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Compartmentalized granular gases: flux model results

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Abstract. A review is given of our previous work on the clustering phenomenon for vibrofluidized granular matter in an array of connected compartments, being a prime example of spontaneous pattern formation in a many-body system far from thermodynamic equilibrium. Experiments show that when the shaking strength is reduced below a certain critical level, the grains cluster together: first into a subset of the compartments and ultimately, on a much longer timescale, into a single compartment. These experimental observations are explained qualitatively and quantitatively by a dynamical flux model.

We discuss several variations on the original system, altering the openings between the compartments, in such a way that the clustering induces convective patterns and directed transport. Here the bifurcational structure becomes more intricate, but is again fully explained by the corresponding flux model.

Keywords: granular matter, coarsening processes (theory)
Compartmentalized granular gases

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1. Introduction

One of the characteristic features of granular gases is their tendency to spontaneously separate into dense and dilute regions [8]–[11]. This property, which makes them fundamentally different from any ordinary molecular gas, can be traced back to the fact that the collisions between the granular particles are inelastic. Every time two particles collide, their relative velocity is reduced proportionally to the coefficient of normal restitution $0 \leq e < 1$. The case $e = 1$ corresponds to a standard elastic gas in which no clustering occurs.

This clustering phenomenon manifests itself in a particularly clear manner in a container that is divided into two identical compartments connected by a slit in the wall that separates the compartments (figure 1(a) [1, 12]). The system is filled with particles and vibrofluidized by shaking the box vertically. With vigorous driving, the particles distribute themselves evenly over the compartments, but when the driving strength is reduced below a critical level, the particles cluster into one of the two compartments. A dynamical equilibrium is established in which the particle flux from the densely populated cluster is balanced by the flux from the diluted compartment. This is possible because per time unit particles in the dense compartment experience many energy-dissipating collisions and will therefore be slow. Likewise, particles in the diluted compartment will be relatively fast, and are thus able to generate the same flux through the slit as the more numerous, but slower, particles in the densely populated compartment.

The two-compartment system is readily generalized to an arbitrary array of $K$ identical compartments, connected by slits in the walls between neighbouring compartments; see figure 1(b) [1]–[3], [12]–[15]. When the driving strength is reduced below the critical level, the particles will now cluster into a subset of the compartments.

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States with multiple clusters can be proven to be unstable \[2\] and therefore, on a much larger timescale, this pattern will coarsen to a final dynamical equilibrium between 1 cluster and \(K - 1\) diluted compartments \[5\]. The analysis of the steady states and the dynamics of this system will be the key objective of this paper.

The clustering and coarsening process can be accurately explained by a flux model, describing the particle flow from each compartment as a function of the shaking strength and the number of particles in the compartment. This flux model, which is essentially a mean-field model, will be described in section 2 and constitutes the foundation of the theoretical analysis presented in this review. As this paper will be dealing with the structures described by this flux model, it will focus predominantly on our own work.

At this point we want to stress that many people have looked into clustering phenomena in other ways as well \[16,17\]: Lipowski, Droz, and co-workers \[14,18,19\] have studied clustering in terms of a modified Ehrenfest urn model. Brey et al \[20\] describe a horizontally vibrated version of the two-compartment system, analysed by means of a hydrodynamic model (cf section 2.1). Cecconi et al \[21\] capture the essential features of the clustering effect in terms of a Kramers-type escape problem, featuring two particles in a double-well potential. The analysis of the two-compartment system has been extended to a bidisperse granular gas consisting of large and small particles of the same material, both for the original system \[13,22\] and for the hydrodynamic version \[23\].

A unifying theme of the various models mentioned above is that the spontaneous symmetry breaking of the particle distribution (i.e., the clustering effect) is related to a function that depends non-monotonically on the particle content of each compartment. Indeed, the compartmentalized granular gas is a prime example of a strongly nonlinear system that only interacts through nearest-neighbour contacts. As such it has many and sometimes profound connections with other systems that possess that property, such as zero range processes \[24\]–\[28\] and asymmetric exclusion processes \[29\]–\[32\].

The paper is organized as follows. In section 2.1 we start with the derivation of the flux function within the framework of granular hydrodynamics. Subsequently, in section 2.2
we discuss the flux model and give the mean-field equations of motion of a granular gas in $K$ connected compartments. We then turn to the time-independent solutions and show that they are all characterized by detailed balance. In section 2.3 we describe the full bifurcation diagram of the $K$-compartment system and in section 2.4 we present the granular fountain, a generalization of the $K$-compartment system featuring convection rolls. This finishes our discussion of the steady-state properties of compartmentalized granular gases, and in section 3 we turn to the dynamics of these systems. We start with a detailed look of the clustering process, with emphasis on the coarsening in larger systems (section 3.1), and then go to the description of the inverse process, i.e., the breakdown of a cluster (section 3.2). In section 4 we discuss a system that breaks the local spatial inversion symmetry and see that this destroys detailed balance: the so-called granular ratchet studied in this section exhibits stable states with a finite net flux through the compartments. Finally, in section 5 we summarize the main points of the paper and make a few concluding remarks.

2. Steady-state analysis

Compartmentalized granular gases can be adequately described by a flux model [1]–[7], [12]. The main ingredient of this model is a non-monotonic flux function, which represents the outflow of particles from a compartment into each of its neighbours in terms of the number of particles in that compartment. Before describing the model in full detail in section 2.2, we will first outline the origins of the flux function, roughly following the derivation by Eggers [12].

2.1. Derivation of the flux function

Imagine a (rectangular) container, mounted on a shaker and filled with small grains. The grains are fluidized by vertically vibrating the set-up with a sawtooth-shaped signal with frequency $f$ and amplitude $a$. The container has a small, rectangular aperture of size $S = w\Delta h$ at a certain height $h$. For the derivation of the flux function the grains are assumed to be frictionless spheres, such that only normal restitution contributes to the dissipation. Furthermore, any collisions with the walls (extending all the way up to infinity) and bottom are assumed to be perfectly elastic.

The objective is to obtain an expression for the particle flux through the aperture as a function of the particle fraction (or number of particles) in the compartment. To this end, we first need to find expressions for the temperature and density near the aperture. We start out from the general (three-dimensional) hydrodynamics equations for a dissipative, low-density granular gas, as they can be derived from Boltzmann’s equation [33]–[39] by taking its first three moments with respect to the velocity distribution. As we will assume that the granular gas in a compartment is always in a steady state and that macroscopic flow is absent\(^4\), these equations reduce to

$$\nabla p = -\rho mg e_z, \quad \nabla \cdot J = I. \quad (1)$$

In the first equation, which expresses momentum conservation, $p$ is the pressure and $\rho$ is the number density of particles. The other parameters are the particle mass, $m$, acceleration of gravity, $g$, and the unit vector in the vertical direction, $e_z$.

\(^4\) This is expected to be the case if the aperture which connects the compartments is sufficiently small.
The second equation, where $J$ and $I$ are the heat flux and the energy dissipation rate per unit volume respectively, represents the energy balance in the system. When integrated over the container volume $V$, it gives with use of Gauss’ theorem

$$\int_{\partial V} J \cdot dA = \int_V I \, dV \quad \text{or} \quad Q_{\text{in}} = Q_{\text{diss}},$$

where $\partial V$ stands for the boundaries of the container, i.e., the walls and the bottom. Since the walls (and bottom) are assumed to be non-dissipative, the only contribution to the left integral stems from the energy input rate ($Q_{\text{in}}$) at the bottom. This should be balanced by the right integral, which can be interpreted as the total dissipation rate in the container due to the inelastic particle collisions.

Using the ideal gas law for dilute granular gases, $p = m\rho T$, where $T$ equals the granular temperature, defined as one third of the velocity fluctuations in the system\(^5\), the first equation of (1) can be integrated, and it gives

$$\rho(z) = \frac{gN_k}{\Omega T(z)} \exp\left\{- \int_0^z \frac{g}{T(\zeta)} \, d\zeta \right\}.$$

(3)

Here $N_k$ is the total number of particles in the compartment and $\Omega$ is its ground area.

The key assumption in this derivation of the flux function is that the temperature is constant within the gas\(^{12}\). For dilute gases this is in fair agreement with molecular dynamics simulations, apart from a small region just above the bottom, where the energy input leads to a local increase of the temperature. With this assumption, the evaluation of the integral in equation (3) results in the barometric height distribution

$$\rho(z) = \frac{gN_k}{\Omega T} \exp\left\{- \frac{gz}{T} \right\}.$$

(4)

To write the temperature as a function of $N_k$, we first need to express $Q_{\text{in}}$ and $Q_{\text{diss}}$ in terms of $N_k$ and $T$. For $Q_{\text{diss}}$ this can be done in various ways, with different levels of sophistication, leading to results with increasing complexity (see e.g.\(^{33,40}\)). The standard way is to assume a local Maxwell–Boltzmann distribution for the velocity, calculate the probability for a collision between two particles at positions $r_1$ and $r_2$, with velocities $v_1$ and $v_2$ respectively, average over the ensemble, and integrate over the container volume. This gives

$$Q_{\text{diss}} = 4\sqrt{\frac{\pi}{\Omega}} m g r N_k^2 (1 - e^2) \sqrt{T}.$$

(5)

More intricate approaches, using the Chapman–Enskog method up to Burnett order, give results which differ from equation (5) by a multiplicative function depending on the coefficient of restitution $e$ only\(^{35,37}\), and are equal to equation (5) in the elastic limit ($e \approx 1$). A similar procedure for the energy input at the bottom gives

$$Q_{\text{in}} = 2mgN_kaf \left(1 + \frac{2af}{\sqrt{\pi T}}\right) \approx 2mgN_kaf.$$

(6)

\(^5\) This means that the arbitrary proportionality constant between the temperature and the kinetic energy fluctuations (corresponding to Boltzmann’s constant in the molecular case) has been set to the particle mass $m$. As a result, $T$ has the units of a squared velocity.
Figure 2. Flux function $F(n_k)$, i.e., the particle flux from compartment $k$ (per unit time) as a function of the fraction $n_k$ contained within it. The horizontal dashed line indicates that the flux from a relatively empty compartment (left intersection point, corresponding to a particle fraction $n_-$) equals the flux from a much more densely filled compartment (right intersection point, with $n_k = n_+$), and that is exactly what happens in the clustered state.

since the typical velocity of a particle in a dilute granular gas ($\sqrt{T}$) must be larger than the velocity of the bottom ($2af$). By equating the expressions equations (5) and (6), we can finally express the temperature in terms of the number of particles in the container:

$$T = \frac{\Omega^2 a^2 f^2}{4\pi(1 - e^2)^2 r^4 N_k^2}. \quad (7)$$

The flux (in particles per second) from the compartment is now found by multiplying the density of particles moving towards the wall ($\rho(z)/2$) at height $h$ with the average velocity normal to the aperture ($\sqrt{2T}/\pi$) and the area of the aperture ($S$):

$$\hat{F}(N_k) = \frac{1}{2} \rho(h) \sqrt{\frac{2T}{\pi}} S = \hat{A} N_k^2 \exp\{-\hat{B} N_k^2\}, \quad (8)$$

where $\hat{A} = \sqrt{2}(1 - e^2)gr^2S/(\Omega^2af)$ and $\hat{B} = 4\pi(1 - e^2)^2[gh/(af)]^2[r^4/\Omega^2]$.

This concludes our derivation of the flux function. Most importantly, the flux $\hat{F}(N_k)$ is a non-monotonic function of the number of particles $N_k$ in the compartment (cf figure 2) which has far-reaching consequences, as will be pointed out in section 2.2.

Before doing so, let us give some attention to the limitations of the flux model. First of all, in order to describe the compartments as containers each filled with a steady-state granular gas, the apertures between containers need to be small. It is instructive to note that in the absence of gravity this restriction can be relaxed considerably due to the fact that equation (3) now leads to the condition that the pressure must be constant throughout the whole system, instead of obeying a barometric height distribution. Brey et al used this property to construct their hydrodynamic Maxwell demon model [20], an elegant variation on the original two-compartment experiment in which the density and temperature profiles in neighbouring compartments are different but their product, i.e., the pressure, is constant. This allows for much larger apertures than the Eggers flux model might be expected to describe.

Secondly, while the sawtooth-shaped driving used in the derivation is very convenient for analytical purposes, it is not the most frequently used driving in experiments, where
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sinusoidal or triangular driving signals are more common. One can show that for these cases the flux function has a slightly different functional form, but its most important characteristics (and in particular its one-humped shape) remain unchanged [41,42].

Finally, the assumptions of a constant temperature profile and the Maxwellian velocity distribution used in deriving the flux function are known to be violated for denser granular gases. Moreover, even for dilute granular gases, the velocity fluctuations turn out to be anisotropic. This could be remedied by using more realistic temperature profiles and estimates on dissipation rate and energy input rate based on non-Gaussian velocity distributions. These are available in the literature (see, e.g., [38,39,41]), but can only be incorporated at the expense of losing the particularly simple functional form of equation (8). Moreover, the phenomena that will be described in the rest of this paper are consequences of the non-monotonicity of the flux function rather than its precise form. Therefore much of what will be said will be true irrespective of the precise form of the flux function, and for clarity of presentation we therefore stick to the simple form of equation (8).

2.2. Flux model

Before turning to the presentation and analysis of the flux model for an arbitrary array of connected compartments, it is convenient to rewrite the flux function \( \hat{F}(N_k) \) of equation (8) in terms of the particle fraction \( n_k \) in that compartment with respect to the uniform distribution. That is, \( n_k = N_k/N_{av} \), where \( N_k \) is the number of particles in compartment \( k \) and \( N_{av} = N_{tot}/K \), with \( N_{tot} \) the total number of particles in the \( K \)-compartment system. We then have (see figure 2)

\[
F(n_k) = An_k^2 \exp(-Bn_k^2),
\]

where the (dimensionless) driving parameter \( B \) is given by [1]

\[
B = \frac{4}{\pi (af)^2} \left(1 - e^2\right)^2 \frac{(\pi r^2 N_{av}/\Omega)^2}{(1 - e^2)^2 (\pi r^2 N_{av}/\Omega)^2}.
\]

This parameter consists of three dimensionless groups: the first group is proportional to the ratio between the energy a grain needs to jump from the bottom to the slit at height \( h \) and the energy it gets from collisions with the vibrating bottom plate. The second group, \((1 - e^2)^2\), is a measure of the dissipation in the gas, with \( e \) being the normal coefficient of restitution of the collisions between the particles. The third group is the square of a filling factor defined as the sum of cross sections of the average number of particles \( \pi r^2 \) divided by the bottom area of a compartment \( \Omega \).

Most importantly, \( B \) is inversely proportional to the driving strength \( (af)^2 \). The factor \( A \) in equation (9) determines the absolute rate of the flux and can simply be incorporated in the timescale.

The time rate of change \( \frac{dn_k}{dt} \) of the particle fraction in the \( k \)th compartment is given by the inflow from its two neighbours minus the outflow from the compartment itself:

\[
\frac{dn_k}{dt} = F(n_{k-1}) - 2F(n_k) + F(n_{k+1}),
\]

with \( k = 1, \ldots, K \). For simplicity, we will use cyclic boundary conditions \( n_{K+1} = n_1 \) (otherwise the end boxes should be treated separately). The above balance equation is
supplemented by the condition \( \sum n_k = K \), related to the conservation of the total number of particles \( N_{\text{tot}} \). Equation (11) for all \( K \) compartments together can conveniently be written in matrix form:

\[
\frac{d\mathbf{n}}{dt} = \mathbf{M} \cdot \mathbf{f} \quad \text{or} \quad \frac{dn_k}{dt} = \sum_{i=1}^{K} M_{ki} F(n_i),
\]

where \( \mathbf{n} \) is a \( K \)-vector containing the fractions \( n_k \), \( \mathbf{f} \) is a vector with components \( F(n_k) \), and \( \mathbf{M} \) is a so-called tri-diagonal \( K \times K \)-matrix with elements \(-2\) on the diagonal, and \( 1 \) on all first off-diagonal positions, as well as on the corners \((1, K)\) and \((K, 1)\). This matrix has the following important properties \[2\]: its rank is \( K - 1 \) and its kernel (corresponding to the eigenvalue zero) is the linear subspace spanned by the eigenvector \( \mathbf{1} = (1, 1, \ldots, 1) \), corresponding to a situation in which the particles are distributed equally over the \( K \) compartments. It is also negative semi-definite, meaning that \( \mathbf{f} \cdot \mathbf{M} \cdot \mathbf{f} < 0 \) unless \( \mathbf{M} \cdot \mathbf{f} = 0 \). As a consequence all other eigenvalues of \( \mathbf{M} \) are negative \[2\].

To find the fixed points of equation (12) we have to solve \( \mathbf{M} \cdot \mathbf{f} = 0 \), from which we can immediately conclude that \( \mathbf{f} \) should be in the kernel of \( \mathbf{M} \), i.e., proportional to \( \mathbf{1} \). This means that all the elements of the flux vector \( \mathbf{f} \) should be equal:

\[
F(n_k) = \text{constant} \quad \text{for all} \quad k = 1, \ldots, K.
\]

At this point there comes into play the most important feature of the flux function equation (9), namely that it is non-monotonic (see figure 2). If \( F(n_k) \) were to be a monotonic function of \( n_k \), the only fixed point of equation (12) would be the uniform distribution in which all \( n_k \) are equal: \( n_k = 1 \). This continues to be a solution of equation (13) for non-monotonic \( F(n_k) \), but there is the additional possibility of creating a fixed point from any distribution of the two different particle fractions \( n_{k,-} \) and \( n_{k,+} \) corresponding to the same value of \( F(n_k) = F(n_{k,-}) = F(n_{k,+}) \), with the only restriction that the total number of particles in the system remains conserved, \( \sum n_k = K \). The procedure by which the fixed points of the \( K \)-compartment system are determined is described in \[2\] and will be reviewed in section 2.3.

Since equation (12) represents a nonlinear evolution equation in \( K - 1 \) dimensions, not only steady states (fixed points) but also periodic or even chaotic solutions might in principle be possible in the long-time limit. For the present system, however, the latter two possibilities are ruled out by the construction of a Lyapunov function \[7,43\].

To see this, we first define the functions

\[
\Gamma(n) := -\int_{\nu=0}^{n} F(\nu) \, d\nu,
\]

\[
L(n) := \sum_{k=1}^{K} \Gamma(n_k).
\]

From the definition of the flux function equation (9) it follows that \( L(n) \leq 0 \) for any vector \( n \) with \( n_k \in [0, 1] \).
Next we compute the time derivative of $L$ along a solution $n(t)$ of the dynamics equation (12):
\[
\frac{d}{dt}L(n(t)) = \sum_{k=1}^{K} \left( \frac{d\Gamma}{dn} \right)_{n=n_k(t)} \left( \frac{dn_k(t)}{dt} \right) \\
= -\sum_{k=1}^{K} \sum_{l=1}^{K} F(n_k(t)) M_{kl} F(n_l(t)) \\
= -\mathbf{f} \cdot \mathbf{M} \cdot \mathbf{f}.
\] (16)

Due to the properties of the tri-diagonal matrix $\mathbf{M}$ mentioned above it follows that $dL(n(t))/dt > 0$, except if $\mathbf{M} \cdot \mathbf{f} = 0$ or, equivalently, if $dn(t)/dt = 0$. In other words, $L(n)$ is a so-called Lyapunov function, implying that $dn(t)/dt \to 0$ for $t \to \infty$, i.e., the only possibility for $n(t)$ for asymptotically long time $t$ is to converge towards a fixed point. Hence, in order to discuss the long-time behaviour of equation (12) we may limit ourselves to steady states. From equation (13) we conclude that the steady states are characterized by detailed balance: the net flux through any of the slits, and hence through the whole system, is zero.

2.3. Stable and unstable fixed points of the $K$-compartment system

The key objective now is to construct the bifurcation diagram (i.e., the fixed points as a function of the driving parameter $B$) for arbitrary $K$. This will be done using the method that was developed in [2] and extended to the more complicated granular fountain in [7] (see section 2.4).

First we note that the $B$-dependence of the flux function $F(n_k)$ (equation (9)) can be transferred to the conservation condition $\sum n_k = K$ by a simple change of variables:
\[
z_k \equiv \sqrt{B} n_k,
\] (17)
which (up to an irrelevant multiplicative constant $A/B$) transforms $F(n_k)$ into $\tilde{F}(z_k) = z_k^2 \exp(-z_k^2)$ (independent of $B$) and $\sum n_k = K$ into $\sum z_k = K\sqrt{B}$. The fixed points of equation (12) can now be found by solving
\[
\tilde{F}(z_k) = \text{constant}, \\
\sum_{k=1}^{K} z_k = K\sqrt{B}.
\] (18)

In words, we have to find all possible sets of fractions $z_k$ (equal or not, but corresponding to the same flux) that add up to $K\sqrt{B}$.

This can be done by constructing $K + 1$ sum-functions $S_i(\tilde{H})$, which represent the distributions containing $i$ dense compartments (filling fraction $z_-$) and $K - i$ dilute compartments $z_+$, that generate the same flux $\tilde{F}$:
\[
S_i(\tilde{F}) = (K - i)z_-(\tilde{F}) + iz_+(\tilde{F}),
\] (19)
for $i = 0, \ldots, K$. For example, for $K = 5$ there are six sum-functions, all of which are depicted in figure 3(a). Given these functions, the steady states are now determined (for
Figure 3. (a) The $K$ sum-functions $S_m(\tilde{F})$, $m = 0, 1, \ldots, K$ for $K = 5$. The points of intersection with the horizontal line $z = K\sqrt{B}$ represent the fixed points for the parameter value $B$. Curves $S_0$ and $S_5$ correspond to the uniform distribution (below and above the critical point $B = 1$, respectively) and the other curves belong to clustered states. (b) Resulting bifurcation diagram for $K = 5$. It has been obtained from figure 3(a) by converting, for all $B$, each $\{z_-, z_+\}$ pair belonging to a point of intersection to an $\{n_-, n_+\}$ pair. Here, $B_0 = 1$ indicates the value at which the uniform distribution becomes unstable and $B_{c,5}$ denotes the first saddle-node bifurcation that generated the stable one-cluster branch.

any value of $B$) by solving

$$S_i(\tilde{F}) = K\sqrt{B}. \quad (20)$$

This is graphically illustrated in figure 3(a), where the steady states for $B = 1.12$ are given by the intersection points of the sum-functions $S_i$ with the horizontal line $K\sqrt{B} = 5.3$.

This information is then translated back to the original fractions $n_k = z_k/\sqrt{B}$ and plotted as the bifurcation diagram of the $K = 5$ depicted in figure 3(b). Solid lines stand for stable states, whereas the dashed lines represent unstable ones. The stability of the states can be determined by means of a linear stability analysis, i.e., from the eigenvalues of the Jacobi matrix $J$ associated with equation (12). This stability analysis is fully discussed in [2] and will not be repeated here.

Following the above procedure one can obtain the bifurcation diagram of the $K$-compartment system for any arbitrary $K$. They all share the following features (cf figure 3(b)). For vigorous driving (small $B$) the uniform distribution is stable, and it becomes unstable at $B = 1$. Here we find the intersection point of all branches of the bifurcation diagram corresponding to non-uniform states that came into existence by a saddle-node bifurcation at lower values of $B$. Only one pair of these branches has a stable portion: this is the branch with exactly one cluster, and $K - 1$ diluted compartments.

2.4. A more complex system: the granular fountain

The procedure given in section 2.3 can be extended to more complicated systems. An example corresponding to a physically realizable system is the so-called granular
Figure 4. (a) One of the convection patterns (a three-cluster state) of the $K = 5$-compartment granular fountain. (b) The modified flux function $H(n_k)$ of the granular fountain (see equation (21)) for $\lambda = 0.02$ and $B = 1$. For certain values of $H$ there are three solutions to $H(n_k) = H$, labelled $n_+$, $n_0$, and $n_-$. (c) Bifurcation diagram of the $K = 5$ granular fountain, evaluated from the flux model with $\lambda = 0.02$. There is a cascade of stable steady states (solid lines) with an increasing number of clusters for increasing $B$. All states corresponding to dashed and dotted lines are unstable.

fountain [6, 7], where the slit at height $h$ is supplemented by a small hole at the bottom (figure 4(a)). This will give rise to a modified flux function

$$H(n_k) = F(n_k) + G(n_k) = An_k^2 e^{-Bn_k^2} + \lambda An_k^2$$

where $G(n_k)$ is the flux through the bottom hole and the additional parameter $\lambda$ is determined by the size ratio of the bottom hole and the slit. As is seen in figure 4(b), this flux function is also non-monotonic, and contains a region in which three different particle fractions ($n_-$, $n_0$, and $n_+$) will give the same flux. Consequently we will have to consider all possible distributions of these three particle fractions, which obviously will result in a more complex bifurcation diagram. Nevertheless, it can be determined following the same procedure outlined above, as is discussed in detail in [7].

For the granular fountain we can also construct a similar Lyapunov function as for the original $K$-compartment system and thus rule out the existence of periodic or chaotic states [7]. The most prominent features of the bifurcation diagram of the $K$-compartment
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system can be discussed using the example of the $K = 5$ granular fountain depicted in figure 4(c): first of all the uniform state is not only stable for vigorous driving (small $B$), but also for very weak shaking. This is because when the particles do not have enough energy to reach the slit any more, the system will equilibrate through the holes at the bottom, leading to a stable uniform distribution for large $B$. Secondly, not only is the one-cluster state found to be stable for some driving interval, but also states with multiple clusters are found to be stable. These stable $k$-cluster states are arranged in a \textit{cascade} of fountain states (with a successively growing number of clusters) acquiring and losing stability. Upon decreasing the shaking strength (increasing $B$) we find that the uniform distribution loses stability in a first-order phase transition. Lowering the shaking strength, the two-, three- and four-clustered fountain states gain stability one after the other. Note that there is an interval around $B = 4–5$ for which the two-, three-, and four-clustered states are simultaneously stable, illustrating that one really has to explore a variety of initial conditions to find all possible stable end states.

Finally, there is another exciting property of the $K$-compartment granular fountain, and that is that all stable clustered states are steady \textit{convective} states. In order to understand why this is true, we turn to figure 4(a): looking, for example, at the rightmost two compartments, we find a cluster in box 4 and a diluted, highly energetic gas in box 5. Since there are many more particles close to the bottom in box 4, there must be a net particle flux through the bottom hole directed towards box 5. And, because of detailed balance in a steady state, the net flux through the slit at height $h$ should be just as large but in the opposite direction. This means that there is a convection roll in each neighbouring pair of compartments with a difference in particle fraction (see figure 4(a)), reminiscent of the famous Rayleigh–Benard convection rolls in a liquid.

3. Dynamics

3.1. Clustering and coarsening

In section 2 we determined the bifurcation diagram containing the steady states and the transient unstable states of the $K$-compartment system. In this section we will turn to the dynamics of the clustering process; i.e., for values of $B$ beyond the clustering threshold, how does the system find its way from the unstable uniform distribution towards the clustered state?

We illustrate this process using the bifurcation diagram of a system consisting of $K = 12$ compartments (figure 5). If the driving parameter $B$ is increased beyond the critical value $B_c = 1$, the uniform distribution ($n_k = 1$ for all $k$) becomes unstable. So if one starts out from a homogeneous distribution at $B > 1$, the system will travel through the dashed curves (marking the transient multi-cluster states), slowly evolving to the solid curve which corresponds to a one-cluster state, being the only stable state beyond $B_c$. It is this trajectory that constitutes the coarsening process of a compartmentalized granular gas.

This coarsening process can be studied in detail by numerically integrating the equations of motion equation (11). In figure 6 we show five snapshots for a cyclic system of $N = 50$ compartments, showing that the coarsening is in fact a two-stage process.

The initial state in the first snapshot ($t = 0$) is nearly uniform. Soon after the shaking is turned on, however, clusters start to form as can be seen in the second snapshot, taken

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Figure 5. Bifurcation diagram for $K = 12$ compartments. For $B > 1$ the uniform distribution is unstable (indicated by a dashed line) and the system evolves through a series of unstable transient states (dashed lines) towards the stable situation of one cluster and 11 empty compartments (the solid lines at $n_k = 1$ and $n_k = 0$).

at $t = 7.2 \times 10^2$. The number of clusters that forms during this first, cluster-formation stage depends on the value of the parameter $B$ and turns out to be proportional to $K(B - B_C)^{0.28}$ [5]. The size of each cluster is determined by the size of the region from which it draws material: a growing cluster simply amasses everything that is in the compartments nearer than half the distance to the neighbouring clusters.

At $t = 2.1 \times 10^3$ there is a clear separation between the slowly growing clusters and an intermediate hilly pattern, which is being drained by the clusters. This marks the start of the second, coarsening stage.

At $t = 1.8 \times 10^6$ the intermediate zones between the four leftmost clusters are depleted to such an extent that the flux from these zones becomes of the same order of smallness as the flux from the clusters. In this new situation material can be transported from one cluster to another, meaning that at least one of them does not grow any more but instead starts to decrease. Indeed, the third cluster from the left eventually collapses and disappears from the system. Note the hilly shape that remains of this cluster at $t = 1.2 \times 10^8$, ready to be depleted relatively quickly by the two neighbouring clusters.

This process (hilly pattern depletion $\rightarrow$ cluster-to-cluster transport $\rightarrow$ collapse of the smallest cluster) repeats itself until only one cluster remains. In figure 7 the time evolution of the particle fractions $n_k$ is plotted on a linear scale (top), showing the sudden collapse and growth of the clusters, and on a logarithmic scale (bottom). The latter reveals what goes on in the dilute regions between the clusters, which are indistinguishable from zero in the linear plot: the decaying structures with slope $-1$ correspond to the hilly pattern, and the constant parts to cluster-to-cluster transport.

Note that the cluster-to-cluster transport phase is much slower than the hilly pattern depletion phase by which it is preceded. The depletion phase can be analysed using a
Figure 6. Five snapshots showing the initial stages of the coarsening process in \( K = 50 \) compartments, as evaluated by the flux model (with time measured in units of \( A^{-1} \)). Note the different scale of the vertical axis in the last two snapshots.

The continuum version of the flux model, from which it follows that \( n_k(t) \propto 1/t \), explaining the slope \(-1\) in the logarithmic plot [5].

Here we will concentrate on the analysis of the second, cluster-to-cluster-transport phase. At a certain moment, the flux from the decaying hill becomes equal to the flux from one of the neighbouring clusters towards the hill, while the flux from the other cluster is still smaller. Now a net flux to the latter cluster will develop and the hilly shape is transformed into a staircase pattern, which soon hardly changes any more: see the snapshot at \( t = 1.8 \times 10^6 \) in figure 6, where staircase patterns are visible between
Figure 7. Time evolution of the fraction $n_k$ in a 20-compartment system on a linear scale (top) and a logarithmic scale (bottom). After an initial phase of cluster formation (which is completed around $t = 2.1 \times 10^2$), the coarsening process sets in. Within the coarsening, two different phases can be discerned: (a) phases during which all clusters grow and the intermediate compartments are depleted as $1/t$ (corresponding to the lines with slope $-1$ in the logarithmic plot), and (b) phases in which one of the clusters is preparing to collapse and transports its material to the neighbouring clusters. The contents of the nearly empty compartments through which this cluster-to-cluster transport takes place remain at a constant level (see the horizontal lines in the logarithmic plot).

The net flux from compartment to compartment $[F(n_{k+1}) - F(n_k)]$ is constant along this staircase profile, with material being transported from the smaller to the larger cluster. This allows us, if compartment $k$ contains a cluster, to rewrite the balance equation as follows:

$$\frac{dn_k}{dt} = [F(n_{k-1}) - F(n_k)] + [F(n_{k+1}) - F(n_k)]$$

$$= \frac{[F(n_{k-\Delta left}) - F(n_k)]}{\Delta left} + \frac{[F(n_{k+\Delta right}) - F(n_k)]}{\Delta right},$$

(22)
where $\Delta$ is the distance (i.e., the number of compartments) between the cluster in $k$ and its neighbouring cluster on the left, and analogously $\Delta$ right denotes the distance from the neighbouring cluster on the right. If we now replace the index $k$ (which runs over all the compartments) with an index $i$, which runs over clusters only, the above equation takes the form

$$
\frac{dn_i}{dt} = \frac{[F(n_{i-1}) - F(n_i)]}{d_{i-1,i}} + \frac{[F(n_{i+1}) - F(n_i)]}{d_{i,i+1}},
$$

where $\Delta$ left is now being denoted as $d_{i-1,i}$, and similarly for $\Delta$ right. This equation, which describes the cluster-to-cluster transport during the second coarsening stage, is a renormalized version of the original compartment-to-compartment equation (11): we have to solve the balance equation only for those compartments that contain clusters, with the net flux $[F(n_{i+1}) - F(n_i)]$ between clusters being rescaled by their distance $d_{i,i+1}$.6

We now come to the central question: how does the fraction $n_i(t)$ of the surviving cluster grow with time? To answer this, we consider the $L$-cluster state and assume that all clusters are equally spaced, so all distances are equal to $d_{i,i+1} = K/L$. Equation (23) then takes the form $dn_i/dt = (L/K)[F(n_{i-1}) - 2F(n_i) + F(n_{i+1})]$. We further assume that the clusters all contain more or less (but not precisely) the same fraction $K/L$. Let us concentrate on the cluster that is going to collapse first (say $n_i$). The neighbouring clusters are a bit larger, which means that their outflux $F$ is considerably smaller; hence for all practical purposes they just act as absorbing boundaries. The cluster-to-cluster equation then reduces to

$$
\frac{dn_i}{dt} = -\frac{2L}{K} F(n_j) = -\frac{2L}{K} A n^2_i e^{-B n_i^2},
$$

or equivalently,

$$
dt = -\frac{K}{2LA} \frac{e^{B n_j^2}}{n^2_j} dn_j.
$$

This can be integrated analytically, from the value $n_j = K/L$ to some much smaller value $n_j = \epsilon$, to give the lifetime $\tau_L$ of the $L$-cluster state:

$$
\tau_L = -\frac{K}{2LA} \int_{K/L}^{\epsilon} \frac{e^{B n_j^2}}{n^2_j} dn_j
= \frac{K}{2LA} \left[ \sqrt{B \pi} \text{erfi}(\sqrt{B n_j}) - \frac{e^{B n_j^2}}{n_j} \right]_{K/L}^\epsilon
\approx \frac{K \sqrt{B}}{LA} \int_0^{K \sqrt{B}/L} e^{u^2} du.
$$

6 In the very dilute limit one may question the validity of the flux-model description leading to equation (23). Indeed, what is the significance of time-averaged fractions $n_i$ corresponding to particle densities far below one particle per compartment? We therefore checked this equation using an alternative approach, in which we suppose that the two clusters are separated by empty compartments, which only very occasionally contain a particle. When a particle leaves cluster $i$ towards the right (which it does at a rate $F(n_i)$) it will perform a random walk until it is absorbed, either by cluster $i$ again or by the neighbouring cluster $i+1$. The probability that it ends up in the neighbouring cluster is equal to the reciprocal distance between the clusters $1/d_{i,i+1}$ (this is known as the gambler’s ruin problem [5, 44]). So the rate at which particles move from cluster $i$ to $i+1$ is equal to $F(n_i)/d_{i,i+1}$. Summing up the four directions of particle transport between clusters $i-1$, $i$, and $i+1$, we arrive at exactly the same equation (23).
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Figure 8. The average cluster size $\bar{n}_j(t)$ versus the logarithm of time $\log_{10} t$, determined by numerically solving the balance equations (11) and (23) for a 30-compartment system; from [5]. It is seen that $\bar{n}_j(t) \propto \sqrt{\log t}$, in agreement with the analytical prediction of equation (28).

Here $\text{erfi}(x)$ is the imaginary error function $[\text{erfi}(x) = (2/\sqrt{\pi}) \int_0^x e^{u^2} du]$, which grows super-exponentially with $x$. In the last step of equation (26) we have only kept the dominant term, i.e., the term with $\text{erfi}(K\sqrt{B/L})$.

For any given value of the shaking parameter $B$, the expression for $\tau_L$ rises steeply for decreasing $L$. Therefore, in the $L$-cluster state, the current lifetime $\tau_L$ takes up by far the largest part of the total elapsed time $t$ until then. Thus we replace $\tau_L$ by $t$, and $K/L$ by $\bar{n}_j(t)$ (the average cluster size), to arrive at

$$t = \frac{\sqrt{B\bar{n}_j(t)}}{A} \int_0^{\sqrt{B\bar{n}_j(t)}} e^{u^2} du,$$

which in leading order gives $t \propto e^{B\bar{n}_j^2(t)}$, or equivalently,

$$\bar{n}_j(t) \propto \sqrt{\log t}.$$

This anomalously slow coarsening behaviour is also found if one numerically solves the balance equation (11); see figure 8. The inset shows $\log \bar{n}_j(t)$ as a function of $\log(\log t)$: all the numerical data lie on a straight line with slope 1/2, in full agreement with equation (28).

3.2. Declustering: the sudden collapse of a cluster

Clustering is a major, usually unwanted effect in the many industries that handle granular matter (mining, food production, pharmaceutical industry, construction works, chemical reactors, etc). It has been estimated that no less than 40% of the capacity of these industries is wasted due to problems related with the transportation and sorting of
granular matter [8]. Therefore, from a practical point of view the opposite process of declustering is even more important than clustering itself.

To get rid of a single cluster in our $K$-compartment system, we simply turn up the shaking intensity again, i.e., we reduce $B$. It is, however, not sufficient to reduce it down to the value $B_0 = 1$ where the uniform distribution becomes unstable, but we need to go below the first saddle-node bifurcation in our diagram, labelled $B_{c,k}$ (cf figure 3). Declustering is by no means the same as clustering in reverse time order: whereas the clustering (cf figure 1(c)) takes place gradually, via transient states in which two or more of the boxes compete for dominance, the declustering occurs quite suddenly and without any transient states. In figure 9, which contains a series of four snapshots of a declustering experiment in five compartments, we see that until 42 s after the beginning of the experiment the cluster is still clearly visible, and then it abruptly collapses within one second [3,4].

The suddenness of the collapse makes it possible to define the so-called cluster lifetime $\tau$. In figure 10(a) we show lifetime data as function of $\tilde{B} = K^2B$, for various numbers of compartments $K$, spanning no less than eight orders of magnitude. For each value of $K$ there are two distinct regimes to distinguish.

(i) Close to the critical value $B_{c,k}$ (different for every value of $K$) the lifetime $\tau$ diverges. This stands to reason, because the cluster becomes a stable equilibrium at $B_{c,k}$, with an infinite lifetime [4].

(ii) Further away from $B_{c,k}$ the lifetimes lie on a universal curve, which increases roughly as $\exp [(3/2)K^2B]$. This reflects the experimental fact that a small reduction of $K^2B$ (i.e., a small increase of the shaking strength) causes an exponential reduction of the cluster lifetime [3].

The existence of the two regimes can be explained by observing how the declustering takes place. In regime (ii), i.e., far away from $B_{c,k}$, the cluster collapses before the particles leaking out of the cluster fill the outermost compartments to any significant level. In this regime the $K$-compartment system is essentially equivalent to a system with an infinite number of compartments [3]. Therefore all data in this regime lie on a single curve.

---

7 For identical systems that only differ in the number of compartments $K$, there is—because of the way the $n_k$ are defined (see section 2.2)—an artificial dependence of the cluster fraction on $K$: $n_k = K$. For a comparison of the collapse of these identical clusters we need therefore to redefine our particle fraction to express this fact, for example, by redefining $\tilde{n}_k = n_k/K$. This goes together with a redefinition of $B$ into $\tilde{B} = K^2B$ which is the natural driving parameter to describe the collapse of a granular cluster.

---
Figure 10. (a) Cluster lifetime $\tau$ versus driving parameter $\tilde{B} \equiv K^2 B$, determined from experiments with $K = 3, 5$ boxes (dots with error bars) and from numerical evaluation of the flux model with $K = 3, 5, 10, 20, 40, 80$ (open symbols). The solid line is the analytical solution for the envelope curve [3], which goes roughly as $\exp[(3/2)K^2 B]$. (b) Doubly logarithmic plot of the cluster lifetime $\tau$ as a function of the distance from the critical $B$-value ($\delta B \equiv B_{c,K} - B$) for the data series of figure 10(a). Both axes have been scaled in order to compare the data for different values of $K$. It is seen that, for $\delta B \to 0$, all data points converge to a straight line with slope $-1/2$, in agreement with the prediction of equation (32).

In contrast, in regime (i), close to $B_{c,K}$, the lifetime $\tau$ is so long that the particles spilling from the cluster have time to reach even the outermost boxes long before the collapse. All this time, the system seems to be heading for a dynamical equilibrium with one well-filled compartment $n_{cl}$ surrounded by $K - 1$ nearly empty compartments. In our analysis we will indeed assume that (just as in the true equilibrium situation to the right of $B_{c,K}$) all nearly empty compartments have an equal fraction $n_k = (K - n_{cl})/(K - 1)$. This means that our description is in fact a mean-field theory, in the same spirit as the Bragg-Williams theory for an Ising ferromagnet, where the actual magnetization of each lattice site is replaced by its mean value [45, 46].

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Only when the critical region has been passed does the system shoot towards the uniform distribution. The lifetime $\tau$, from the start of the experiment until this sudden collapse, can be calculated from the flux model:

$$\tau = \int_{n_{\text{cl}} = K}^{n_{\text{thr}}} dt,$$

(29)

where $dt$ is understood to be expressed in terms of $n_{\text{cl}}$. Because of the suddenness of the collapse, the exact value for the threshold $n_{\text{thr}}$ is rather arbitrary. In practice $n_{\text{thr}} = K/3$ gives good results for all $K \geq 3$.

Note from the shape of the curve that at the critical point \(\{B_{c,K}, n_{c,+}\}\) (cf figure 3(b)) not only $\partial n_{\text{cl}}/\partial t = 0$ but also its derivative $\partial^2 n_{\text{cl}}/\partial n_{\text{cl}} \partial t$ should vanish\(^8\), i.e., if we perform a Taylor expansion of $\partial n_{\text{cl}}/\partial t$ around \(\{B_{c,K}, n_{c,+}\}\), we have, up to first order in $\delta B \equiv B_{c,K} - B$ and up to second order in $\delta n \equiv n_{\text{cl}} - n_{c,+}$,

$$\frac{d(\delta n)}{dt} \approx -L\delta B - M(\delta n)^2 + X\delta n\delta B,$$

(30)

where the (positive) coefficients $L$, $M$, and $X$ can be calculated from the flux function $F(n_k)$ and depend only on the number of compartments $K$ \([4]\). By a simple transformation $\widehat{\delta n} \equiv \delta n - X\delta B/(2M)$ we obtain $d(\delta \widehat{n})/dt \approx -L\delta B - M(\delta \widehat{n})^2$, and thus

$$dt \approx -\frac{\delta \widehat{n}}{L\delta B + M(\delta \widehat{n})^2},$$

(31)

which is then integrated directly to give the lifetime $\tau$ as defined by equation (29). Since the system lingers a long time in the neighbourhood of the critical point, the main contribution to $\tau$ comes from an interval between $\delta \widehat{n} = \epsilon$ (when $n_{\text{cl}}$ is just above the critical fraction) and $\delta \widehat{n} = -\epsilon$ (when it is just below it). So

$$\tau \approx -\int_\epsilon^{\epsilon} \frac{d(\delta \widehat{n})}{L\delta B + M(\delta \widehat{n})^2} = \frac{2 \arctan\left(\epsilon \sqrt{M/(L\delta B)}\right)}{\sqrt{ML\delta B}} \approx \frac{\pi}{\sqrt{ML\delta B}}.$$

(32)

The last approximation uses the fact that, sufficiently close to the critical point, the argument $\epsilon(M/L)^{1/2}$ becomes a large number. Thus our mean-field theory predicts that, close to the critical point, $\tau$ is proportional to $(\delta B)^{-1/2}$. This is in good agreement with the lifetime data obtained from the flux model, as can be seen in the doubly logarithmic plot of figure 10(b): all data points for $\delta B/B_{c,K} \lesssim 0.05$ lie on a straight line with slope $-1/2$. These data correspond to the upward strokes of $\tau(B)$ in figure 10(a) near each critical value $B_{c,K}$. For $\delta B/B_{c,K} > 0.05$ the lifetimes start to deviate from the straight line (they are considerably smaller) and we enter the regime of the universal envelope curve in figure 10(a).

\(^8\) The curve $B(n_{\text{fixed point}})$ has a minimum in the critical point.
4. Breaking spatial inversion symmetry: the granular ratchet

In sections 2 and 3 we have concentrated on periodic systems that were locally symmetric under spatial inversion, i.e., the position and size of the apertures towards the left and right neighbours are the same when seen from each compartment. As a result the outflow of particles to the right and to the left is also the same, namely $F(n_k)$. We have shown that for this type of system it is possible to derive a Lyapunov function from which time-periodic and chaotic states can be excluded. In this section we will briefly review a system that breaks this local symmetry in a special way, the so-called granular ratchet [6].

The granular ratchet is inspired by a special $K/2$-cluster state in the $K$-compartment granular fountain (with cyclic boundary conditions, and for even $K$), namely the one in which each compartment alternately contains a low and a high particle fraction; see figure 11(a). Now if we alternately remove either the slit or the bottom hole from the set-up, we expect to create a system in which the same distribution of particles will lead to a net current through the system (figure 11(b)). Obviously the magnitude of the net flux through this new system will be exactly the same as that of the convective flow in the corresponding granular fountain, so the fountain distribution should also be a fixed point of the ratchet. It turns out that there is an interval of the driving strength for which this fixed point is actually stable. The name granular ratchet refers to the existence of precisely such a steady solution with a net flow through the hole system, and is an example of a Brownian motor, i.e., a system characterized by the ratchet effect in which isotropic noise is converted into a directed motion [47]. It is in fact a very special type of Brownian motor, since the directed motion does not arise from any intrinsic asymmetry in the system but from the spontaneous symmetry breaking due to the clustering effect [6, 48, 49].

The description of the granular ratchet system in terms of the flux model follows directly from the balance equations for the granular fountain equations (11) and (21):

\[
\begin{align*}
\frac{dn_k}{dt} &= G(n_{k+1}) + F(n_{k-1}) - F(n_k) - G(n_k), & \text{even } k, \\
\frac{dn_k}{dt} &= F(n_{k+1}) + G(n_{k-1}) - F(n_k) - G(n_k), & \text{odd } k,
\end{align*}
\]  

Figure 11. (a) A $K/2$-cluster state in the $K$-compartment granular fountain, here for $K = 4$, with a special distribution in which clusters and diluted compartments alternate. The boundary conditions are cyclic. (b) By alternately deleting either the slit or the bottom hole from the set-up we expect to create a system in which the same distribution leads to a finite flux through all compartments: the granular ratchet.
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Figure 12. Bifurcation diagram for the granular ratchet with $K = 4$ compartments and $\lambda = 0.05$ (cf equation (21)). At $B = 1$ the uniform $n_k = 1$ distribution becomes unstable and gives way to a stable fluxless clustered state through a pitchfork bifurcation (red curves). A second pitchfork bifurcation (at $B \approx 1.2$) generates the (at that moment still unstable) ratchet state, with a finite net flux (black curves). The ratchet state stabilizes through a third pitchfork bifurcation at $B \approx 1.25$ in which a completely asymmetric, unstable state is created (blue curves); it destabilizes again at $B \approx 3.8$ when it recombines with this same asymmetric state. Solid lines denote stable solutions, whereas dashed and dotted lines are unstable.

with cyclic boundary conditions. Note that if $\{n_-, n_+\}$ is a steady state of the two-compartment granular fountain, then the net flux through the bottom hole should balance the net flux through the slit, i.e., $G(n_+) - G(n_-) = -(F(n_+) - F(n_-))$. It now follows directly that the alternating distribution $\{n_+, n_-, n_+, \ldots, n_-\}$ is a fixed point of equation (33), confirming the more intuitive line of reasoning in the previous paragraph. Less obvious is the stability of this solution and the possible existence of further stable solutions, for which we have to consider the full bifurcation diagram of the system.

Figure 12 shows the bifurcation diagram for the smallest conceivable ‘true’ granular ratchet, namely for $K = 4$. As its most prominent new feature it displays two different types of coexisting stable states with spontaneously broken symmetry for not too large $B$-values:

(i) **Ratchet states** with alternating dense and dilute compartments and a resulting finite directed net particle flux. The finite value of this flux is the hallmark of a spontaneous ratchet effect (black solid curves in figure 12).

(ii) **Fluxless clustered states** with alternating pairs of dense and dilute compartments, with detailed balance and no net particle flux (red solid curves in figure 12).

Note that the $K = 2$ granular ratchet is fully equivalent to the $K = 2$ granular fountain (cf section 2.4).

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Starting from the uniform distribution at vigorous driving (low $B$), at $B = 1$ a fluxless clustered state comes into existence through a pitchfork bifurcation, rendering the uniform state unstable. For slightly higher $B$ ($\approx 1.2$), the uniform state becomes even more unstable and the ratchet state comes into existence through a second pitchfork bifurcation. At birth, this ratchet state is also unstable, but it is soon stabilized by the generation of an unstable, completely asymmetric state. There now follows an interval in which the ratchet state and the fluxless clustered state are both stable, until at $B \approx 3.8$ the ratchet state and the completely asymmetric state recombine into an unstable ratchet state, which eventually disappears through a saddle-node bifurcation at $B \approx 4.1$. From $B \approx 3.8$ on, the fluxless clustered state is the only stable state in the system.

Clearly, if we write equation (33) in matrix form we obtain a more complicated expression than the one we obtained for the compartmentalized system and the granular fountain (equation (12)), namely

$$\frac{dn}{dt} = M_L \cdot f + M_R \cdot g,$$

(34)

in which $n$ and $f$ are as defined in section 2.2, and $g$ is a vector with components $G(n_k)$. $M_L$ is a symmetric tri-diagonal $K \times K$-matrix with elements $-1$ on the diagonal and with both first off-diagonals containing alternately 1 and 0. The matrix $M_R$ has the same structure but shifted, such that on the first off-diagonals there is a 1 where $M_L$ contains a zero, and vice versa. Note that $M_L + M_R = M$, with $M$ as defined previously in section 2.2.

Because of the structure of the balance equation there is no Lyapunov function similar to the one defined in equation (14). The existence of a different type of Lyapunov function or functional for the granular ratchet is highly unlikely, but cannot be excluded based on the bifurcation diagram for $K = 4$: the only possible solutions of $dn/dt = 0$ in equation (34) were found to be fixed points. This changes when we turn to the bifurcation diagram of the $K = 6$ ratchet (figure 13).

Again, the uniform distribution is stable below $B = 1$, but now it loses stability in a different way: the stable fluxless clustered state comes into existence through a saddle-node bifurcation before the uniform distribution becomes unstable. The similarity to the bifurcation diagrams of the $K$-compartment systems (cf figure 3(b)) is by no means coincidental: in fact, the bifurcation diagram of the $K = 6$ granular ratchet contains the ‘ordinary’ $K = 3$ bifurcation diagram as a subset (red curves in figure 13)! This can be understood by realizing that the system contains $K/2$ slits, over which clustering can take place. The bottom holes in the other $K/2$ walls merely transport particles between equally filled compartments, which they will do no matter how high $n_k$ is. The set of fixed points for the $K$-compartment ratchet therefore contains the fixed points for a system consisting of $K/2$ compartments connected by slits as a subset.

For a slightly higher value of $B$ ($\approx 1.2$) the uniform distribution goes through a pitchfork bifurcation, in which the unstable ratchet state is created (black curves in figure 13). The ratchet state in turn stabilizes (at $B \approx 1.3$) through a Hopf bifurcation in which an unstable limit cycle is born (blue curves in figure 13). At $B \approx 1.7$ this limit cycle ceases to exist and two unstable asymmetric states are formed. Finally these states recombine with the ratchet state at $B \approx 3.7$, and for slightly higher $B$ the now unstable ratchet state recombines with the uniform state.
Figure 13. Bifurcation diagram for the granular ratchet with $K = 6$ compartments and $\lambda = 0.05$ (cf equation (21)). The main differences with the $K = 4$ bifurcation diagram are the way in which the uniform distribution becomes unstable (at $B = 1$) and the Hopf bifurcation (at $B \approx 1.7$) in which an unstable limit cycle is formed.

In figure 14 we explore the limit cycle in somewhat greater detail for $B = 1.39$. The schematic structure of the limit cycle is depicted in figure 14(a): there is a three-dimensional stable manifold in which the trajectories all lead towards the limit cycle, represented by the straight arrows leading towards the plane in which the limit cycle lies. The limit cycle itself is unstable, and we have schematically drawn the corresponding two-dimensional unstable manifold. Starting from the limit cycle one either spirals inward, towards the ratchet state which is located in the centre of this manifold, or one spirals outward and eventually ends up in one of the fluxless clustered states.

Figure 14(b) shows three trajectories from equation (34) starting close to the limit cycle. The red and the blue trajectories are on either side of the unstable manifold: the red trajectory spirals towards the ratchet in the centre, while the blue trajectory goes outward to end up in a fluxless clustered state. The third, black curve starts extremely close to the manifold and stays close to the limit cycle for the complete duration of the integration.

For $K > 6$ the granular ratchet bifurcation diagram develops more and more limit cycles and antisymmetric states, but there always remains a finite $B$-interval for which the ratchet is stable.

5. Concluding remarks

In this review we have shown how the flux model describes both the steady states (i.e., the bifurcation diagram) and the dynamics of compartmentalized granular gases. The emphasis here has been on the theory, but it should be mentioned that there is excellent agreement between the theoretical predictions on the one hand and experiments and particle simulations on the other. Experiments confirming the bifurcation diagram for $K = 2, 3$ compartments can be found in [1], and for the fountain and ratchet systems
Figure 14. (a) Schematic structure of the unstable limit cycle of the $K = 6$ ratchet: the limit cycle has two unstable directions corresponding to spiraling paths leading to (i) the ratchet state in the centre and (ii) the fluxless clustered states outside the cycle. The arrows perpendicular to the surface of the limit cycle represent stable directions. (b) Three trajectories of the $K = 6$ ratchet very close to the limit cycle (calculated from equation (33) for $B = 1.39$): to obtain this clear view of the limit cycle the trajectories (in the five-dimensional system space) have been projected onto the $(n_1 - n_2, n_3 - n_4)$-plane. The red trajectory spirals inward towards the stable ratchet state at $(n_1 - n_2, n_3 - n_4) = (0.615, 0.615)$; the blue curve spirals outward and will eventually go towards a fluxless clustered state; the black curve is so close to the limit cycle that it has hardly started to deviate from it during the whole integration time.

... Molecular dynamics simulations on the bifurcation structure of the $K$-compartment granular fountain are given in [7].

Also the dynamics of the clustering process has been tested experimentally. In figure 15 we show an experiment for $K = 12$ compartments (taken from [50]) illustrating the slow coarsening process leading to the final state with a single cluster. Experiments for the reverse process of declustering, when the shaking strength is turned up above the critical value and the cluster breaks down (see figure 9), have been reported in [3].

Experiments and simulations have also been performed on the $K = 4$ ratchet system, and the measured net flow through the ratchet was found to agree accurately with the model prediction [6]. In fact, our $K = 4$ ratchet set-up was the first experimental realization of a ‘stochastic’ Brownian motor without any asymmetry in its geometry, a concept that had been predicted several years before on theoretical grounds (see [47] for a review): the symmetry breaking, necessary for the generation of a net flux, here comes about spontaneously as a collective clustering effect of the many stochastically colliding particles.

One interesting aspect of the compartmentalized gases that we have not touched upon in this review is the fact that they are intrinsically noisy, since they contain many fewer than the standard $10^{23}$ particles of textbook statistical systems. Thus, granular gases are very natural and suitable systems to study small-number statistical fluctuations, and in particular the influence of these fluctuations on critical phenomena like the clustering transition. The fluctuations are not taken into account in the flux model, it being a...
Figure 15. Coarsening experiment: top view of a 12-compartment set-up, vertically shaken with amplitude $a = 1.0$ mm and frequency $f = 40$ Hz. Starting from a uniform distribution, the system is seen to go through transient states with four, three, and two clusters (with increasing lifetimes) before finally reaching the stable one-cluster state (from [50]).

mean-field model, but the role of fluctuations in the $K = 2$ system was touched upon by Eggers [12] and Lipowski and Droz [18], and worked out in detail in [51]. It was found that the noise can be incorporated into the flux model via a Gaussian white noise term in the balance equation (11). Furthermore, corroborated by molecular dynamics simulations, it was found that already for $N_{\text{tot}} = 300$ particles (i.e., many fewer than $10^{23}$) the mean-field description of the flux model holds accurately [51]. Only for smaller $N_{\text{tot}}$ does the finite-number noise start to dominate and the mean-field description break down.

It would be interesting to extend these studies to more compartments, to see how the different bifurcation structures for $K \geq 2$, or for the ratchet system, are affected by the finite-number noise. Based on the result for $K = 2$ compartments it may be anticipated that the mean-field results of the present review will hold as long as the total number of particles is of the order of $100K$ or more.

In section 1 we have already discussed the connection of the compartmentalized granular gases to zero-range processes and asymmetric exclusion processes. Let us conclude with another intriguing connection, namely, the traffic jam problem. There exists a strong analogy between traffic flow on a highway and the flux of granular particles from one compartment to the next: if one divides the highway into imaginary cells (with a length of say 1 km per cell), and $\rho_k$ denotes the density of cars in cell $k$, the traffic flow from one cell to the next is given by a flux function $F(\rho_k)$ that shows a marked resemblance to the granular flux function in figure 2. Among traffic analysts, the function $F(\rho_k)$ is known as the ‘fundamental diagram’ [30], and it has been measured innumerable times in real-life situations. At low densities, the cars flow freely from cell to cell at their desired velocity and $F(\rho_k)$ is an increasing function of $\rho_k$, whereas at high densities the cars hinder each other (i.e., the interaction between them is dissipative, just as in the case of the granular particles) and therefore $F(\rho_k)$ decreases. As we have seen in the present paper, clustering (traffic jam formation) is adequately described with this type of flux function. Indeed, in a case study along the highway between the Dutch cities of Breda and Eindhoven, given the traffic situation at a certain moment, our flux model—with proper adaptations—has proven to be able to predict in which cells and at which moment traffic jams are most likely to form within the next half hour [52].
Compartmentalized granular gases

In conclusion, compartmentalized granular gases provide a beautiful example of a many-particle system far from equilibrium, showing pattern formation, coarsening, sudden collapse, a stochastic ratchet effect, and numerous other physical phenomena that are interesting from a fundamental point of view and at the same time important for practical applications. They are related to a multitude of non-equilibrium systems with similar properties. What makes the granular gases stand out is their direct accessibility to experiment, numerical simulations, and theory. Thanks to this, all phenomena can be studied by an integral approach in which these three go hand in hand.

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