 1, \dots, l-1, \quad (\text{A.20})$$

$$q_0 = p_0, \quad q_{m_l} = (-1)^l p_l. \quad (\text{A.21})$$

Fourier representations of a_j are of particularly simple form,

$$\mathcal{F}[a_j](\omega) = \frac{\sinh\left(q_j \frac{\gamma}{2} \omega\right)}{\sinh\left(\frac{\pi}{2} \omega\right)}, \quad (\text{A.22})$$

where in our convention Fourier transform and its inverse read

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} d\lambda e^{-i\lambda\omega} f(\lambda), \quad \mathcal{F}^{-1}[f](\lambda) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\lambda\omega} f(\omega). \quad (\text{A.23})$$

Despite there is only m_l physical strings, it is advantageous to define scattering phase shifts for the string types which *do not* occur in the string content,

$$a_{(n,v)} = -\frac{1}{2\pi i} \partial_\lambda \log S_{(n,v)}(\lambda) = \frac{1}{2\pi} \frac{2 \sin(\gamma n)}{v \cosh(2\lambda) - \cos(\gamma n)}. \quad (\text{A.24})$$

Fourier space counterparts are obtained after exploiting π -periodicity in the imaginary direction, performing an elementary contour integration around the path

$$[-\tau, \tau] \cup [\tau, \tau + i\tau] \cup i[\tau + i\tau, -\tau + i\tau] \cup [-\tau + i\tau, -\tau], \quad (\text{A.25})$$

while sending $\tau \rightarrow \infty$, using invariance under reflection $\lambda \mapsto -\lambda$, and finally picking residua of $a_{(n,v)} \exp(-i\omega\lambda)$ on the imaginary interval $[0, i\pi]$. This yields

$$\mathcal{F}[a_{(n,+)}](\omega) = \frac{\sinh\left((\kappa_+ - np_0^{-1}) \frac{\pi}{2} \omega\right)}{\sinh\left(\frac{\pi}{2} \omega\right)}, \quad \kappa_+ = 2 \left\lfloor \frac{n}{2p_0} \right\rfloor + 1, \quad (\text{A.26})$$

$$\mathcal{F}[a_{(n,-)}](\omega) = \frac{\sinh\left((\kappa_- - np_0^{-1}) \frac{\pi}{2} \omega\right)}{\sinh\left(\frac{\pi}{2} \omega\right)}, \quad \kappa_- = 2 \left\lfloor \frac{n + p_0}{2p_0} \right\rfloor, \quad (\text{A.27})$$

supplemented with $a_{(n,v)} = 0$ when $n\gamma = \pi/2$. Above we introduced the *mode numbers* $\kappa_\pm \in \mathbb{Z}_{\geq 0}$. By virtue of the identity

$$S_{(n,v)} S_{(\ell_2 - n, (-1)^{\ell_1} v)} = (-1)^{\ell_1}, \quad (\text{A.28})$$

one might exploit a ‘shortening condition’ for strings of lengths larger than $n > \lfloor \frac{\ell_2}{2} \rfloor$, which allow to be interpreted as strings of length $n \rightarrow \ell_2 - n$ if their parity gets transformed as $v \rightarrow (-1)^{\ell_1} v$. Our calculations were made with the first option.

A.3. String Bethe equations

Under the assumption that $\lambda_\alpha - \lambda_\beta = \mathcal{O}(N^{-1})$ for large N , the set of quantum numbers I_α^j in Bethe equations (A.5) can be smoothly interpolated. This enables to introduce particle (hole) densities by counting the number of Bethe roots (vacancies) which form a string of length j on the interval $[\lambda, \lambda + d\lambda]$. The spectrum can then be represented by an infinite set of smooth root distributions $\rho_j(\lambda)$,

$$\rho_j + \bar{\rho}_j = a_j - a_{j,k} \star \rho_k. \quad (\text{A.29})$$

This set of equations is referred to as the string Bethe equations for the densities. Notice that each density ρ_j is assigned a complementary variable $\bar{\rho}_j$. Here the scattering kernels $a_{j,k}$ are of the form

$$a_{j,k}(\lambda) = \frac{1}{2\pi i} \partial_\lambda \log S_{j,k}(\lambda). \quad (\text{A.30})$$

Bethe equations for the strings given by equation (A.29) can be cast in a universal local form. This can be achieved most elegantly by operating with a matrix kernel $(a + \delta)^{-1}$, which component-wise reads

$$(a + \delta)_{j,k}^{-1} = \delta_{j,k} \delta - s I_{j,k}, \quad (\text{A.31})$$

with the incidence matrix I reading

$$I_{j,k} = \delta_{j-1,k} + \delta_{j+1,k}. \quad (\text{A.32})$$

Here we use simultaneously the Dirac delta function $\delta(\lambda)$ and the Kronecker delta symbol $\delta_{i,j}$. Convolution a function $f_j(\lambda)$ (analytic inside *physical region* equation (2.14)) with $(a + \delta)_{j,k}^{-1}$ yields

$$(a + \delta)_{j,k}^{-1} \star f_k = f_j - s \star (f_{j-1} + f_{j+1}). \quad (\text{A.33})$$

Additionally, we have the identity

$$(a + \delta)_{j,k}^{-1} \star (a_{k,m} + \delta_{k,m} \delta) = \delta_{j,m} \delta. \quad (\text{A.34})$$

By operating with $(a + \delta)^{-1} \star$ on the raw form of string Bethe equations (see equation (A.29)) produces a local (nearest-neighbour) coupled system of equations,

$$\rho_j + \bar{\rho}_j = s \star (\bar{\rho}_{j-1} + \bar{\rho}_{j+1}). \quad (\text{A.35})$$

A.3.1. Gapless regime. In the gapless regime the Bethe equations for strings in the raw format get modified by the presence of parity, resulting in a finite number of m_l coupled equations [27, 37]

$$\text{sign}(q_j)(\rho_j + \bar{\rho}_j) = a_j - a_{j,k} \star \rho_k. \quad (\text{A.36})$$

Appendix B. Derivation of the Green function

Here we provide some extra details on establishing the central identity (3.5). The convolution kernels $G_{j,k}$, which determined the matrix Green function (see equation (3.4) in the main text), are directly expressible as linear combinations in terms of a_j . Two notable symmetry properties of G -kernels are

$$G_{j,k} = G_{k,j}, \quad G_{j,k}(\lambda) = G_{j,k}(-\lambda). \quad (\text{B.1})$$

Similarly it can be shown that the kernels $a_{j,k}$ are composite objects and split as

$$a_{j,k} = G_{j-1,k} + G_{j+1,k}. \quad (\text{B.2})$$

Using the ‘quasi-d’Alembertian’ relation for kernels $a_{j,k}$,

$$a_{j,k} = s \star (a_{j-1,k} + \delta_{j-1,k} \delta) + s \star (a_{j+1,k} + \delta_{j+1,k} \delta), \quad (\text{B.3})$$

we arrive at a compact representation

$$G_{j,k} = s \star (a_{j,k} + \delta_{j,k} \delta), \quad (\text{B.4})$$

The fact that G is the Green’s function of the wave operator \square can be readily explicitly verified. This amounts to show that

$$(\square G)_{i,k} \equiv (s^{-1} - I\delta)_{i,j} \star G_{j,k} = \delta_{i,k} \delta. \quad (\text{B.5})$$

with the boundary conditions $G_{0,k} \equiv 0$ implicitly assumed. A brief calculation (using equation (B.4)) shows

$$[(s^{-1} - I\delta) \star G]_{j,k} = G_{j,k}^+ + G_{j,k}^- - (G_{j-1,k} + G_{j+1,k}) \quad (\text{B.6})$$

$$= (a_{j,k} + \delta_{j,k} \delta) - (G_{j-1,k} + G_{j+1,k}) = \delta_{j,k} \delta. \quad (\text{B.7})$$

Appendix C. Simple examples

C.1. Principal roots of unity

As an explicit example we treat a sequence of the so-called *principal* roots of unity. They belong to values $\ell_1 = 1$ and $\ell_2 \geq 3$. Subsequently we put $p_0 = \ell_2 \rightarrow \ell$. Restriction to the principal points is rather standard in the literature, the reason being that the set of non-linear integral equations which are formulated in the scope of Thermodynamic Bethe Ansatz techniques close at a finite level and in addition takes the simplest analytic form [37–40].

Auxiliary numbers are given by

$$q_j = \ell - j, \quad j = 1, 2, \dots, \ell - 1, \quad q_\ell = p_1 = 1, \quad (\text{C.1})$$

while the Bethe equations for the strings read

$$\rho_j + \bar{\rho}_j = s_1 \star (\bar{\rho}_{j-1} + \bar{\rho}_{j+1}), \quad j = 1, 2, \dots, \ell - 2, \quad (\text{C.2})$$

$$\rho_{\ell-2} + \bar{\rho}_{\ell-2} = s_1 \star (\bar{\rho}_{\ell-3} + \bar{\rho}_{\ell-1} + \rho_\ell), \quad (\text{C.3})$$

$$\rho_{\ell-1} + \bar{\rho}_{\ell-1} = \rho_\ell + \bar{\rho}_\ell = s_1 \star \bar{\rho}_{\ell-2}. \quad (\text{C.4})$$

There are ℓ -many distinct string types in the string content

$$(j, +), \quad j = 1, 2, \dots, \ell - 1, \quad \text{and} \quad (1, -). \quad (\text{C.5})$$

Strings with positive parity are ordered according to their increasing lengths $n_j = j$, whereas the only negative-parity string is placed at the end. It is crucial to bare in mind that the last two strings (i.e. the boundary pair) play a very special role. In particular, these two string scatter inversely (see equation (A.28))

$$S_{(1,-)}S_{(\ell-1,+)} = -1, \quad (\text{C.6})$$

while with respect to the remaining strings one finds

$$S_{(1,+),(1,+)} = S_{(1,-),(1,-)}, \quad (\text{C.7})$$

$$S_{(1,-),(j,+)} = S_{(\ell-1,+),(j,+)}, \quad j = 1, 2, \dots, \ell - 1. \quad (\text{C.8})$$

From this analysis we can conclude that only one string from the boundary pair carries dynamical information, while the other is in this respect redundant and can be eliminated. On the other hand, d’Alembertian relation among the ‘bulk nodes’ remains unaffected,

$$\frac{S_j\left(\lambda + \frac{i\gamma}{2}\right)S_j\left(\lambda - \frac{i\gamma}{2}\right)}{S_{j-1}(\lambda)S_{j+1}(\lambda)} = 1, \quad j = 1, 2, \dots, \ell - 1. \quad (\text{C.9})$$

Truncation effect yields

$$\frac{S_{\ell-1}\left(\lambda + \frac{i\gamma}{2}\right)S_{\ell-2}\left(\lambda - \frac{i\gamma}{2}\right)}{S_{\ell-2}(\lambda)} = S_{\ell} = -1. \quad (\text{C.10})$$

D’Alembert identities among scattering matrices for strings at principal roots of unity $\Delta = \cos(\pi/\ell)$ essentially indicate that this discrete set of points very closely resembles the situation at a generic value of anisotropy in the gapped regime. The difference is merely visible as the ‘truncation effect’ which gives rise to an exceptional structure at the boundary (see [27] and [38–40]).

C.2. Non-trivial example: roots (ν_1, ν_2)

The number of strings is here equal to $m_2 = \nu_1 + \nu_2$, with lengths given by

$$n_j = \begin{cases} j & 1 \leq j \leq \nu_1 - 1 \\ 1 + (j - \nu_1)\nu_1 & \nu_1 \leq j \leq m_2 - 1. \\ \nu_1 & j = m_2 \end{cases} \quad (\text{C.11})$$

Parities should be assigned as $v_1 = 1$, $v_{\nu_1} = -1$, while for the remaining indices one should use prescription given by equation (A.15). There are only two relevant kernels, namely s_1 and s_2 , which are determined in terms of p -numbers in accordance with auxiliary numbers

$$p_0 = \frac{1 + \nu_1\nu_2}{\nu_2}, \quad p_1 = 1, \quad p_2 = \frac{1}{\nu_2}. \quad (\text{C.12})$$

Bethe equations for densities read

$$\rho_j + \bar{\rho}_j = s_1 \star (\bar{\rho}_{j-1} + \bar{\rho}_{j+1}), \quad 1 \leq j \leq \nu_1 - 2, \quad (\text{C.13})$$

$$\rho_{\nu_1-1} + \bar{\rho}_{\nu_1-1} = s_1 \star \bar{\rho}_{\nu_1-2} + \tilde{s}_1 \star \bar{\rho}_{\nu_1-1} - s_2 \star \bar{\rho}_{\nu_1}, \quad (\text{C.14})$$

$$\rho_j + \bar{\rho}_j = s_1 \star (\bar{\rho}_{j-1} + \bar{\rho}_{j+1}), \quad \nu_1 \leq j \leq \nu_1 + \nu_2 - 2, \quad (\text{C.15})$$

$$\rho_{\nu_1+\nu_2-2} + \bar{\rho}_{\nu_1+\nu_2-2} = s_2 \star (\bar{\rho}_{\nu_1+\nu_2-3} + \bar{\rho}_{\nu_1+\nu_2-1}) + s_2 \star \rho_{\nu_1+\nu_2}, \quad (\text{C.16})$$

$$\rho_{\nu_1+\nu_2-1} + \bar{\rho}_{\nu_1+\nu_2-1} = \rho_{\nu_1+\nu_2} + \bar{\rho}_{\nu_1+\nu_2} = s_2 \star \bar{\rho}_{\nu_1+\nu_2-2}. \quad (\text{C.17})$$

The relation between the charge indices and their respective spin and parity labels can be explicitly encoded as

$$X_j = X_{(j,+)}, \quad j = 1, 2, \dots, \nu_1 - 1, \quad (\text{C.18})$$

$$X_{m_1+m_2-j} = X_{(\bar{j}, \bar{u})}, \quad j = \nu_1, \nu_1 + 1, \dots, \nu_1 + \nu_2 - 1, \quad (\text{C.19})$$

where $m_1 = \nu_1, m_2 = \nu_1 + \nu_2$, and where (\bar{j}, \bar{u}) are ‘mirror indices’ defined by identifications

$$\bar{j} = \ell_2 - j, \quad \bar{u} = (-1)^{\ell_1} u. \quad (\text{C.20})$$

Below we provide two concrete examples. For instance, the string content for $(\nu_1, \nu_2) = (2, 2)$ ($p_0 = \frac{5}{2}$) is

$$\{(1, +), (1, -), (3, +), (2, +)\}, \quad (\text{C.21})$$

whereas the corresponding charge content is

$$\{X_{(1,+)}, X_{(2,+)}, X_{(4,-)}\}. \quad (\text{C.22})$$

The last two strings, i.e. $(3, +)$ and $(2, +)$, constitute the boundary pair and do not carry an independent dynamical content, meaning that their distributions are simultaneously fixed upon specifying $X_{(2,+)}$. Let us consider another example, e.g. $(\nu_1, \nu_2) = (2, 3)$ ($p_0 = \frac{7}{3}$). Here the string content reads

$$\{(1, +), (1, -), (3, +), (5, -), (2, +)\}, \quad (\text{C.23})$$

while the charges are

$$\{X_{(1,+)}, X_{(2,+)}, X_{(4,-)}, X_{(6,+)}\}. \quad (\text{C.24})$$

Appendix D. Magnetization density sum rule

D.1. Gapped regime

Given a state $|\Psi\rangle$, summing over all integrated density distributions should amount precisely to the magnetization density

$$m = \lim_{N \rightarrow \infty} N^{-1} \langle \Psi | S^z | \Psi \rangle. \quad (\text{D.1})$$

By supposing that $|\Psi\rangle$ is characterized by densities ρ_j , equation (D.1) is cast as

$$m = \left[\int_{-\pi/2}^{\pi/2} d\lambda \sum_{j=1}^{\infty} j \rho_j(\lambda) \right] - \frac{1}{2}. \quad (\text{D.2})$$

To simplify the notation we write integrations over the fundamental period shortly as $1 \star f \equiv \int_{-\pi/2}^{\pi/2} d\lambda f(\lambda)$. We furthermore introduce the integrated particle distributions $1 \star \rho_j$ and the integrated charge densities $1 \star X_j$. First we state a few useful identities,

$$1 \star s = \frac{1}{2}, \quad 1 \star a_j = 1, \quad 1 \star s \star f = (1 \star s)(1 \star f) = \frac{1}{2}(1 \star f). \quad (\text{D.3})$$

Let $\square^{(t)}$ designate a *truncated* version of \square by retaining only densities ρ_j up to $j \leq t$, and charges $X_j(\mu)$ up to $j \leq t+1$. This allows us to approximate the density of magnetization up to degree t as $m_t = -\frac{1}{2} + \sum_{j=1}^t n_j(1 \star \rho_j)$. The exact magnetization density is obtained from the limit $m = \lim_{t \rightarrow \infty} m_t$. By applying $1 \star$ on $\square^{(t)}$ we readily obtain

$$m_t = [(t+1)(1 \star X_t) - t(1 \star X_{t+1})] - \frac{1}{2}, \quad (\text{D.4})$$

whence under the assumption that the limit $\lim_{t \rightarrow \infty} (1 \star X_t) = 1 \star X_\infty$ exists, the self-consistent solution of equation (D.4) corresponds to large- j limit of the integrated charge eigenvalue $m = 1 \star X_\infty - \frac{1}{2}$.

D.2. Gapless regime

D.2.1. Principal roots. Here we show that the identification of the string densities for the boundary pair as in equation (3.18) implies vanishing magnetization, $m = 0$. Now we employ the integrated variables,

$$\mathbf{n}_j \equiv 1 \star \rho_j, \quad \bar{\mathbf{n}}_j \equiv 1 \star \bar{\rho}_j, \quad \mathfrak{J}_j \equiv 1 \star X_j, \quad \mathbf{a}_j \equiv 1 \star a_j, \quad \mathbf{a}_{j,k} \equiv 1 \star a_{j,k}. \quad (\text{D.5})$$

As a quick non-trivial consistency check we derive the sum rule for the principal points ($p_0 = \ell$). By assuming that a state ρ associated to $|\Psi\rangle$ is unpolarized, i.e. $m = 0$, we imposing the identification of the boundary particle types in the form $\rho_{(1,-)} \equiv \rho_\ell = \bar{\rho}_{\ell-1}$, and readily show that

$$m = -\frac{1}{2} + \bar{\mathbf{n}}_{\ell-1} + \sum_{j=1}^{\ell-1} j \mathbf{n}_j = \ell(\mathfrak{J}_{\ell-1} + \bar{\mathbf{n}}_{\ell-1}) - \frac{1}{2} = \frac{\ell}{4} \mathbf{a}_{\ell-2} - \frac{1}{2} = 0. \quad (\text{D.6})$$

In the above calculation we accounted for that fact that at the principal points the integrated convolution kernels obey

$$\mathbf{a}_j = \frac{\ell - j}{\ell}, \quad j \leq \ell. \quad (\text{D.7})$$

using $\bar{\mathbf{n}}_{\ell-1} = \frac{1}{4} \mathbf{a}_{\ell-2} - \mathfrak{J}_{\ell-1}$ and $\bar{\rho}_j = a_j - s^{-1} \star X_j$ which is valid for nodes in the range $j = 1, 2, \dots, \ell - 2$.

D.2.2. Generic roots of unity. The situation with generic roots of unity is more involved due to the presence of exceptional nodes. It proves useful to split the entire contribution into the regular part $m_{\text{reg}} = \sum_{j=1}^{m_l-2} n_j \mathbf{n}_j$, and finally adding the contribution of the boundary nodes $m_b = n_{m_l-1} \mathbf{n}_{m_l-1} + n_{m_l} \mathbf{n}_{m_l}$,

$$m = m_{\text{reg}} + m_b = \sum_{j=1}^{m_l} n_j \mathbf{n}_j = \ell_2(\bar{\mathbf{n}}_{m_l-1} - (-1)^l(\mathfrak{J}_{m_l-1-1} - \mathfrak{J}_{m_l-1})), \quad (\text{D.8})$$

The remaining integrated hole distribution which needs to be determined is $\bar{\mathbf{n}}_{m_l-1}$. This one can be obtained by combining raw string Bethe equation at node $m_l - 2$,

$$\begin{aligned} & \bar{\mathbf{n}}_{m_l-2} + \text{sign}(q_{m_l-2})(\mathbf{a}_{m_l-2, m_l-1} + \mathbf{a}_{m_l-2, m_l})\bar{\mathbf{n}}_{m_l-1} \\ &= \text{sign}(q_{m_l-2}) \left[\mathbf{a}_{m_l-2} - \mathbf{a}_{m_l-2, m_l-1}(\mathbf{n}_{m_l-1} - \mathbf{n}_{m_l}) - \sum_{k=1}^{m_l-2} \mathbf{a}_{m_l-2, k} \mathbf{n}_k \right] - \mathbf{n}_{m_l-2}, \end{aligned} \quad (\text{D.9})$$

with the local Bethe equation found at node $m_l - 1$,

$$\bar{\mathbf{n}}_{m_l-1} = \frac{1}{4} \bar{\mathbf{n}}_{m_l-2} - \frac{1}{2}(\mathbf{n}_{m_l-1} - \mathbf{n}_{m_l}), \quad (\text{D.10})$$

along with $\mathbf{n}_{m_l-1} - \mathbf{n}_{m_l} = (-1)^{l-1}(\mathfrak{J}_{m_l-1} - \mathfrak{J}_{m_l-2} - \mathfrak{J}_{m_l-1-1})$ and the integrated kernels

$$\mathbf{a}_{(n,+)} = 1 - \frac{n}{p_0} + 2 \left\lfloor \frac{n}{2p_0} \right\rfloor, \quad \mathbf{a}_{(n,-)} = -\frac{n}{p_0} + 2 \left\lfloor \frac{n}{2p_0} + \frac{1}{2} \right\rfloor. \quad (\text{D.11})$$

Notice that by virtue of $m = 0$ we have $\mathbf{n}_{m_l} = \bar{\mathbf{n}}_{m_l-1}$.

References

- [1] Deutsch J 1991 Quantum statistical mechanics in a closed system *Phys. Rev. A* **43** 2046
- [2] Srednicki M 1994 Chaos and quantum thermalization *Phys. Rev. E* **50** 888
- [3] Ilievski E, De Nardis J, Wouters B, Caux J-S, Essler F H and Prosen T 2015 Complete generalized Gibbs ensembles in an interacting theory *Phys. Rev. Lett.* **115** 157201
- [4] Ilievski E, Medenjak M and Prosen T 2015 Quasilocal conserved operators in the isotropic Heisenberg spin-1/2 chain *Phys. Rev. Lett.* **115** 120601
- [5] Rigol M, Dunjko V and Olshanii M 2008 Thermalization and its mechanism for generic isolated quantum systems *Nature* **452** 854–8
- [6] Cassidy A C, Clark C W and Rigol M 2011 Generalized thermalization in an integrable lattice system *Phys. Rev. Lett.* **106** 140405
- [7] Polkovnikov A, Sengupta K, Silva A and Vengalattore M 2011 Colloquium: nonequilibrium dynamics of closed interacting quantum systems *Rev. Mod. Phys.* **83** 863
- [8] Gogolin C and Eisert J 2015 Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems arXiv:1503.07538
- [9] Langen T, Erne S, Geiger R, Rauer B, Schweigler T, Kuhnert M, Rohringer W, Mazets I E, Gasenzer T and Schmiedmayer J 2015 Experimental observation of a generalized Gibbs ensemble *Science* **348** 207–11
- [10] Brockmann M, Wouters B, Fioretto D, De Nardis J, Vlijm R and Caux J-S 2014 Quench action approach for releasing the Néel state into the spin-1/2 XXZ chain *J. Stat. Mech.* **P12009**
- [11] Mestyán M, Pozsgay B, Takács G and Werner M 2015 Quenching the XXZ spin chain: quench action approach versus generalized Gibbs ensemble *J. Stat. Mech.* **P04001**
- [12] Krichever I, Lipan O, Wiegmann P and Zabrodin A 1997 Quantum integrable models and discrete classical Hirota equations *Commun. Math. Phys.* **188** 267–304

- [13] Zabrodin A 1997 Discrete Hirota's equation in quantum integrable models *Int. J. Mod. Phys. B* **11** 3125–58
- [14] Korepin V E, Bogoliubov N M and Izergin A G 1997 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
- [15] Faddeev L 1996 How algebraic Bethe Ansatz works for integrable model [hep-th/9605187](#)
- [16] Klümper A and Pearce P A 1992 Conformal weights of RSOS lattice models and their fusion hierarchies *Physica A* **183** 304–50
- [17] Kuniba A, Nakanishi T and Suzuki J 1994 Functional relations in solvable lattice models I: functional relations and representation theory *Int. J. Mod. Phys. A* **9** 5215–66
- [18] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 Integrable structure of conformal field theory II. Q -operator and DDV equation *Commun. Math. Phys.* **190** 247–78
- [19] Gromov N, Kazakov V and Vieira P 2009 Finite volume spectrum of 2D field theories from Hirota dynamics *J. High Energy Phys.* **JHEP09(2009)060**
- [20] Kazakov V, Sorin A and Zabrodin A 2008 Supersymmetric Bethe Ansatz and Baxter equations from discrete Hirota dynamics *Nucl. Phys. B* **790** 345–413
- [21] Kazakov V and Leurent S 2010 Finite size spectrum of SU(N) principal chiral field from discrete Hirota dynamics [arXiv:1007.1770](#)
- [22] Baxter R 1973 Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors *Ann. Phys.* **76** 1–24
- [23] Bazhanov V V, Iukowski T, Meneghelli C and Staudacher M 2010 A shortcut to the Q -operator *J. Stat. Mech.* **P11002**
- [24] Korff C 2005 A Q -operator identity for the correlation functions of the infinite XXZ spin-chain *J. Phys. A: Math. Gen.* **38** 6641
- [25] Takahashi M 1971 One-dimensional Heisenberg model at finite temperature *Prog. Theor. Phys.* **46** 401–15
- [26] Gaudin M 1971 Thermodynamics of the Heisenberg-Ising ring for $\Delta > 1$ *Phys. Rev. Lett.* **26** 1301
- [27] Takahashi M 2005 *Thermodynamics of One-Dimensional Solvable Models* (Cambridge: Cambridge University Press)
- [28] Volin D 2012 String hypothesis for spin chains: a particle/hole democracy *Lett. Math. Phys.* **102** 1–29
- [29] Bethe H 1931 Zur theorie der metalle *Z. Phys.* **71** 205–26
- [30] Prosen T 2011 Open XXZ spin chain: nonequilibrium steady state and a strict bound on ballistic transport *Phys. Rev. Lett.* **106** 217206
- [31] Prosen T and Ilievski E 2013 Families of quasilocal conservation laws and quantum spin transport *Phys. Rev. Lett.* **111** 057203
- [32] Prosen T 2014 Quasilocal conservation laws in XXZ spin-1/2 chains: open, periodic and twisted boundary conditions *Nucl. Phys. B* **886** 1177–98
- [33] Pereira R, Pasquier V, Sirker J and Affleck I 2014 Exactly conserved quasilocal operators for the XXZ spin chain *J. Stat. Mech.* **P09037**
- [34] Prosen T 1998 Quantum invariants of motion in a generic many-body system *J. Phys. A: Math. Gen.* **31** L645
- [35] Prosen T 1999 Ergodic properties of a generic nonintegrable quantum many-body system in the thermodynamic limit *Phys. Rev. E* **60** 3949
- [36] Doyon B 2015 Thermalization and pseudolocality in extended quantum systems [arXiv:1512.03713](#)
- [37] Takahashi M and Suzuki M 1972 One-dimensional anisotropic Heisenberg model at finite temperatures *Prog. Theor. Phys.* **48** 2187–209
- [38] Kuniba A, Sakai K and Suzuki J 1998 Continued fraction TBA and functional relations in XXZ model at root of unity *Nucl. Phys. B* **525** 597–626
- [39] Arutyunov G, de Leeuw M and van Tongeren S J 2012 The quantum deformed mirror TBA I *J. High Energy Phys.* **JHEP10(2012)090**
- [40] Arutyunov G *et al* 2013 The quantum deformed mirror TBA II *J. High Energy Phys.* **JHEP02(2013)012**
- [41] Fagotti M and Essler F H 2013 Stationary behaviour of observables after a quantum quench in the spin-1/2 Heisenberg XXZ chain *J. Stat. Mech.* **P07012**
- [42] Fagotti M, Collura M, Essler F H and Calabrese P 2014 Relaxation after quantum quenches in the spin-1/2 Heisenberg XXZ chain *Phys. Rev. B* **89** 125101
- [43] Wouters B, De Nardis J, Brockmann M, Fioretto D, Rigol M and Caux J-S 2014 Quenching the anisotropic Heisenberg chain: exact solution and generalized Gibbs ensemble predictions *Phys. Rev. Lett.* **113** 117202
- [44] Majumdar C K and Ghosh D K 1969 On next-nearest-neighbor interaction in linear chain. I *J. Math. Phys.* **10** 1388–98
- [45] Caux J-S and Essler F H 2013 Time evolution of local observables after quenching to an integrable model *Phys. Rev. Lett.* **110** 257203

- [46] Pozsgay B 2014 Overlaps between eigenstates of the XXZ spin-1/2 chain and a class of simple product states *J. Stat. Mech.* **P06011**
- [47] Pozsgay B, Mestyán M, Werner M, Kormos M, Zaránd G and Takács G 2014 Correlations after quantum quenches in the XXZ spin chain: failure of the generalized Gibbs ensemble *Phys. Rev. Lett.* **113** 117203
- [48] Brockmann M, De Nardis J, Wouters B and Caux J-S 2014 A Gaudin-like determinant for overlaps of Néel and XXZ Bethe states *J. Phys. A: Math. Theor.* **47** 145003
- [49] Cherednik I V 1987 An analogue of the character formula for Hecke algebras *Funct. Anal. Appl.* **21** 172–4
- [50] Bazhanov V and Reshetikhin N 1990 Restricted solid-on-solid models connected with simply laced algebras and conformal field theory *J. Phys. A: Math. Gen.* **23** 1477
- [51] Klümper A 1993 Thermodynamics of the anisotropic spin-1/2 Heisenberg chain and related quantum chains *Z. Phys. B* **91** 507–19
- [52] Klümper A and Sakai K 2002 The thermal conductivity of the spin-1/2 XXZ chain at arbitrary temperature *J. Phys. A: Math. Gen.* **35** 2173
- [53] Göhmann F, Klümper A and Seel A 2004 Integral representations for correlation functions of the XXZ chain at finite temperature *J. Phys. A: Math. Gen.* **37** 7625
- [54] Mestyán M and Pozsgay B 2014 Short distance correlators in the XXZ spin chain for arbitrary string distributions *J. Stat. Mech.* **P09020**
- [55] Calabrese P and Cardy J 2006 Time dependence of correlation functions following a quantum quench *Phys. Rev. Lett.* **96** 136801
- [56] Calabrese P, Essler F H L and Fagotti M 2011 Quantum quench in the transverse-field Ising chain *Phys. Rev. Lett.* **106** 227203
- [57] Fioretto D and Mussardo G 2010 Quantum quenches in integrable field theories *New J. Phys.* **12** 055015
- [58] Calabrese P, Essler F H L and Fagotti M 2012 Quantum quench in the transverse field Ising chain: I. Time evolution of order parameter correlators *J. Stat. Mech.* **P07016**
- [59] Calabrese P, Essler F H L and Fagotti M 2012 Quantum quenches in the transverse field Ising chain: II. Stationary state properties *J. Stat. Mech.* **P07022**
- [60] Kormos M, Shashi A, Chou Y-Z, Caux J-S and Imambekov A 2013 Interaction quenches in the one-dimensional Bose gas *Phys. Rev. B* **88** 205131
- [61] Sotiriadis S and Calabrese P 2014 Validity of the GGE for quantum quenches from interacting to noninteracting models *J. Stat. Mech.* **P07024**
- [62] Bertini B, Schuricht D and Essler F H L 2014 Quantum quench in the sine-Gordon model *J. Stat. Mech.* **P10035**
- [63] Essler F H L, Mussardo G and Panfil M 2015 Generalized Gibbs ensembles for quantum field theories *Phys. Rev. A* **91** 051602
- [64] Collura M, Calabrese P and Essler F H L 2015 Quantum quench within the gapless phase of the spin- $\frac{1}{2}$ Heisenberg XXZ spin chain *Phys. Rev. B* **92** 125131
- [65] Bonnes L, Essler F H L and Läuchli A M 2014 ‘Light-cone’ dynamics after quantum quenches in spin chains *Phys. Rev. Lett.* **113** 187203
- [66] Boos H E, Göhmann F, Klümper A and Suzuki J 2007 Factorization of the finite temperature correlation functions of the XXZ chain in a magnetic field *J. Phys. A: Math. Theor.* **40** 10699
- [67] Trippé C, Göhmann F and Klümper A 2010 Short-distance thermal correlations in the massive XXZ chain *Eur. Phys. J. B* **73** 253–64
- [68] Nardis J D, Piroli L and Caux J-S 2015 Relaxation dynamics of local observables in integrable systems *J. Phys. A: Math. Theor.* **48** 43FT01
- [69] Zadnik L, Medenjak M and Prosen T 2016 Quasilocal conservation laws from semicyclic irreducible representations of $U_q(\mathfrak{sl}_2)$ in XXZ spin-1/2 chains *Nucl. Phys. B* **902** 339–53
- [70] Zotos X, Naef F and Prelovsek P 1997 Transport and conservation laws *Phys. Rev. B* **55** 11029
- [71] Zotos X 1999 Finite temperature Drude weight of the one-dimensional spin-1/2 Heisenberg model *Phys. Rev. Lett.* **82** 1764
- [72] Benz J, Fukui T, Klümper A and Scheeren C 2005 On the finite temperature Drude weight of the anisotropic Heisenberg chain *J. Phys. Soc. Japan* **74** 181–90
- [73] Ilievski E and Prosen T 2013 Thermodynamic bounds on Drude weights in terms of almost-conserved quantities *Commun. Math. Phys.* **318** 809–30