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To cite this article: Moongyu Park and John H Cushman J. Stat. Mech. (2009) P02039

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Operator-stable Lévy motions and renormalizing the chaotic dynamics of microbes in anisotropic porous media

Moongyu Park\textsuperscript{1} and John H Cushman\textsuperscript{2,3}

\textsuperscript{1} Department of Mathematical Sciences, University of Alabama in Huntsville, Huntsville, AL 35899, USA
\textsuperscript{2} Department of Earth and Atmospheric Sciences, Purdue University, West Lafayette, IN 47907, USA
\textsuperscript{3} Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
E-mail: mp0002@uah.edu and jcushman@purdue.edu

Received 6 November 2008
Accepted 24 December 2008
Published 16 February 2009

Abstract. Self-motile particles, such as microbes, that can preferentially orient their microscale movement are embedded in a multiscale anisotropic porous medium. Such an environment is consistent with most natural geological systems. The preferential directional orientation and position of the self-motile particles at the microscale is accounted for with an operator-stable Lévy process. The anisotropy of the porous medium on the mesoscale is accounted for with an operator-stable velocity with either finite or infinite second moments. Upscaling is accomplished with generalized central limit theorems which can be shown to be equivalent to a renormalized group approach. If the mesoscale drift is operator-stable Lévy (fractal), then the macroscale Fokker–Planck equation has time-dependent diffusion tensor and fractional derivatives which are directionally dependent.

Keywords: stochastic particle dynamics (theory), diffusion
1. Introduction

The transport of colloids through porous media is of fundamental importance to any science involving chromatography, as well as the environmental and pharmaceutical sciences. Here we focus on the special class of colloids that are self-motile such as microbes; however the work is just as applicable, in a slightly simplified form, to non-motile particles. As will be seen, the non-motile case is simply the limit as the motility, represented by the stability parameter $\alpha \in (0, 2]$, goes to 2. While microbes can propel themselves by many means, the driving mechanism most commonly studied [1] is a series of rotary motors that spin flagella in phase in a common direction. Periodically, one of these motors will reverse direction and its flagella will entangle the remaining flagella thus stopping and rotating the bug. After the motors have realigned themselves, the microbe heads off in a random direction. Runs are generally longer in the direction of an energy (light, chemical, heat, magnetic) source, and thus effectively the microbe appears to swim towards its food. The resultant swimming process looks like a random walk in three dimensions. As in [2], we assume that the motility of a microbe can be described in terms of an operator-stable Lévy motion on the pore scale (microscale). This assumption allows the bug to preferentially run down energy gradients within the pore. Lévy processes are characterized by very heavy tails with infinite second moment. In addition, their probability densities generally have to be defined via their characteristic functions (i.e., the Fourier transform of the probability density) and consequently most of the subsequent analysis is carried out in Fourier space. The mathematical theory of operator-stable Lévy motions can be found in [3–7], and their use in portfolio modeling can be found in [5], [8–10]. The operator-stable Lévy diffusion is coupled with a microscale drift that represents the mean pore water velocity, which is assumed stationary, ergodic and Markovian.

We restrict the analysis to the non-absorptive case, but generalize previous work [2, 11] to handle anisotropic porous media. This generalization is straightforward, but of critical importance, because almost all natural porous media are anisotropic in three dimensions. It has been known for over a century that most geological formations are both
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heterogeneous and anisotropic and this has been graphically illustrated by recent large-scale tracer tests [12, 13]. Most recently, investigators have suggested that heterogeneity can be adequately modeled using a fractal analysis [2, 11, 14]. The main difference between the approach presented here and the previous analysis is that here we allow components of the mesoscale Lagrangian velocity to have different fractal dimensions in the principal directions of flow. This effectively allows for the representation of fractal anisotropy in the Darcy conductivity. Specifically, we assume that dispersion on the mesoscale (the regional homogenization of pore and solid space) can be represented by the asymptotics of the microscale convection plus pore diffusion, coupled to the Lagrangian Darcy drift velocity. To set the stage for the fractal mesoscale velocity, we first illustrate the renormalization process in the non-fractal case.

The macroscale porous formation is assumed periodic and transport of the colloid on this scale is given by the renormalization of the mesoscale drift with the mesoscale dispersive flux. There is no macroscale drift that is distinct from the macroscale dispersive flux. The goal is to be able to write Fokker–Planck equations on each of the three scales considered. This is done by first deriving the stochastic ordinary differential equations (SODEs) on the mesoscale and on the macroscale via application of generalized central limit theorems. This approach can be shown to be equivalent to a rigorous renormalization group approach. The Fokker–Planck equations follow directly once the SODEs are obtained on each scale. These equations are in general fractional with different fractal dimensions in the principal directions.

2. Operator-stable Lévy processes

An infinitely divisible distribution $\mu$ [15] on $\mathbb{R}^d$ is called operator-stable [3, 7, 15] if it has the property that for any $a > 0$, there is an invertible $d \times d$ matrix $Q = a^G$ and $c \in \mathbb{R}^d$ such that

$$\hat{\mu}(k)^a = \hat{\mu}(Q^T k) e^{ic \cdot k}, \quad (2.1)$$

where $a^G$ is defined by the series $\sum_{m=0}^{\infty} (\ln a)^m G^m / (m!)$, $Q^T$ is the transpose of the matrix $Q$, $c \cdot k$ is the dot product in $\mathbb{R}^d$, and $\hat{\mu}(k)$ is the characteristic function of $\mu$:

$$\hat{\mu}(k) = \int_{\mathbb{R}^d} e^{ik \cdot x} \mu(dx), \quad k \in \mathbb{R}^d.$$ 

The matrix $G$ is called an exponent for $\mu$. If $G = (1/\alpha) I$ with the $d \times d$ identity matrix $I$, then $\mu$ is a $d$-dimensional $\alpha$-stable distribution [15, 16]. A stochastic process \{V(t) | t ≥ 0\} is called an operator-stable Lévy process if it is a Lévy process [15] and its increments have operator-stable distributions. Because of (2.1), an operator-stable Lévy process $V(t)$ has a scaling law: for any $a > 0$ and $t > 0$ there is a $d \times d$ matrix $G$ and $c(t) \in \mathbb{R}^d$ such that

$$V(at) \overset{d}{=} a^G V(t) + c(t), \quad (2.2)$$

where $\overset{d}{=} \text{means equal in distribution.}$
where \( \{ V(r) \} \) and \( \{ L(t) \} \) are operator-stable Lévy processes. Subsequently, this will be our model of microbial dynamics at the mesoscale for an anisotropic fractal porous medium. We present a brief analysis of this Lagrangian equation and take a look at its Eulerian counterpart. Let \( \tilde{V}(t) = V(t) - V(0) \) and \( \tilde{X}(t) = X(t) - X(0) \). The characteristic function \( \phi_v \) of \( \tilde{V}(t) \sim (tA_v, t\nu_v, t\mu_v) \), called the generating triplet \([3,15]\) of \( \phi_v \), is
\[
\phi_v(t,k) = \exp \left[ -\frac{t}{2} k \cdot A_v \cdot k + t \hat{B}(k) + it k \cdot \mu_v \right],
\]
where \( t\mu_v = E[\tilde{V}(t)] \) is the expected value of \( \tilde{V}(t) \), \( A_v \) is a symmetric non-negative definite \( 3 \times 3 \) matrix, \( \nu_v \) is the Lévy measure \([3,15]\) for \( \phi_v \), and
\[
\hat{B}(k) = \int_{S_3} \int^\infty_0 \left( e^{ik \cdot (rGs)} - 1 - i k \cdot (rGs) \right) \frac{dr}{r^2} \nu_v(ds).
\]
\( S_3 = \{ s \in \mathbb{R}^3 : |s| = 1 \} \) and \( \nu_v \) is the finite measure \([3,15]\) given by \( \nu_v(F) \equiv \nu_v(\{ rGs : r \in [1, \infty), s \in F \}) \) for a Borel subset \( F \) of \( S_3 \). We will use the index, \( \ell \), to represent the characteristic function of \( (L(t) - L(0)) \). If \( \nu_v = 0 \), then \( \tilde{V}(t) \) is Brownian, which is a special case in \([11,14]\). We assume \( \nu_v \neq 0 \) in this paper.

Let
\[
Y(t) = \int_0^t V(r) dr,
\]
and decompose \( Y(t) \) as
\[
Y(t) = \int_0^t \left( V(0) + \tilde{V}(r) \right) dr = tV(0) + \tilde{Y}(t),
\]
where \( \tilde{Y}(t) = \int_0^t \tilde{V}(r) dr \). Since Lévy processes have a countable number of discontinuities \([17]\), the time integral of \( \tilde{Y}(t) \) may be considered in the Riemann sense:
\[
\tilde{Y}(t) = \lim_{n \to \infty} \sum_{j=1}^n \frac{t}{n} \tilde{V}(r_j),
\]
where \( 0 = r_0 < r_1 < \cdots < r_n = t \) and \( r_j - r_{j-1} = (t/n) \) for \( j = 1, 2, \ldots, n \). Following \([14]\), we compute the characteristic function \( \phi_y(t,k) = E[\exp(i k \cdot \tilde{Y}(t))] \) of \( \tilde{Y}(t) \):
\[
\phi_y(t,k) = \exp \left[ -\frac{t^3}{6} k \cdot A_v \cdot k + t \hat{H}(t,k) + \frac{t^2}{2} ik \cdot \mu_v \right],
\]
where
\[
\hat{H}(t,k) = \int_{S_3} \int^\infty_0 \left( \frac{e^{ik \cdot (rGs)} - 1}{it k \cdot (rGs)} - 1 - \frac{1}{2} ik \cdot (rGs) \right) \frac{dr}{r^2} \nu_v(ds).
\]
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Let us next look at the Eulerian perspective, i.e. the Fokker–Planck equations for $\tilde{V}(t)$ and $\tilde{Y}(t)$. Let $f_v(t, x)$ and $f_y(t, x)$ be the density functions of $\tilde{V}(t)$ and $\tilde{Y}(t)$, respectively. By differentiating $\hat{f}_y$ with respect to $t$, we have

$$\frac{\partial \hat{f}_y}{\partial t} = \left[ -\frac{t^2}{2} \mathbf{k} \cdot \mathbf{A}_v \cdot \mathbf{k} + \hat{H}(t, \mathbf{k}) + t \frac{\partial \hat{H}}{\partial t}(t, \mathbf{k}) + t i \mathbf{k} \cdot \mathbf{\mu}_v \right] \hat{f}_y(t, \mathbf{k}).$$

It is now easy to show that $\hat{H}(t, \mathbf{k}) + t(\partial \hat{H}/\partial t)(t, \mathbf{k}) = \hat{B}(t \mathbf{k})$ and so by the inverse Fourier transform we arrive at

$$\frac{\partial f_y}{\partial t} = \left( \frac{t^2}{2} \nabla \cdot \mathbf{A}_v \cdot \nabla + \mathcal{J} \right) f_y + t \mathbf{\mu}_v \cdot \nabla f_y, \quad (2.8)$$

where the operator $\mathcal{J}$ is defined in terms of the Fourier transform:

$$\mathcal{J} \hat{f}_y(t, \mathbf{k}) \equiv \hat{B}(t \mathbf{k}) \hat{f}_y(t, \mathbf{k}). \quad (2.9)$$

Similarly we find

$$\frac{\partial f_v}{\partial t} = \left( \frac{1}{2} \nabla \cdot \mathbf{A}_v \cdot \nabla + \mathcal{B} \right) f_v + \mathbf{\mu}_v \cdot \nabla f_v, \quad (2.10)$$

with $\mathcal{B} f_v(t, \mathbf{k}) \equiv \hat{B}(\mathbf{k}) \hat{f}_v(t, \mathbf{k})$.

3. Upscaling from microscale to mesoscale

On the microscale we assume that the behavior of microbes depends on the direction and so the integrated SODE is of the form

$$\mathbf{X}^{(0)}(t) = \mathbf{X}^{(0)}(0) + \int_0^t \mathbf{V}^{(0)}(r) \, dr + \mathbf{L}^{(0)}(t), \quad t \geq 0, \quad (3.1)$$

where $\mathbf{V}^{(0)}(t)$ is assumed stationary, ergodic and Markovian with the mean $\tilde{\mathbf{V}}^{(0)}$, and $\mathbf{L}^{(0)}(t)$ is an operator-stable Lévy process ($\mathbf{L}^{(0)}(t) \sim (t \mathbf{A}_v^{(0)}, t \mathbf{v}_v^{(0)}, t \mathbf{\mu}_v^{(0)})$) which corresponds to the microbes’ motility. Here, the superscript $(0)$ means microscale which corresponds to the pore scale. (We use $(1)$ for mesoscale and $(2)$ for macroscale.)

The following convergence is proved in [2]:

$$\frac{\mathbf{X}^{(0)}(t) - \mathbf{X}^{(0)}(0) - t \mathbf{\mu}_v^{(0)} + \mathbf{L}_1^{(0)}(t)}{\sqrt{t}} \overset{d}{\to} \mathbf{L}_1^{(1)}(t), \quad (3.2)$$

whose characteristic function, $\psi_{\mathbf{L}_1}$, is

$$\psi_{\mathbf{L}_1}(t, \mathbf{k}) = \exp \left[ -\frac{t}{2} \mathbf{k} \cdot \mathbf{A}_v^{(1)} \cdot \mathbf{k} - \frac{t}{2} \int_0^\infty \int_{S_3} \int_0^\infty (\mathbf{k} \cdot \mathbf{r}_s^G) \frac{d \mathbf{r}_s}{\mathbf{r}_s^2} M_{\mathbf{L}_1}(d \mathbf{s}) \right]. \quad (3.3)$$

doi:10.1088/1742-5468/2009/02/P02039
4. Upscaling from the mesoscopic to the macroscopic in the non-fractal limit

Before proceeding to the case of most interest, we analyze a simpler problem: that of a non-fractal mesoscale velocity. We assume the random drift velocity at the mesoscale, \( \mathbf{V}^{(1)}(t) \), to be operator-stable with finite second moment and constant mean:

\[
E(\mathbf{V}^{(1)}(t)) = E(\mathbf{V}^{(1)}(0)) = \mathbf{1}^{(1)}
\]

for any \( t \geq 0 \). \hfill (4.1)

Then since \( E(\mathbf{V}^{(1)}(t)) = 0 \) for each time \( t \), \( \boldsymbol{\mu}_v = 0 \). Let

\[
|c_{jm}| \equiv \left| \int_{R^3} u_j u_m \nu_v(du) \right| < \infty, \quad j, m = 1, 2, 3,
\]

which means \( E(|\mathbf{V}^{(1)}(t)|^2) < \infty \). We upscale \( \mathbf{Y}^{(1)}(t) \) first and subsequently \( \mathbf{X}^{(1)}(t) \). Let \( t = \lambda t' \) so that

\[
\mathbf{Y}^{(1)}(\lambda t') - \lambda t' \mathbf{1}^{(1)} = \lambda^{-1/2} t'(\mathbf{V}^{(1)}(0) - \mathbf{1}^{(1)}) + \lambda^{-3/2} \mathbf{1}^{(1)}(\lambda t'). \quad (4.3)
\]

Since \( \lambda^{-1/2} t'(\mathbf{V}^{(1)}(0) - \mathbf{1}^{(1)}) \) converges to zero almost surely (a.s.) as \( \lambda \to \infty \), it also converges in distribution to zero. For the second term on the right-hand side of (4.3) we compute the natural logarithm of its characteristic function:

\[
\ln E \left[ \exp \left( \mathbf{i} \lambda^{-3/2} \mathbf{k} \cdot \mathbf{Y}^{(1)}(\lambda t') \right) \right] = -\frac{(t')^3}{6} \mathbf{k} \cdot \mathbf{A}_v \cdot \mathbf{k}
\]

\[
+ t' \int_{S_{d_1}} \int_0^\infty \lambda \left( \frac{\exp(i\lambda^{-1/2} t' \mathbf{k} \cdot (r G s)) - 1}{i\lambda^{-1/2} t' \mathbf{k} \cdot (r G s)} - 1 \right) \frac{dr}{r^2} M_\nu(ds)
\]

\[
- t' \int_{S_{d_1}} \int_0^\infty i\lambda \left( \frac{\lambda^{-1/2} t' \mathbf{k} \cdot (r G s)}{2} \right) \frac{dr}{r^2} M_\nu(ds)
\]

\[
= -\frac{(t')^3}{6} \mathbf{k} \cdot \mathbf{A}_v \cdot \mathbf{k} + t' \int_{S_{d_1}} \int_0^\infty \lambda G_1(\lambda, \mathbf{k}, r, s) \frac{dr}{r^2} M_\nu(ds)
\]

\[
- it' \int_0^\infty \int_{\{k \cdot (r G s) > 0\}} \lambda G_2(\lambda, \mathbf{k}, r, s) M_\nu(ds) \frac{dr}{r^2}
\]

\[
- it' \int_0^\infty \int_{\{k \cdot (r G s) < 0\}} \lambda G_2(\lambda, \mathbf{k}, r, s) M_\nu(ds) \frac{dr}{r^2},
\]

where

\[
G_1(\lambda, \mathbf{k}, r, s) = \frac{\sin(\lambda^{-1/2} t' \mathbf{k} \cdot (r G s))}{\lambda^{-1/2} t' \mathbf{k} \cdot (r G s)} - 1,
\]

\[
G_2(\lambda, \mathbf{k}, r, s) = \frac{\cos(\lambda^{-1/2} t' \mathbf{k} \cdot (r G s))}{\lambda^{-1/2} t' \mathbf{k} \cdot (r G s)} - 1 + \frac{1}{2} \lambda^{-1/2} t' \mathbf{k} \cdot (r G s).
\]

Observe that the integrands \( \lambda G_1(\lambda, \mathbf{k}, r, s) \) and \( \lambda G_2(\lambda, \mathbf{k}, r, s) \) are monotone functions. Therefore, by the monotone convergence theorem, \( \ln E[\exp(\mathbf{i} \lambda^{-3/2} \mathbf{k} \cdot \mathbf{Y}^{(1)}(\lambda t'))] \) converges to

\[
-\frac{(t')^3}{6} \mathbf{k} \cdot \mathbf{A}_v \cdot \mathbf{k} + \frac{(t')^3}{6} \int_{S_{d_1}} \int_0^\infty (\mathbf{i} \mathbf{k} \cdot (r G s))^2 \frac{dr}{r^2} M_\nu(ds)
\]
as \( \lambda \to \infty \). Thus

\[
\frac{\bar{Y}^{(1)}(\lambda t') - \lambda t' \bar{V}^{(1)}}{\lambda^{3/2}} \overset{d}{\longrightarrow} W(t'), \quad \lambda \to \infty,
\]

(4.4)

where the characteristic function \( \phi_w \) of \( W(t') \) is

\[
\phi_w(t', k) = \exp \left[ -\frac{(t')^3}{6} \left( k \cdot \mathbf{A}_v \cdot k + \int_{S_d} \int_0^\infty (k \cdot (r^G s))^2 \frac{dr}{r^2} M_v(ds) \right) \right].
\]

Let \( \bar{Y}^{(2)}(t') = \lim_{\lambda \to \infty} \bar{Y}^{(1)}(\lambda t') \) so that

\[
\bar{Y}^{(2)}(t') \approx t' \bar{V}^{(1)} + \bar{Y}^{(2)}(t').
\]

(4.5)

We assume periodicity of \( V^{(1)}(t) = V^{(1)}(t, \xi^{(1)}(0)) \) in the initial data,

\[
V^{(1)}(t, \xi^{(1)}(0) + \nu^{(1)}) = V^{(1)}(t, \xi^{(1)}(0)),
\]

where \( \nu^{(1)} = (\nu_1^{(1)}, \nu_2^{(1)}, \nu_3^{(1)}) \) for integers \( \nu_j^{(1)} \), \( j = 1, 2, 3 \), and \( \xi^{(1)}(\cdot) = \xi^{(1)}(\cdot, \omega) \) is a random particle path in the porous medium with \( \omega \) an elementary path. Hence, for each \( \lambda \to \infty \), we can approximate the stochastic process \( \tilde{X}^{(1)}(t) = \tilde{X}^{(1)}(0) + \int_0^t \mathbf{V}(\xi^{(0)}, \omega) \cdot d\xi^{(0)}(s) \) as

\[
\tilde{X}^{(1)}(t) \approx \tilde{X}^{(1)}(0) + \int_0^t \mathbf{V}(\xi^{(0)}(\omega)) \cdot d\xi^{(0)}(s).
\]

(4.6)

Equation (4.4) gives the convergence of the second term in (4.7). The first term obviously converges to zero. As in [2, 14], we can also show that the third term converges to zero in distribution. Therefore, \( Z^{(1)}(t') \) converges to \( W(t') \) in distribution and hence for each \( t > 0 \), we can approximate the stochastic process \( X^{(2)}(t) = \lim_{\lambda \to \infty} \bar{X}^{(1)}(\lambda t) \) as

\[
\bar{X}^{(2)}(t) \approx t([V^{(1)}] + \mu^{(1)}(t)) + \bar{Y}^{(2)}(t).
\]

(4.8)

From an Eulerian perspective we derive the Fokker–Planck (advection–dispersion) equation that the transition density \( f(t, x) \) for \( X^{(2)}(t) \) satisfies:

\[
\frac{\partial f}{\partial t} = -\mathbf{v}^{(2)} \cdot \nabla f + \nabla \cdot \mathbf{D}_v(t) \cdot \nabla f,
\]

(4.9)

where \( \mathbf{v}^{(2)} = [V^{(1)}] + \mu^{(1)}(t) \) and

\[
\mathbf{D}_v(t) = \frac{t^2}{2} (A_v^{(1)} + C_v).
\]

(4.10)

Each entry \( c_{jm} \) of the \( 3 \times 3 \) matrix \( C_v \) is defined in (4.2). The assumption of the finite second moment for the Lagrangian velocity process generates the classical Fokker–Planck equation with the time-dependent dispersion tensor \( D_v(t) \) at the macroscale.
5. Upscaling operator-stable Lévy drift velocity with infinite second moment

Here we consider the case where the porous medium induces a fractal velocity field. Let
\[ \mathbf{V}^{(1)}(t) = (V_1^{(1)}(t), V_2^{(1)}(t), V_3^{(1)}(t))^T \]
be an operator-stable Lévy process with constant mean, \( E(\mathbf{V}^{(1)}(t)) = \mathbf{V}^{(1)} \)
for any \( t \geq 0 \) and an exponent \( G = \text{diag}(1/\alpha_1, 1/\alpha_2, 1/\alpha_3) \).
This process was studied in [18]–[21]. Kunita [20] called it a dilation-stable process.
Pruitt and Taylor [21] and Hendricks [18, 19] considered the case where the components
\( V_1^{(1)}(t), V_2^{(1)}(t), V_3^{(1)}(t) \) are independent. Each component \( V_j^{(1)}(t) \) is an
\( \alpha_j \)-stable Lévy process [15] with divider dimension \( \alpha_j \). The characteristic function of
\( \tilde{\mathbf{V}}^{(1)}(t) = (\tilde{V}_1^{(1)}(t), \tilde{V}_2^{(1)}(t), \tilde{V}_3^{(1)}(t))^T \)
is
\[
E[\exp(\mathbf{i} \mathbf{k} \cdot \tilde{\mathbf{V}}^{(1)}(t))] = E \left[ \exp \left( \sum_{j=1}^{3} k_j \tilde{V}_j^{(1)}(t) \right) \right] = \prod_{j=1}^{3} E[\exp(\mathbf{i} k_j \tilde{V}_j^{(1)}(t))]
= \prod_{j=1}^{3} \exp \left[ -t \sigma_j^2 |k_j|^{\alpha_j} \left( 1 - i \beta_j \text{sgn}(k_j) \tan \left( \frac{\pi \alpha_j}{2} \right) \right) \right].
\]

Note that \( \tilde{\mathbf{V}}^{(1)}(t) \) has infinite second moment [15] because \( \sigma_j = \frac{\alpha_j}{2} < 2 \). We can compute
the natural logarithm of the characteristic function of \( \tilde{\mathbf{Y}}^{(1)}(t) \) in the same fashion as in section 2:
\[
\ln E[\exp(\mathbf{i} \mathbf{k} \cdot \tilde{\mathbf{Y}}^{(1)}(t))] = \sum_{j=1}^{3} \left[ -\frac{t^{1+\alpha_j} \sigma_j^{\alpha_j}}{1+\alpha_j} |k_j|^{\alpha_j} \left( 1 - i \beta_j \text{sgn}(k_j) \tan \left( \frac{\pi \alpha_j}{2} \right) \right) \right].
\]

Letting \( t = \lambda t' \), then we can show that
\[
Y^{(1)}(\lambda t') - \lambda t' \tilde{\mathbf{V}}^{(1)}(\lambda t) \xrightarrow{d} \Xi(t'),
\]
as \( \lambda \to \infty \), where the natural log of the characteristic function \( \phi_\Xi \) of \( \Xi(t') \) is
\[
\ln \phi_\Xi(t', \mathbf{k}) = \sum_{j \in I_o} \left[ -\frac{t^{1+\alpha_j} \sigma_j^{\alpha_j}}{1+\alpha_j} |k_j|^{\alpha_j} \left( 1 - i \beta_j \text{sgn}(k_j) \tan \left( \frac{\pi \alpha_j}{2} \right) \right) \right],
\]
with \( I_o = \{ j \mid \alpha_j = \alpha_o, \ j = 1, 2, 3 \} \). Let \( Y^{(2)}(t) = \lim_{\lambda \to \infty} Y^{(1)}(\lambda t) \); then for \( t > 0 \) we have
\[
Y^{(2)}(t) \approx t \tilde{\mathbf{V}}^{(1)} + \Xi(t).
\]
We also assume periodicity of \( \tilde{\mathbf{V}}^{(1)}(t) = \tilde{\mathbf{V}}^{(1)}(t, \xi^{(1)}(0)) \) in the initial data and use
\( [\tilde{\mathbf{V}}^{(1)}] \) in (4.6) to upscale from mesoscale to macroscale. Let
\[
Z^{(1)}(\lambda t') \equiv \frac{\dot{X}^{(1)}(\lambda t') - ([\mathbf{V}^{(1)}] + \mu^{(1)}_t) \lambda t'}{\lambda^{1+1/\alpha_o}} = \frac{t'([\tilde{\mathbf{V}}^{(1)}] + \Phi^{(1)} + \mathbf{L}^{(1)}(\lambda t') - \lambda t' \mu^{(1)}_t)}{\lambda^{1+1/\alpha_o}},
\]
where \( \Phi^{(1)} = \mu^{(1)}_t - [\mathbf{V}^{(1)}] \).

\[\text{doi:10.1088/1742-5468/2009/02/P02039}\]
Therefore, \( Z^{(1)}_\lambda(t') \) converges to \( \Xi(t) \) in distribution. For each \( t > 0 \), we can approximate the stochastic process which we label \( \tilde{X}^{(2)}(t) = \lim_{\lambda \to \infty} \tilde{X}^{(1)}(\lambda t) \) as

\[
\tilde{X}^{(2)}(t) \approx t([V^{(1)}] + \mu^{(1)}_i) + \Xi(t).
\] (5.4)

If \( \beta_j = 1 \) for \( j \in I_o \), then we can derive from (5.2) the fractional Fokker–Planck equation that the transition density \( f(t, x) \) for \( \tilde{X}^{(2)}(t) \) satisfies:

\[
\frac{\partial f}{\partial t} = -v^{(2)} \cdot \nabla f + \sum_{j \in I_o} D_j^{(2)}(t) \frac{\partial^{\alpha_j} f}{\partial x_j^{\alpha_j}},
\] (5.5)

where \( v^{(2)} = [V^{(1)}] + \mu^{(1)}_i \), \( D_j^{(2)}(t) = -t^{\alpha_1} \sigma_j^{\alpha_j} / \cos(\pi \alpha_j/2) \), and \( \partial^{\alpha_j} f / \partial x_j^{\alpha_j} \) is defined by

\[
\frac{\partial^{\alpha_j} f}{\partial x_j^{\alpha_j}}(t, k) = \left[ -k_j^{\alpha_j} \exp \left( i \operatorname{sgn}(-k_j) \pi \alpha_j/2 \right) \right] \tilde{f}(t, k).
\] (5.6)

Note that \( -k_j^{\alpha_j} \exp(i \operatorname{sgn}(-k_j)\pi \alpha_j/2) \) is one value of the multiple-valued \((-ik_j)^a\). We observe that in the case of infinite second moments, the classical derivatives seen earlier are replaced by fractional derivatives. The fractional order is the divider dimension of the velocity process in the principal directions.

Remark 5.1. Our approach can be extended to an operator-stable Lévy process \( V(t) \) with a diagonalizable exponent \( G = PDP^{-1} \), where \( P \) is an invertible matrix and \( D = \operatorname{diag}(1/\alpha_1, 1/\alpha_2, 1/\alpha_3) \), if \( P^{-1} V(t) \) has independent components, because for any \( a > 0 \) and \( t > 0 \),

\[
P^{-1} V(at) = a^D P^{-1} V(t) + P^{-1} c(t).
\] (5.7)

Remark 5.2. When \( L^{(0)}(t) = (L^{(0)}_1(t), L^{(0)}_2(t), L^{(0)}_3(t))^T \) is an operator-stable Lévy process with an exponent \( G_i = \operatorname{diag}(1/\alpha_{i1}, 1/\alpha_{i2}, 1/\alpha_{i3}) \) \((1 < \alpha_{ij} \leq 2, j = 1, 2, 3)\) and independent components at the microscale we can also upscale \( X^{(0)}(t) \) in (3.1) with a stationary, ergodic, and Markovian process, \( V^{(0)}(t) \), with the mean \( \tilde{V}(t) \) from microscale to mesoscale and from mesoscale to macroscale in the same fashion, and obtain the same Fokker–Planck equation (5.5).

6. Conclusion

While microbes can propel themselves by many means, the driving mechanism most commonly studied is a series of rotary motors that spin flagella in phase in a common direction. Periodically, one of these motors will reverse direction and its flagella will entangle the remaining flagella, thus stopping and rotating the bug. After the motors have realigned themselves, the microbe heads off in a random direction. Runs are generally longer in the direction of an energy (light, chemical, heat, magnetic) source, and thus effectively the microbe appears to swim towards its food. The resultant swimming process looks like an operator-stable Lévy motion on the pore scale (microscale). Lévy processes are characterized by very heavy tails with infinite second moment. The operator-stable Lévy diffusion was coupled with a microscale drift that represents the pore water velocity, which was assumed stationary, ergodic and Markovian.
The novelty of this work is that anisotropic fractal porous media are modeled on the mesoscale. This generalization is straightforward, but of critical importance, because almost all natural porous media are anisotropic in three dimensions. The main difference between the approach presented here and the previous analysis is that here we allow components of the mesoscale Lagrangian velocity to have different fractal dimensions in the principal directions of flow. This effectively allows for the representation of fractal anisotropy in the Darcy conductivity. Specifically, we assumed that dispersion on the mesoscale is represented by the asymptotics of the microscale convection plus pore diffusion, coupled to the Lagrangian Darcy drift velocity.

The macroscale porous formation was assumed periodic and transport of the colloid on this scale was given by the renormalization of the mesoscale drift with the mesoscale dispersive flux. There was no macroscale drift that is distinct from the macroscale dispersive flux.

Fokker–Planck equations were obtained on each of the three scales considered. This was accomplished by first deriving the stochastic ordinary differential equations (SODEs) on the mesoscale and macroscale via application of generalized central limit theorems. This approach can be shown equivalent to a rigorous renormalization group approach. The Fokker–Planck equations are in general fractional with different fractal dimensions in the principal directions.

Acknowledgment

JHC wishes to acknowledge the National Science Foundation for support under contract EAR 0620460.

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doi:10.1088/1742-5468/2009/02/P02039
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