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LETTER

Small-world of communities: communication and correlation of the meta-network

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Abstract. Given a network and a partition in \(n\) communities, we address the issues of ‘how communities influence each other’ and ‘when do two given communities communicate’. We prove that, for a small-world network, a simple superposition principle applies among communities and each community plays the role of a microscopic spin governed by a sort of effective TAP (Thouless, Anderson and Palmer) equation. The relative susceptibilities derived from these equations calculated at finite or zero temperature (where the method provides an effective percolation theory) give us the answers to the above issues. As for the already studied case \(n = 1\), these equations are exact in the paramagnetic regions (at \(T = 0\) this means below the percolation threshold) and provide effective approximations in the other regions. However, unlike the case \(n = 1\), asymmetries among the communities may lead, via the TAP-like structure of the equations, to many metastable states whose number, in the case of negative short-cuts among the communities, may increase exponentially fast with \(n\) and glassy scenarios with a remarkable number of abrupt jumps take place. Furthermore, as a by-product, a natural and efficient method for detecting the community structure of a generic network emerges from the relative susceptibilities.

Keywords: disordered systems (theory), spin glasses (theory), exact results, random graphs, networks
1. Introduction

Recently, in network science the issue of finding the ‘optimal’ community structure that should be present in a given random graph (a network) \((\mathcal{L}, \Gamma)\), \(\mathcal{L}\) and \(\Gamma\) being the set of the vertices and bonds, respectively, has received much attention. The general idea behind the concept of community structure comes from the observation that in many situations real data show an intrinsic partition of the vertices of the graph into \(n\) groups, called communities, \((\mathcal{L} = \bigcup_{l=1}^{n} \mathcal{L}(l), \Gamma = \bigcup_{l,k=1}^{n} \Gamma^{(l,k)})\), such that between any two communities there are a relatively small number of bonds compared with the number of bonds present in each community. The partition(s) can be used to build a higher-level meta-network where the meta-nodes are now the communities (cells, proteins, groups of people, ...) and play important roles in unveiling the functional organization inside the network. In order to detect the community structure of a given network, many methods have been proposed and particular progress has been made by mapping the problem for identifying community structures to optimization problems [1,2], by looking for \(k\)-clique sub-graphs [3], and by looking for clustering desynchronization [4]. In general there is no unique criterion. However, given a structure in communities, whatever the method used, and assuming that the found partition \((\bigcup_{l=1}^{n} \mathcal{L}(l), \bigcup_{l,k=1}^{n} \Gamma^{(l,k)})\) sufficiently well represents the intrinsic community structure of the given network [5], there still remains the fundamental issue about the true relationships among these communities. Under which conditions and how much two given communities, also in the presence of other communities, exchange information; how they influence each other, positively or negatively; what is the typical state of a single community; what is the expected behavior for large \(n\), etc are all issues that cannot be addressed by simply using the above methods to detect the community structure. In fact, all these methods, with the exception of [1] and [4], are essentially based only on a topological (and, in most cases, local) analysis of the network. To uncover the real communication among the communities we have to impose over the \((\mathcal{L}, \Gamma)\) graph a minimal model in which the vertices assume at least two states, and analyze their correlations. Confining the problem to the equilibrium case we have hence to use Gibbs–Boltzmann statistical mechanics. Given a community structure of size \(N\), if each node is associated with a spin \(\sigma_i\), \(i = 1, \ldots, N\), then each meta-node is associated with the
meta-spin \( s^{(l)} = \sum_{i \in L^{(l)}} \sigma_i / \sum_{i \in L^{(l)}} 1 \), \( l = 1, \ldots, n \), where the sums run only over the nodes of the \( l \)th community. We show then that the average magnetizations \( m^{(l)} \) of the meta-spins obey special effective equations as if they were microscopic spins immersed in a ferro or glassy material. From these equations it will then be simple to derive the relative susceptibilities \( \chi^{(l,k)} \), \( l, k = 1, \ldots, n \) among the communities, revealing in an unambiguous way the communication and correlation properties of the meta-network.

2. A minimal model

If a Gibbs–Boltzmann distribution, \( \exp(-\beta H) \), has been assumed, \( H \) being some effective Hamiltonian, one can obtain the (adimensional) couplings \( \beta J_{i,j}^{(l,k)} \) from the data of the given graph by isolating the two vertices \( i, j \) from all the others, and by measuring the correlation function of the obtained isolated dimer \( \langle \sigma_i \sigma_j \rangle' \), where \( \langle \cdot \rangle' \) stands for the Gibbs–Boltzmann average of the isolated dimer. The general problem is actually more complicated, due to the presence of two sources of disorder since both the set of bonds \( \Gamma \) and the single couplings \( \{ J_{i,j}^{(l,k)} \} \) may change with time. Assuming that the timescale over which these changes take place is much larger than that of the thermal vibrations of the spins, we have then to consider a disordered Ising model with quenched disorder. Here we specialize this general problem to the case of Poissonian disorder of the graph, while we leave the disorder of the couplings arbitrary. We formulate the problem in terms of Ising models on generic small-world graphs [6]: given an arbitrary graph \(( L_0, \Gamma_0)\), the pure graph, and an associated community structure \( (C_i^n_{l=1} L^{(l)}_0, C_i^{l \leq k = 1} \Gamma^{(l,k)}_0) \), in which each community has an arbitrary size, we consider a generic Ising Hamiltonian \( H_0 \) defined on this nonrandom structure, the pure model, characterized by arbitrary couplings \( J_{i,j}^{(l,k)} \) and, between any two sites \( i, j \), with \( i \in L^{(l)} \) and \( j \in L^{(k)} \), we add some random connections (short-cuts) having average connectivities \( c^{(l,k)} \) (more precisely, if \( |L^{(k)}_0| \) is the size of the \( k \)th community, we introduce the directed random variables \( c_{i,j}^{(l,k)} \) taking the values \( 0, 1 \) with probabilities \( 1 - c^{(l,k)} / |L^{(k)}_0| \) and \( c^{(l,k)} / |L^{(k)}_0| \), respectively). Then, along with these random connections, random couplings \( J^{(l,k)}_{i,j} \) with a quenched disorder are imposed, and we study the corresponding random Ising model, the random model, having Hamiltonian \( H \) (see figure 1). In a compact way \( H \) is therefore given by

\[
H \left( \{ \sigma_i \}_{i=1}^N \right) = H_0 \left( \{ \sigma_i \}_{i=1}^N \right) + \Delta H \left( \{ \sigma_i \}_{i=1}^N \right) - \sum_{l=1}^n h^{(l)} \sum_{i \in L^{(l)}_0} \sigma_i.
\] (1)

In equation (1) \( H_0 \) is the nonrandom part of the Hamiltonian having nonrandom couplings \( J_{i,j}^{(l,k)} \) (typically short-range couplings but not necessarily) whereas \( \Delta H \) is the random part of the Hamiltonian involving only the long-range couplings \( c_{i,j}^{(l,k)} J_{i,j}^{(l,k)} \); finally \( h^{(l)} \) is an arbitrary external field acting only on the \( l \)th community.

3. Equations of the meta-network

In [7] we established a new general method for analyzing critical phenomena in the small-world models represented by equation (1) for the case in which there is one single community, \( n = 1 \): given any arbitrary initial graph \(( L_0, \Gamma_0)\), if we increase the average

connectivity by $c$ through the addition of $Nc$ bonds randomly spread over the initial
graph, under the condition that $c > 0$, we find an effective field theory that generalizes
the Curie–Weiss mean-field theory via the equation

$$m^{(\Sigma)} = m_0 \left( \beta J^{(\Sigma)}_0, \beta J^{(\Sigma)}J^{(\Sigma)}_0 m^{(\Sigma)} \right),$$

which is able to take into account both the infinite and finite dimensionality simultaneously
present in small-world models. In equation (2) $m_0(\beta J_0; \beta h)$ represents the magnetization

Figure 1. An example with $n = 2$, $|\mathcal{L}_0^{(1)}| = 6$, $|\mathcal{L}_0^{(2)}| = 4$, $|\Gamma_0^{(1,1)}| = 5$, $|\Gamma_0^{(2,2)}| = 4$,
$|\Gamma_0^{(1,2)}| = 1$. Continuous lines represent short-range couplings $J_0^{(l,k)}$ of the pure
model, whereas dashed and dot–dashed lines represent the additional random
couplings $J_{i,j}^{(l,k)}$, with $i \in \mathcal{L}_0^{(l)}$ and $j \in \mathcal{L}_0^{(k)}$, and are present only in the random
model (panel (c)). Panels (a) and (b) differ in the position of one single coupling
$J_0^{(1,2)}$; nevertheless the example in (a), compared with the case in (b), at least
at finite temperature (where the lengths of paths are important), benefits clearly
from better communication (higher betweenness).
of the pure model, i.e. without short-cuts, known as a function of the short-range coupling $J_0$ and an arbitrary external field $h$, whereas the symbol $\Sigma$ stands for the ferro-like solution, $\Sigma = F$, or the spin-glass-like solution, $\Sigma = SG$. $J_0^{(\Sigma)}$ and $J^{(\Sigma)}$ are effective couplings. Here we generalize this result to the present case of $n$ communities of arbitrary sizes and interactions; short-range and long-range (or short-cuts) couplings. We show that, among the communities, a natural superposition principle applies and we find that the $n$ order parameters, $F$- or $SG$-like, obey a system of equations which, apart from the absence of the Onsager reaction term [8], can be seen as an $n \times n$ effective system of TAP (Thouless, Anderson and Palmer) equations [9] in which each community plays the role of a single ‘microscopic’ spin through its own order parameter, $m^{(\Sigma;l)}$, $l = 1, \ldots, n$ [10]. In particular, in the simpler case in which there are no short-range couplings among different communities ($J^{(l,k)}_0 = 0$ for $l \neq k$) these self-consistent equations take the form

$$m^{(\Sigma;l)} = m^{(l)}_0 \left( \beta J^{(\Sigma;l)}_0; \beta H^{(\Sigma;l)} + \beta h^{(l)} \right),$$

$$\beta H^{(\Sigma;l)} = \sum_{k=1}^{n} \beta J^{(\Sigma;l,k)} m^{(\Sigma;k)},$$

where $m^{(l)}_0 (\beta J^{(l)}_0; \beta h^{(l)})$ is the magnetization of the pure model for the $l$th community, and the effective couplings $J^{(F;l,k)}_0$, $J^{(SG;l,k)}_0$, $J^{(F;l)}_0$ and $J^{(SG;l)}_0$ are given by

$$\beta J^{(F;l,k)}_0 \overset{\text{def}}{=} \frac{d}{d\mu^{(l,k)}(J)} \tanh(\beta J),$$

$$\beta J^{(SG;l,k)}_0 \overset{\text{def}}{=} \frac{d}{d\mu^{(l,k)}(J)} \tanh^2(\beta J),$$

$$J^{(F;l)}_0 \overset{\text{def}}{=} J^{(l)}_0, \quad \beta J^{(SG;l)}_0 \overset{\text{def}}{=} \frac{1}{\tanh(\beta J^{(l)}_0)}.$$
Small-world of communities: communication and correlation of the meta-network

Figure 2. Solutions of equations (3) with $n = 2$ in the case $J^{(l,k)}_0 = 0$ (a generalized Viana–Bray model) with long-range couplings $J^{(1,1)} = 1$, $J^{(2,2)} = -1$ and $J^{(1,2)} = 1$ ($J^{(1,2)} = -1$ also leads to the same plot), and connectivities $c^{(1,1)} = c^{(2,2)} = 2$, and $c^{(1,2)} = 0$ or $c^{(1,2)} = 1$. $L$ stands for the free energy term associated with each solution.

Viana–Bray model [11]), concern only models where short loops in the graph are absent, while in our approach the graph $(\mathcal{L}_0, \Gamma_0)$ may be an arbitrary lattice, regular or not, and having loops of any length. The possibility of improving the theory off the P region is a formidable task to be investigated in the future; however, as we show below, the very interesting phase-transition scenario emerging from equation (3) should already be clear.

In [7] ($n = 1$) we established the general scenario of the critical behavior coming from equation (2), stressing the differences between the cases $J_0 \geq 0$ and $J_0 < 0$, the former only being able to give second-order phase transitions with classical critical exponents, whereas the latter are able to give rise, for a sufficiently large connectivity $c$, to multicritical points in principle also with first-order phase transitions. The same scenario essentially takes place for $n \geq 2$, provided that the $J_0$s and the $J$s are almost the same for all the communities, otherwise many other situations are possible. In particular, unlike the case $n = 1$, relative antiferromagnetism between two communities $l$ and $k$ is possible as soon as the $J_{i,j}^{(l,k)}$ have negative averages, while internal antiferromagnetism inside a single community, say the $l$th one, due to the presence of negative couplings $J^{(l)}_0 < 0$, is never possible as soon as disorder is present [12]. Less intuitively and quite interestingly, if we try to connect randomly with some added connectivity $c^{(l,k)}$ the $l$th community, having only positive couplings inside (‘good’), to the $k$th community, having only negative couplings inside (‘bad’), not only does the bad community gain a nonzero order but even the already good community gains improved order. In figure 2 we report an example. However, with respect to the case $n = 1$, another peculiar feature to take into account is the presence of many metastable states. In fact, this is a general mechanism of the TAP-like structure of the equations: as we consider systems with an increasing number of communities, the number of metastable states grows with $n$ and may grow exponentially fast in the case of negative short-cuts. A metastable state can be made virtually stable (or, more precisely,
Small-world of communities: communication and correlation of the meta-network

Figure 3. Solutions of equations (3) with \( n = 3 \) in the case \( J_0^{(l,k)} \equiv 0 \) and \( c^{(l,k)} \to \infty \) (a generalized Curie–Weiss model), with \( J^{(l,l)} \equiv 3/c^{(l,l)} \) and \( J^{(l,k)} \equiv -0.5/c^{(l,k)} \), for \( l \neq k \). We also plot the associated free energy term \( L \).

leading) by forcing the system with appropriate initial conditions, by fast cooling, or by means of suitable external fields. As a result, with respect to variations of the several parameters of the model (couplings, connectivities, sizes of the communities), the presence of metastable states may lead itself to first-order phase transitions even when the \( J_0 \)'s are all nonnegative. This general mechanism has already been studied in the simplest version of these models, namely the \( n = 2 \) Curie–Weiss model (\( J_0^{(l,k)} \equiv 0 \) and \( c^{(l,k)} \to \infty \)), where a first-order phase transition was observed to be tuned by the relative sizes of the two communities and by the external fields [13]. Moreover, first-order phase transitions have been observed in simulations of a two-dimensional small-world model with directed short-cuts [14]. In particular, in a system of many communities, \( n \gg 1 \), a remarkable and natural presence of first-order phase transitions (tuned by the several parameters) is expected which, if \( J_0 \)'s or \( J_0 \)s is negative, reflects on the fact that the communities, at sufficiently low temperature, behave as spins in an effective glassy state [15,16]. Figure 3 concerns the Curie–Weiss case with \( n = 3 \); as is evident, even for small \( n \), the number of metastable states is rapidly growing.

4. Communication properties of the meta-network

In general, how much two communities influence each other is encoded in the matrix

\[
\tilde{\chi}^{(l,k)} \overset{\text{def}}{=} \frac{\partial m^{(\Sigma)} / \partial (\beta h^{(k)})}{\partial (\beta h^{(k)})},
\]

the (adimensional) susceptibility of the random model which tells us how the \( l \)th community reacts to a small variation occurring only (initially) in the \( k \)th community. From equation (3), or from its most general form that also includes nonzero short-range couplings among different communities, we have

\[
\tilde{\chi}^{(\Sigma)} = \left( 1 - \tilde{\chi}_0 \cdot \beta J^{(\Sigma)} \right)^{-1} \cdot \tilde{\chi}_0.
\]

where we have introduced the matrix of the effective long-range couplings \( \beta J^{(\Sigma,l,k)} \), and \( \tilde{\chi}^{(l,k)}_0 \), the adimensional susceptibility of the pure model. Note that in the case \( J^{(l,k)}_0 = 0 \) for \( l \neq k \), \( \tilde{\chi}^{(l,k)}_0 \) is a diagonal matrix whereas \( \tilde{\chi}^{(\Sigma)} \) is not. By looking for the points where \( \tilde{\chi}^{(\Sigma)} \) becomes singular, from equation (4) we see immediately that the critical surface \( \beta_c^{(\Sigma)} \) where the second-order transition takes place is a solution of the following exact equation

\[
\det \left( 1 - \tilde{\chi}^{(l,k)}_0 \cdot \beta J^{(\Sigma)} \right) = 0. \tag{5}
\]

By sending \( \beta \to \infty \), the theory can be projected in particular at zero temperature where, for positive couplings, a natural effective percolation theory arises. Then, by using \( \lim_{\beta \to +\infty} \beta J^{(\Sigma)} = c \), in the limit \( \beta \to \infty \) equation (4) becomes

\[
\mathcal{E} = (1 - \mathcal{E}_0 \cdot c)^{-1} \cdot \mathcal{E}_0, \tag{6}
\]

where we have introduced

\[
\mathcal{E}_0^{(l,k)} \equiv \lim_{\beta \to +\infty} \tilde{\chi}^{(l,k)}_0 \left( \{ \beta J^{(l',k')}_0 \}; 0 \right). \tag{7}
\]

Equation (6) tells us how \( \mathcal{E} \) changes as we vary \( c \), being an exact equation when \( c \) belongs to the P region, i.e. below the percolation threshold \( c_c \), which is solution of the following exact equation \(^3\)

\[
\det(1 - \mathcal{E}_0 \cdot c) = 0. \tag{8}
\]

At \( T = 0 \) there is no thermal dissipation and it is easier to analyze the communication properties. Given the arbitrary pure graph \( (\mathcal{L}_0, \Gamma_0) \), and a community structure assignment which splits the set of the bonds \( \Gamma_0 \) in \( n(n-1)/2 \) sets, \( \Gamma_0 = \cup_{l,k} \Gamma^{(l,k)}_0 \), in the pure graph the communities \( l \) and \( k \) communicate if and only if \( \mathcal{E}_0^{(l,k)} \neq 0 \), whereas in the random graph they communicate if and only if \( \mathcal{E}^{(l,k)} \neq 0 \). We can understand the communication process by observing that the characteristic time \( t^{(l,k)}_0 \) to exchange a unit of information between the two communities \( l \) and \( k \) in the pure model grows as \( t^{(l,k)}_0 \propto (\mathcal{E}_0^{(l,k)})^{-1} \) and, similarly, for the random model as \( t^{(l,k)} \propto (\mathcal{E}^{(l,k)})^{-1} \). From equation (6) we see that in the pure model, if, for \( l \neq k \), \( \mathcal{E}_0^{(l,k)} = 0 \), the two communities \( l \) and \( k \) cannot communicate \( (t^{(l,k)}_0 \to \infty) \), but for any arbitrary small \( c^{(l,k)} > 0 \) they communicate and the characteristic time decays with \( c \) approximately as (it is easy to see that for any \( l \) is always \( \mathcal{E}_0^{(l,l)} \geq 1 \)) \(^4\)

\[
t^{(l,k)} \propto (\mathcal{E}_0^{(l,l)} c^{(l,k)} \mathcal{E}^{(k,k)}_0 + O(c^2))^{-1}, \tag{9}
\]

whereas, at higher order in \( c \), equation (6) takes into account that the communities \( l \) and \( k \) can also communicate indirectly via chains of other intermediate communities. In general, if the pure graph has its own dimension \( d_0 \) (possibly fractal) that is sufficiently

\(^3\) In equation (7) it is understood that we are considering only graphs \( (\mathcal{L}_0, \Gamma_0) \) for which the pure model has no critical temperature even for \( T = 0 \). In fact, if this is not the case, as occurs for instance if \( (\mathcal{L}_0, \Gamma_0) \) is a \( d_0 \)-dimensional lattice with \( d_0 \geq 1 \), the P region is shrunk to the single trivial point \( c^{(l,k)} \equiv 0 \) and we can effectively take \( \mathcal{E}_0^{(l,k)} = +\infty \) [10]. More precisely, if \( \beta_0 < \infty \) or even if \( \beta_0 = 0 \), as occurs in scale free networks with the exponent \( \gamma \leq 3 \) [17], one should use equations (7) keeping \( N \) finite.

\(^4\) Analogous relations also hold at finite \( T \), provided the source of the signal is sufficiently slow.
high, $d_0 \geq 1$, one has $E_0^{(l,k)} \rightarrow \infty$ for $N \rightarrow \infty$, so that we have $t_0^{(l,k)} \rightarrow 0$ and then also $t^{(l,k)} \rightarrow 0$; i.e. the communities communicate instantaneously (they percolate). However, in the random model, even if $E_0^{(l,k)}$ is finite when $c$ approaches the percolation threshold surface $c_c$, then we have $t^{(l,k)} \rightarrow 0$, a peculiar feature which is possible only in the random model. We stress again that the graph $(\mathcal{L}_0, \Gamma_0)$ is completely arbitrary. So, for example, for $n = 1$ it is easy to check the consistency of our effective percolation theory (i.e. we recover the same critical surface) with the recent model introduced in [18] where the classical Erdős–Rényi random graph [19] is generalized to include finite clustering. However, whereas in [18] the percolation analysis is performed by ab initio calculations starting directly from graph theory elements (at least this seems in principle possible for graphs having regular loops), our effective percolation theory requires us to perform simulated annealing toward $T = 0$ of the nonrandom model defined over the pure graph $(\mathcal{L}_0, \Gamma_0)$ and immersed in a small external field. At each small but finite $T$ we make the simulation, then, once the relevant quantities like $\chi_0$ or $m_0$ are obtained for small $T$, the percolation properties of the random graph $(\mathcal{L}, \Gamma)$ are easily calculated. Therefore, in our effective percolation theory, in equation (6) the matrix $E_0$ represents an input datum. In general, it can be sampled efficiently by simple simulated annealing procedures using equation (7), since the problem is mapped to an unfrustrated Ising model ($\beta J_0^{(l,k)} \geq 0$).

The matrix $E_0$ (or, more generally, at finite $T$ the matrix $\chi$), leads also to a natural new criterion to detect community structures: given a hypothetical value $n$, by a suitable generalization of the modularity introduced by Girvan and Newman [2] which makes use of $E_0$ rather than that of the adjacency matrix, we can define a ‘measure’ which takes into account paths of arbitrary length rather than links, and the resulting community structure coincides with that partition such that communication among all the communities is minimal [10]. In the algorithm proposed in [2], given the network, one removes the link having the highest betweenness, where the concept of betweenness, a measure of the centrality of the given link, can be defined in several ways. In particular the betweenness of a link can be defined as the number of geodesic paths (the shortest path connecting two sites) passing through it. After deleting many times the links having the highest betweenness, a partition of the original network in communities can be obtained and a measure of the quality of assignment in communities is given as

$$Q = Q_1 = \sum_l \left[ e_1^{(l,l)} - (a_1^{(l)})^2 \right],$$

(10)

where $e_1^{(l,k)}$ is the fraction of all bonds connecting the communities $l$ and $k$, and $a_1^{(l)}$ is defined as $a_1^{(l)} \equiv \sum_k e_1^{(l,k)}$. The term $(a_1^{(l)})^2$ in equation (10) represent the expected fraction of bonds falling inside the community $l$ when their ends are connected randomly. Thanks to the presence of the term $(a_1^{(l)})^2$ in equation (10), $Q_1$ gives a measure of 0 when one considers the trivial case in which $\Gamma_0$ is a single community ($n = 1$), and partitions that maximize $Q_1$ correspond to the best community structures. However, we can consider other similar measures that take into account not only bonds, but also, for example, paths of two consecutive bonds. In general we can define

$$Q_h = \sum_l \left[ e_h^{(l,l)} - (a_h^{(l)})^2 \right],$$

(11)

$$\text{doi:10.1088/1742-5468/2009/08/L08004}$$
we have found

However, for simplicity, in the following we will consider only the case

path affects the susceptibility, while at

more important and interesting at finite temperature, where the length of each single

with each other randomly with inter-average connectivities

that of Newman and Girvan for the case of geodesic betweenness (this can be seen from

the combinatorial meaning of the matrices $E_l^{(l,k)}$ or $E_{(l,k)}$ [10]), but the measure associated

with the found partition is given by using $Q_\infty$, and not $Q_1$: essentially $Q_1$ measures the

relative lack of links among different communities, while $Q_\infty$ measures the relative lack of

communication among different communities.

We point out that, in the pure model, having some bonds between the $l$th and $k$th

communities does not guarantee that the condition $\mathcal{E}_l^{(l,k)} > 0$ is satisfied. In fact, it is not
difficult to see that to have $\mathcal{E}_l^{(l,k)} > 0$ it is necessary that the number of paths between

the $l$th and the $k$th communities be at least of order $N$ [10]. Note also that such a

requirement does not exclude the possibility that even a single bond between the two

communities is enough, provided that through this bond there pass at least $O(N)$ paths

(high betweenness, or centrality; see figure 1). It should then be clear that measures

based on a local analysis, and on elementary use of the adjacency matrix, cannot capture

the real communication properties. We illustrate now in a simple example, analytically

feasible in our approach, how remarkable the differences can be in taking into account

just links or, more properly, paths of any length. Such differences actually become much

more important and interesting at finite temperature, where the length of each single path

affects the susceptibility, while at $T = 0$ the length of a path does not play any role.

However, for simplicity, in the following we will consider only the case $T = 0$. Suppose

we have found $n$ Erdős–Rényi sets, having intra-average connectivities $c^{(l,l)}$ and connected

with each other randomly with inter-average connectivities $c^{(l,k)}$, $l \neq k$. In this case, we

have $x_0 = 1$; therefore, from equations (6) and (7), we get immediately $\mathcal{E} = (1 - c)^{-1}$,

which is very different from the adjacency matrix $c$. This can be well understood by

observing that, below the percolation threshold surface given by equation (8), it holds

$\mathcal{E} = (1 - c)^{-1} = 1 + c + c^2 + \ldots$, where each term in the sum takes into account the

presence of paths of length 0, 1, 2, ..., respectively. Let us apply this result to the following

six examples with $n = 2$ and 3 communities:

c_1 = \begin{pmatrix} 0.35 & 0.3 \\ 0.3 & 0.35 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0.4 & 0.3 \\ 0.3 & 0.3 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0.6 & 0.3 \\ 0.3 & 0.1 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.0 \end{pmatrix},

and

c_5 = \begin{pmatrix} 0.4 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{pmatrix}, \quad c_6 = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.0 \end{pmatrix}.

In all the above examples the matrix $c$ is always normalized, so that we always have $e = c$. The matrix $e_\infty$ coincides with the matrix $\mathcal{E} = (1 - c)^{-1}$ to be normalized to 1.

The results corresponding to the above six examples, together with the modularities $Q$ and $Q_\infty$, and the maximum eigenvalue $\lambda$ of the matrix $c$, give, respectively:

e_{\infty,1} = \begin{pmatrix} 0.406 & 0.094 \\ 0.094 & 0.406 \end{pmatrix}, \quad Q = 0.2, \quad Q_\infty = 0.312, \quad \lambda = 0.650.

Small-world of communities: communication and correlation of the meta-network

\[ e_{\infty;2} = \begin{pmatrix} 0.437 & 0.094 \\ 0.094 & 0.375 \end{pmatrix}, \quad Q = 0.195, \quad Q_{\infty} = 0.310, \quad \lambda = 0.654, \]

\[ e_{\infty;3} = \begin{pmatrix} 0.562 & 0.094 \\ 0.094 & 0.250 \end{pmatrix}, \quad Q = 0.007, \quad Q_{\infty} = 0.264, \quad \lambda = 0.740, \]

\[ e_{\infty;4} = \begin{pmatrix} 0.625 & 0.094 \\ 0.094 & 0.187 \end{pmatrix}, \quad Q = -0.045, \quad Q_{\infty} = 0.217, \quad \lambda = 0.810, \]

and

\[ e_{\infty;5} = \begin{pmatrix} 0.333 & 0.023 & 0.042 \\ 0.023 & 0.249 & 0.032 \\ 0.042 & 0.032 & 0.221 \end{pmatrix}, \quad Q = 0.320, \quad Q_{\infty} = 0.463, \quad \lambda = 0.487, \]

\[ e_{\infty;6} = \begin{pmatrix} 0.436 & 0.027 & 0.049 \\ 0.027 & 0.191 & 0.025 \\ 0.049 & 0.025 & 0.172 \end{pmatrix}, \quad Q = 0.160, \quad Q_{\infty} = 0.418, \quad \lambda = 0.640. \]

From these examples we observe the following: (i) as expected \( Q \) and \( Q_{\infty} \) tend to be ‘parallel’ measures, but they are quite far from being proportional; (ii) from the point of view of communication, these six networks do not present very strong differences (we expect stronger differences in other kinds of networks embedded in some geometries; as in the case depicted in figure 1); (iii) as we pass from case 1 to the case 4, and similarly from case 5 to case 6, i.e. toward more and more asymmetric cases, both \( Q \) and \( Q_{\infty} \) take lower and lower values, meaning that the partition has a poor meaning (\( Q \) small), or that the global communication among the communities is higher (\( Q_{\infty} \) small). This latter observation is less trivial and deserves attention: as a rule, asymmetric situations benefit from better communication. Consider in particular cases 5 and 6 and compare the matrix \( e \) (or \( e_{\infty} \)) with the matrix \( e_{\infty} \): despite \( e_{(1,2)}^{(1)} \) and \( e_{(1,3)}^{(1)} \) being equal, we have that \( e_{(1,3)}^{(1)} \) is almost twice \( e_{(1,2)}^{(1)} \). This is due to the fact that, between the second and third communities, the latter benefits from a stronger asymmetry with respect to the first community, which in turn has the largest density of intrabonds. For the same reason, in case 6 we note also that the matrix element \( e_{(3,3)}^{(3)} \) is of the same order of magnitude as \( e_{(3,3)}^{(2)} \), despite \( e_{(3,3)}^{(3)} \) being equal to 0.

5. Conclusions

Real-world networks present an intrinsic partition in communities. However, despite important progress, the absence of universality of the many proposed techniques makes community detection an ‘art’ rather than a solid science [20]. One weak point of these techniques lies in the fact that in most cases only the topology (i.e. the structure of the graph)—and often only a local topology—is taken into account and correlations are never introduced. In particular, the real communication properties among the communities cannot rely on a local analysis. On the other hand, starting from real data it is possible to define in an unambiguous way a minimal model, a disordered Ising model, able to take into account all the correlations, both short- and long-range, present on the given network. We then discover that, whatever the given community structure, the exact relationship of the meta-nodes is regulated by a quite universal form of effective TAP equations which, through F- and SG-like order parameters and then the matrix \( \chi \) of the
relative susceptibilities (in general completely different from the adjacency matrix), give rise to a rich variety of configuration and communication scenarios which are analytically and/or numerically feasible. In particular, by simulated annealing procedures applied to the nonrandom model \((c = 0)\), equations (6)–(9) allow us to analyze how the percolation and communication properties of the system change when it becomes random \((c \neq 0)\). We can also use the same technique to analyze the nonrandom graphs themselves for both studying the communication properties and the community structure. We stress that there is no limitation to the choice of the nonrandom graph \((L_0, \Gamma_0)\), and in particular, as occurs in real-world networks, it can contain loops of any length, which are situations where the traditional mean-field theories developed for disordered systems (that suppose a tree-like structure of the graph) cannot be applied. Our approach is exact in the \(P\) region (where, however, we recall correlations of two spin neighbors of the same spin are not zero due to the presence of short loops) and on its boundaries (the critical surface). The possibility to improve the method off the \(P\) region is under investigation. We finally anticipate here that it is possible to generalize all the results to cases of small-world models with free-scale intra- and inter-connections among communities [21].

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