



LETTER TO THE EDITOR • **OPEN ACCESS**

On the dimension of angles and their units

To cite this article: Peter J Mohr *et al* 2022 *Metrologia* **59** 053001

View the [article online](#) for updates and enhancements.

You may also like

- [Angles are inherently neither length ratios nor dimensionless](#)
Paul Quincey, Peter J Mohr and William D Phillips
- [Reply to Comment on 'Angles in the SI: a detailed proposal for solving the problem'](#)
Paul Quincey
- [On the status of plane and solid angles in the International System of Units \(SI\)](#)
M I Kalinin

On the dimension of angles and their units

Peter J Mohr^{1,*}, Eric L Shirley¹, William D Phillips^{1,2} and Michael Trott³

¹ National Institute of Standards and Technology, Gaithersburg, MD 20899, United States of America

² Joint Quantum Institute, University of Maryland, College Park, MD 20742, United States of America

³ Wolfram Inc., 100 Trade Center Drive Champaign, IL 61820-7237, United States of America

E-mail: mohr@nist.gov, eric.shirley@nist.gov, william.phillips@nist.gov and trott@wolfram.com

Received 1 April 2022, revised 22 June 2022

Accepted for publication 23 June 2022

Published 1 September 2022



Abstract

We show the implications of angles having their own dimension, which facilitates a consistent use of units as is done for lengths, masses, and other physical quantities. We do this by examining the properties of complete trigonometric and exponential functions that are generalizations of the corresponding functions that have dimensionless numbers for arguments. These generalizations provide functions of angles with the dimension of angle as arguments, but with no reference to units. This parallels most equations in physics which are valid for any units. This property also provides a consistent framework for including quantities involving angles in computer algebra programs without ambiguity that may otherwise occur. This is in contrast to the conventional practice in scientific applications involving trigonometric or exponential functions of angles where it is assumed that the argument is the numerical part of the angle when expressed in units of radians. That practice also assumes that the functions are the corresponding radian-based versions. These assumptions allow angles to be treated as if they had no dimension and no units, an approach that can lead to important difficulties such as incorrect factors of 2π , which can be avoided by assigning an independent dimension to angles.

Keywords: radian, SI, dimension, angle, units

(Some figures may appear in colour only in the online journal)

1. Introduction

Angle is a familiar concept that needs no formal definition. When two lines cross in a plane, two pairs of vertical angles are formed. Unless the lines are perpendicular, one pair of angles is acute and the other pair is obtuse. In trigonometry, the geometric properties of interior angles of right triangles may be used to derive general relations between various trigonometric functions. For example, the formula for the sine of the sum of two angles in terms of sine and cosine functions of each of the angles, does not depend on a quantitative specification of the angles. That is, when we write $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ we do not specify whether α and β are measured in degrees or radians or grads, nor do we specify that

the appropriate functions be used, according to the units of the arguments. This and other features of angles lead to the often-repeated (and often misunderstood) assertion that angles do not have units and therefore do not have a dimension (or that their dimension is unity).

By contrast, when angles appear in physical measurements, it is essential to specify the unit of measure for the angle. For example, it would be of no use to specify the angle subtended by the distance between two stars, or the angle between two laser beams, or the rotation angle of a wheel without giving units like arcseconds or milliradians or full rotations.

All of these comments about plane angles apply equally to phase angles. For example, the amplitude of an electromagnetic plane wave of fixed frequency and propagation vector (wave vector) is proportional to a sine function of an argument, a phase angle, whose value changes linearly with time and distance. If we want to report the difference in phase angle between two spacetime points, for example, we must specify the units—degrees or radians or cycles (full rotations).

* Author to whom any correspondence should be addressed.



Original content from this work may be used under the terms of the [Creative Commons Attribution 4.0 licence](https://creativecommons.org/licenses/by/4.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Expressions for the frequencies of periodic phenomena are particularly important examples of the difficulties associated with units for angles. Frequency is the time rate of change of the phase angle associated with a periodic series of events. As such, it is just as important that the units of frequency reflect the units of angles as that they express the units of time. Frequencies are usually given in units of radians per second, or hertz, meaning cycles (full rotations or revolutions of the phase angle) per second. Less often, frequencies might be given in degrees per second or revolutions per minute. Unfortunately, sometimes frequencies are simply expressed in inverse seconds, leading to confusion about what the unit really is.

If we agree that angles have units, we must also agree that they have a dimension, the dimension of angle, which is independent of other dimensions like length, time, or mass. Sometimes it is argued that angles do not have an independent dimension, or that the dimension is unity, because the dimension of angle is length divided by length. The justification given for this is that one may measure angles as arc length divided by radius vector length. However, this ratio gives only the numerical factor of the quantity expression for the angle. To completely specify the angle in this case, the unit radian must be included. Without a unit, the numerical factor could be the number of degrees of the angle, or anything else. Moreover, that is only one way in which angle might be measured [1]. In fact, the most common way of measuring angles is with a protractor, whereby unknown angles are compared to known angles. This is exactly analogous to measuring an unknown length by comparing it to known lengths. Angles and lengths are manifestly different kinds of physical quantities, and each has its associated dimension.

While we argue that angles are physical quantities distinct from other physical quantities or combinations of such quantities, and that they therefore have their own dimension and unit, we recognize, as have many others, that this represents a choice, and that that choice is made in part for the sake of convenience. The number of base units in the metric system (now the SI, the International System of Units) has changed over time, because of changes in what was considered to be most useful and convenient. The scientific community could have decided to have a unit system in which temperature is measured in joules, but we find it to be more convenient to measure temperature in kelvins. In some sense we have the same kind of choice here with angles, and we argue that the advantage achieved by giving angles a separate dimension and unit outweigh any difficulties.

Our examination of the question of how to deal with units for angles is of interest not only for the important goal of a consistent, unambiguous use of units in physics, but also for systematically including units for angles in algebraic computer software applications. The current practice of sometimes ignoring the units and resolving any resulting ambiguity by human judgment is not suitable for the digital age.

There is a long history of discussions of the role of units for angles [1–59]. The underlying ideas in this paper have, in

essence, been considered in a multitude of earlier works. Here, we have synthesized many of those ideas, while paying particular attention to making the proposed reforms compatible with modern computer processing.

While it would be impractical for every argument in a numerical computation of trigonometric functions to explicitly carry an angle, having radians, degrees, ..., in general (compound) units that describe physical quantities, such as angular velocity or differential cross sections, are crucial for automated symbolic computations where results from one computation have to be dimensionally consistent with the next computation step. For instance, *Mathematica* [60] considers angles to have an angle dimension. But in the absence of a well-defined standard, dimensional inconsistencies resulting from conversions such as ‘1 Hz = 1/s’ are unavoidable and currently require hard-coded heuristics rather than a fully deterministic algorithmic treatment. The issue of the consistent presence of an angle dimension becomes even more amplified at the level of equations that involve physical quantities (see sections 5.4 and 5.5 below).

Although there is a diversity of opinions on the question, the overwhelming majority of papers on the subject acknowledge that angles should be regarded as having an independent dimension and associated units. The difficulty of consistently implementing this proposal is also often cited as a reason for maintaining the acknowledged unsatisfactory status quo.

2. Unit notation

Following Maxwell [4] and the general practice of international metrology, we specify a physical quantity Q by a coefficient and a unit as

$$Q = \{Q\}[Q], \quad (1)$$

where $\{Q\}$ is a real or complex number as in Maxwell’s definition (or more generally an operator, matrix, etc), and $[Q]$ is the unit. Or to be more precise, we may write

$$Q = \{Q\}_{[Q]}[Q], \quad (2)$$

because the value of $\{Q\}$ depends on the unit $[Q]$.

For example, a length of 3 metres, $L = 3 \text{ m}$, corresponds to $\{L\} = 3$ and $[L] = \text{m}$. The choice of units in equation (1) is not unique, so we may have

$$Q = \{Q\}_{[Q]_1}[Q]_1 = \{Q\}_{[Q]_2}[Q]_2. \quad (3)$$

For the previous example, the length L could (ill-advisedly) also be expressed in inches as

$$L = 3 \text{ m} = 118.11 \dots \text{ in}, \quad (4)$$

where $[L]_1 = \text{m}$, $\{L\}_m = 3$, $[L]_2 = \text{in}$, and $\{L\}_{\text{in}} = 118.11 \dots$. The quantity Q is the same physical quantity, regardless of the unit in which it is expressed.

3. Units for angles

When we consider angles as measurable physical quantities, as with other such quantities, an angle θ may be expressed as

$$\theta = \{\theta\}_{[\theta]}[\theta], \quad (5)$$

where $[\theta]$ is one of any number of possible units. The list includes degree, minute, second, radian, revolution or cycle, grad, among other possible choices, although those mentioned are commonly used. We focus on degree, radian, and revolution (i.e., cycle or period) as examples, with the understanding that generalization to other units is always possible.

An example of a particular angle, expressed in different units, is

$$\theta = 45 \text{ deg} \quad \text{where } [\theta] = \text{deg} \quad \text{and } \{\theta\}_{\text{deg}} = 45, \quad (6)$$

$$\theta = \frac{\pi}{4} \text{ rad} \quad \text{where } [\theta] = \text{rad} \quad \text{and } \{\theta\}_{\text{rad}} = \frac{\pi}{4}, \quad (7)$$

$$\theta = \frac{1}{8} \text{ rev} \quad \text{where } [\theta] = \text{rev} \quad \text{and } \{\theta\}_{\text{rev}} = \frac{1}{8}, \quad (8)$$

where the notation $45 \text{ deg} \equiv 45^\circ$ is used, and rev is the abbreviation for revolution. In these examples, the unit provides necessary information about the angle. If an angle were simply given as a number, say ' $\theta = 45$ ', with no unit, it could taken to be 45 rad, which is a perfectly respectable angle, although it may not be the one the author had in mind.

The relationship between plane right triangles and the trigonometric functions is clear. For a given interior angle, the sine of this angle is the ratio of the length of the side opposite the angle to the length of the hypotenuse. This is a real number between 0 and 1. (Here we limit the discussion to real angles, although the extension to complex angles is considered below.) Thus the properties of trigonometric functions are based on geometrical angles, whereas the functions themselves are usually defined to have numbers as the argument. This apparent inconsistency will be addressed in section 4.

If we are working in an environment where only numbers are being used, such as a scientific calculator or traditional FORTRAN, then the unit ambiguity is avoided if it is recognized that the sine function itself depends on what units are being used. As is well known, looking up the sine of a number representing an angle depends on its unit, and vice versa. There are different trigonometric tables for angles in degrees and for angles in radians. Similarly, when using a calculator to find the sine or inverse sine of a number, it is necessary to specify which unit is being assumed for the input or expected as output by, for example, touching the Deg/Rad key first. This dependence on the type of sine function being used can be denoted by writing

$$\sin_{\text{deg}}(45) = \frac{1}{\sqrt{2}}, \quad (9)$$

$$\sin_{\text{rad}}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad (10)$$

$$\sin_{\text{rev}}\left(\frac{1}{8}\right) = \frac{1}{\sqrt{2}}, \quad (11)$$

where the subscript on the name of the function indicates the unit being assumed for the angle. Similarly, for the inverse functions, we have

$$\arcsin_{\text{deg}}\left(\frac{1}{\sqrt{2}}\right) = 45, \quad (12)$$

$$\arcsin_{\text{rad}}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \quad (13)$$

$$\arcsin_{\text{rev}}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{8}, \quad (14)$$

where the particular arcsine function being used determines the value for the inverse. However, even if the type of arcsine function is specified, the result on the right-hand side of equations (12)–(14) has lost that information. As a result, the numbers 45, $\pi/4$, or $1/8$ could be degrees, radians, revolutions, or anything else. As humans, we can look back at our notes and figure it out, but it is useful to retain that information along with the number, either to provide complete information about the result or so that it can unambiguously be used for the next step in a series of calculations. This is the rationale for using units in the first place, and it is important for information to be processed by computers.

To accomplish this, specification of the unit is added to the 'output' of equations (12)–(14). Thus for the angle $\theta = 45 \text{ deg} = \pi/4 \text{ rad} = 1/8 \text{ rev}$, we have

$$\theta = \arcsin_{\text{deg}}\left(\frac{1}{\sqrt{2}}\right) \text{ deg} = 45 \text{ deg}, \quad (15)$$

$$\theta = \arcsin_{\text{rad}}\left(\frac{1}{\sqrt{2}}\right) \text{ rad} = \frac{\pi}{4} \text{ rad}, \quad (16)$$

$$\theta = \arcsin_{\text{rev}}\left(\frac{1}{\sqrt{2}}\right) \text{ rev} = \frac{1}{8} \text{ rev}. \quad (17)$$

Note that equations (9)–(14) are mapping of numbers to numbers without dimensions, but equations (15)–(17) provide angles with units included. These equations may be generalized to an angle $\theta = \{\theta\}_{[\theta]}[\theta]$ expressed in any unit $[\theta]$, as in equation (5), by writing

$$w = \sin_{[\theta]}(\{\theta\}_{[\theta]}) \quad (18)$$

and

$$\{\theta\}_{[\theta]} = \arcsin_{[\theta]}(w). \quad (19)$$

In general, because the sine function is periodic, an infinite number of angles map into a particular value of the sine function, so we assume that the formula for the inverse gives angles in the range: $|\{\theta\}_{[\theta]}| \leq 90, \pi/2, 1/4$, as appropriate.

In this section, we have considered the purely numerical sine and arcsine functions. That is, functions that have real numbers for both their domains and ranges. The extension to a sine function of angles with dimensions and any units and to the corresponding arcsine function that has dimensional angles as its range is examined in the next section. Of course, this generalization applies to the cosine and exponential functions and their inverses as well.

4. Unit-independent transcendental functions

Here, we consider the concept of ‘complete’ functions [6]. That is, they are independent of the units in which the arguments are expressed. This is a property of trigonometric functions based on plane geometry, which may be derived with no reference to units for the angles. On the other hand, for physical applications, it is useful to specify units for both the arguments of the functions and for the functions themselves. This has been spelled out in detail above for the three examples of units of degrees, radians, and revolutions or cycles. For this purpose, it is useful to make a clear distinction between trigonometric functions that have angles with dimensions as arguments and trigonometric functions that have real numbers as arguments, as considered in the preceding sections. In the latter case, the real number is the numerical factor of the angle when expressed in a particular unit. The terminology ‘geometrical angle’ and ‘analytical angle’ for these differing representations for angles was applied by Romain [8], although only radian units for the analytic form were considered at the time. This terminology has been repeated in many works since then. Here we examine the relation between complete trigonometric functions and the explicit unit forms considered in the foregoing.

4.1. Trigonometric functions

To describe complete transcendental functions, it is necessary to consider generalizations of their derivatives. Angles may be represented as quantities with the dimension of angle, but with an arbitrary unit. Infinitesimal changes in an angle correspond to infinitesimal changes in the ratios of the associated sides of the triangle that includes the angle. This relationship provides a quantitative relation between changes of the angle and changes of the trigonometric functions. However, the conventional formula for the derivative,

$$\left(\frac{d}{d\theta} \sin(\theta) = \cos(\theta) \right), \quad (20)$$

is problematic, because the equation is not dimensionally consistent. Both sine and cosine are dimensionless, as they are ratios of lengths of the sides of a triangle, so the left-hand side of the equation has the dimension of the inverse of an angle, and the right-hand side is dimensionless. As mentioned previously, angles are not intrinsically dimensionless. The consequence of angles not being dimensionless is that the standard formulas which assume particular units are being used are not sufficiently general, and it is necessary to reexamine the relevant formulas to take the dimensionality into account.

To address this, we calculate the derivative as the defining limit by writing

$$\frac{d}{d\theta} \sin(\theta) = \lim_{\Delta\theta \rightarrow 0} \frac{\sin(\theta + \Delta\theta) - \sin(\theta)}{\Delta\theta}. \quad (21)$$

From the conventional angle sum identity, based on the geometric properties of triangles and the notation in figure 1, we have

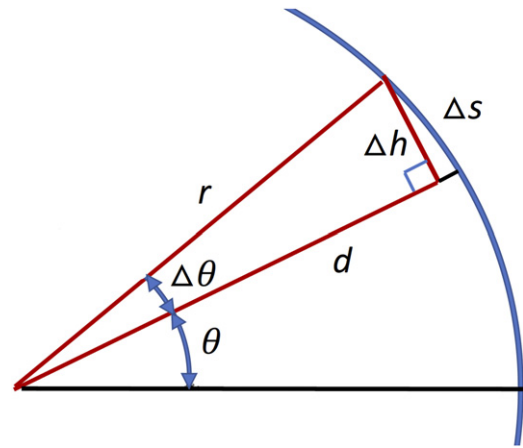


Figure 1. The triangle (in red) relevant to equations (21)–(24) with angle $\Delta\theta$. The hypotenuse has length r , the adjacent side has length d and the opposite side has length Δh . The arc length of the circle of radius r subtended by $\Delta\theta$ is Δs .

$$\begin{aligned} \sin(\theta + \Delta\theta) &= \sin(\theta) \cos(\Delta\theta) + \cos(\theta) \sin(\Delta\theta) \\ &= \sin(\theta) \frac{d}{r} + \cos(\theta) \frac{\Delta h}{r}. \end{aligned} \quad (22)$$

Also from figure 1, it is evident that

$$\lim_{\Delta\theta \rightarrow 0} \frac{d}{r} = 1, \quad (23)$$

$$\lim_{\Delta\theta \rightarrow 0} \frac{\Delta h}{\Delta s} = 1. \quad (24)$$

Thus,

$$\frac{d}{d\theta} \sin(\theta) = C \cos(\theta), \quad (25)$$

where

$$C = \lim_{\Delta\theta \rightarrow 0} \frac{\Delta s}{r \Delta\theta} = \frac{1}{r} \frac{ds}{d\theta}. \quad (26)$$

Note that this provides a dimensionally consistent result for the derivative. The coefficient C can be evaluated by the integration

$$C \int_0^\Theta d\theta = \frac{1}{r} \int_0^{2\pi r} ds, \quad (27)$$

where Θ is the angle of a complete revolution or period of the sine function

$$\sin(\theta + \Theta) = \sin(\theta), \quad (28)$$

and $2\pi r$ is the circumference of a circle of radius r , which yields

$$C = \frac{2\pi}{\Theta}. \quad (29)$$

Similarly, for the cosine function, we have

$$\frac{d}{d\theta} \cos(\theta) = \lim_{\Delta\theta \rightarrow 0} \frac{\cos(\theta + \Delta\theta) - \cos(\theta)}{\Delta\theta}, \quad (30)$$

where

$$\begin{aligned} \cos(\theta + \Delta\theta) &= \cos(\theta) \cos(\Delta\theta) - \sin(\theta) \sin(\Delta\theta) \\ &= \cos(\theta) \frac{d}{r} - \sin(\theta) \frac{\Delta h}{r}, \end{aligned} \quad (31)$$

which gives

$$\frac{d}{d\theta} \cos(\theta) = -C \sin(\theta). \quad (32)$$

4.2. Exponential function

The complete exponential function of an angle can be defined in terms of the complete trigonometric functions. This generalization recognizes that the exponential function expressed as Euler's number e raised to a power is not a complete function. In that case the argument is the numerical part of the angle expressed in radian units, as shown in the next section.

The complete-function definition is

$$\exp(i\theta) = \cos(\theta) + i \sin(\theta), \quad (33)$$

where $i^2 = -1$. That is, we do not specify what number is raised to a power. The derivative follows from the derivatives of the cosine and sine functions as

$$\begin{aligned} \frac{d}{d\theta} \exp(i\theta) &= -C \sin \theta + i C \cos \theta \\ &= i C \exp(i\theta) \\ &= i \frac{2\pi}{\Theta} \exp(i\theta). \end{aligned} \quad (34)$$

We also have $\exp(0) = 1$. With this generalization of the derivative of the exponential function, one obtains the power series

$$\exp(i\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2\pi i \theta}{\Theta} \right)^n, \quad (35)$$

or

$$\exp(i\theta) = e^{2\pi i \theta / \Theta}. \quad (36)$$

The expressions on the right-hand side of equations (35) and (36) are well-defined, because the ratio θ/Θ is just a number with no unit. Thus the derivation starting from equation (21) through equation (36) provides an expression for the complete exponential function with a dimensional argument in terms of an analytic dimensionless function with no dependence on units.

The power series for the complete cosine and sine functions, the real and imaginary parts of equation (35), are

$$\cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{2\pi\theta}{\Theta} \right)^{2n}, \quad (37)$$

$$\sin(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{2\pi\theta}{\Theta} \right)^{2n+1}. \quad (38)$$

If

$$z = \exp(i\theta) \quad (39)$$

then the complete logarithmic function is

$$\log(z) = i\theta + ik\Theta, \quad (40)$$

where θ is the dimensional geometric angle and k is an integer. Here the name 'log' with no label is meant to signify the complete logarithmic function, as for the trigonometric and exponential function names in this paper.

To gain some perspective, it is useful to consider differentiation and integration of the complete logarithmic function to check the consistency of viewing it as having the dimension of angle. From equation (34), we have

$$\frac{dz}{d\theta} = \frac{2\pi i}{\Theta} z, \quad (41)$$

so that

$$\frac{d}{dz} \log(z) = \frac{d}{dz} i\theta = \frac{\Theta}{2\pi} \frac{1}{z}. \quad (42)$$

Thus the derivative of the logarithmic function also has the correct dimension of angle. (Recall that z is a dimensionless number.) This can be extended to the power series of $\log(1-z)$ by noting that

$$\frac{d}{dz} \log(1-z) = -\frac{\Theta}{2\pi} \frac{1}{1-z}, \quad (43)$$

$$\left(\frac{d}{dz} \right)^n \log(1-z) = -\frac{\Theta}{2\pi} \frac{(n-1)!}{(1-z)^n}, \quad (44)$$

which gives

$$\log(1-z) = -\frac{\Theta}{2\pi} \sum_{n=1}^{\infty} \frac{z^n}{n}. \quad (45)$$

To check the integral of $\log(z)$, we write

$$\frac{d}{dz} z \log(z) = \log(z) + \frac{\Theta}{2\pi} = \log(z) + \frac{d}{dz} \frac{\Theta}{2\pi} z, \quad (46)$$

or

$$\frac{d}{dz} \left(z \log(z) - \frac{\Theta}{2\pi} z \right) = \log(z) \quad (47)$$

so that

$$\int \log(z) dz = z \log(z) - \frac{\Theta}{2\pi} z + \text{constant}, \quad (48)$$

where the constant has the dimension of angle. Thus the integral of the logarithmic function also has the proper unit of angle. Of course, this expression reduces to the special case commonly used, where the logarithmic function is assumed to have the base e and the radian is replaced by 1. That is, $\log \rightarrow \ln$ and $\Theta \rightarrow 2\pi$. In this form, the indefinite integral of $\ln(z)$ is $z \ln(z) - z$.

The exponential function in equation (35) may be analytically continued to real values of the argument to give

$$\exp(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2\pi\phi}{\Theta} \right)^n, \quad (49)$$

which provides the complete hyperbolic functions \cosh and \sinh as

$$\cosh(\phi) = \frac{\exp(\phi) + \exp(-\phi)}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{2\pi\phi}{\Theta} \right)^{2n}, \quad (50)$$

$$\sinh(\phi) = \frac{\exp(\phi) - \exp(-\phi)}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\frac{2\pi\phi}{\Theta} \right)^{2n+1}. \quad (51)$$

4.3. Explicit unit expressions

4.3.1. Arbitrary units, $[\theta] = A$. To connect to the earlier sections in which particular units are considered, the expressions in the previous section can be given for a particular, but arbitrary, choice of unit $[\theta] = A$. In this case, we employ the relations $\Theta = \{\Theta\}_A A$ and $\theta = \{\theta\}_A A$ to write

$$\frac{\theta}{\Theta} = \frac{\{\theta\}_A}{\{\Theta\}_A}, \quad (52)$$

a ratio that is independent of units. Equation (36) may be written as

$$\exp(i\theta) = \exp_A(i\{\theta\}_A) = b_A^{i\{\theta\}_A}, \quad (53)$$

where

$$b_A = e^{2\pi/\{\Theta\}_A} \quad (54)$$

is the base of the exponential function for the unit A . The corresponding logarithmic function is

$$\log_{b_A}(z) = i\{\theta\}_A + ik\{\Theta\}_A, \quad (55)$$

where

$$z = b_A^{i\{\theta\}_A}. \quad (56)$$

This can be contrasted to the complete logarithmic function in equation (40), which has the dimension of angle as its value, whereas in equation (55) the function is dimensionless. Note that $\text{Re} \log_A(b_A) = 1$.

We also have

$$\cos_A(\{\theta\}_A) = \frac{b_A^{i\{\theta\}_A} + b_A^{-i\{\theta\}_A}}{2}, \quad (57)$$

$$\sin_A(\{\theta\}_A) = \frac{b_A^{i\{\theta\}_A} - b_A^{-i\{\theta\}_A}}{2i}. \quad (58)$$

4.3.2. Radian unit, $A = \text{rad}$. If A is the radian, we have $\{C\}_{\text{rad}} = 2\pi/\{\Theta\}_{\text{rad}} = 1$, $b_{\text{rad}} = e$, and

$$\exp(i\theta) = \exp_{\text{rad}}(i\{\theta\}_{\text{rad}}) = e^{i\{\theta\}_{\text{rad}}}, \quad (59)$$

which is the conventional result. If

$$z = e^{i\{\theta\}_{\text{rad}}}, \quad (60)$$

then

$$\log_e(z) = \ln(z) = i\{\theta\}_{\text{rad}} + ik2\pi. \quad (61)$$

Also

$$\cos_{\text{rad}}(\{\theta\}_{\text{rad}}) = \frac{e^{i\{\theta\}_{\text{rad}}} + e^{-i\{\theta\}_{\text{rad}}}}{2}, \quad (62)$$

$$\sin_{\text{rad}}(\{\theta\}_{\text{rad}}) = \frac{e^{i\{\theta\}_{\text{rad}}} - e^{-i\{\theta\}_{\text{rad}}}}{2i}. \quad (63)$$

As is evident from equations (59)–(63), where $\{C\}_{\text{rad}} = 1$, the conventional trigonometric and exponential functions are implicitly assumed to be based on the radian unit. For example, the relation that the cosine function is the derivative of the sine function with no numerical coefficient assumes that

the argument is the numerical factor of the angle expressed in the radian unit. Similarly, when the exponential function is written as the base e raised to the power of the argument, it is assumed that the argument is the numerical factor of the angle expressed in the radian unit.

4.3.3. Revolution or cycle unit, $A = \text{rev}$. In this case, $\{C\}_{\text{rev}} = 2\pi/\{\Theta\}_{\text{rev}} = 2\pi$, $b_{\text{rev}} = e^{2\pi}$, and

$$\exp(i\theta) = \exp_{\text{rev}}(i\{\theta\}_{\text{rev}}) = e^{2\pi i\{\theta\}_{\text{rev}}}. \quad (64)$$

It is important to be aware of the units being considered, because it is also often implicitly assumed that the revolution (i.e., cycle or period), rather than the radian, is the suppressed unit, particularly where the angle denotes phase. For example, wavelength λ is often taken to be the distance over which a spatially periodic function undergoes a phase change corresponding to one revolution or cycle rather than one radian. In the latter case, it is often called the reduced wavelength λ . The risk of not taking this into account can and does lead to errors of 2π . Similarly, frequencies, are commonly expressed in Hz or cycles s^{-1} rather than radians s^{-1} , which can also lead to the same error.

4.3.4. Dimensionless functions. Various forms of the exponential and logarithmic functions are described above in terms of units for the dimension of angle or phase. For purely numerical (dimensionless) applications, the radian-base forms are commonly used with no reference to angle units. In such applications, we have

$$y = e^x, \quad (65)$$

and

$$x = \ln(y), \quad (66)$$

where x and y are dimensionless real numbers.

Another commonly used dimensionless base is 10, where

$$y = 10^x, \quad (67)$$

and

$$x = \log_{10}(y). \quad (68)$$

In any of these applications, there is no reference to angle or phase.

5. Applications

The use of complete transcendental functions can resolve some apparent mismatches of angle units that appear in commonly used equations in physics. To examine this, it is useful to distinguish between ‘unit analysis’ and ‘dimensional analysis’. As considered here, dimensional analysis is more general in the sense that a geometrical angle or phase angle may be expressed in various units, such as degrees, radians, and revolutions or cycles, but in all cases, it has the dimension of angle, which we denote by A . This can be expressed by writing $\langle\theta\rangle = A$, where θ is an angle and the brackets are used to denote dimension, in contrast to the notation for the unit $[\theta] = \text{rad}$, for example. Other dimensions that we will consider here are time, length, mass, and charge denoted by T , L , M ,

and \mathbf{Q} , respectively. (While the ampere, with dimension ‘current’ is one of the traditional base units of the SI, the definition of the ampere involves the defining charge of the electron, Hence, it is natural, when considering electric units, to focus on the unit coulomb and its associated dimension, electrical charge.) In all cases, the dimensions on either side of an equation must be the same.

In the following, examples are given where a dimensional analysis based on complete functions resolves an apparent angle unit ambiguity.

5.1. Centripetal acceleration

A familiar relation in physics is the equation for centripetal acceleration a_c of a mass, in uniform circular motion, conventionally given by

$$a_c = \frac{v^2}{r} = r\omega^2, \quad (69)$$

where v is the linear velocity of the mass, r is the radius of the path, and ω is the rotational frequency of the mass. The SI units of the two terms on the right-hand end of that equation are m s^{-2} and $\text{m rad}^2 \text{s}^{-2}$. There is an obvious mismatch between these two terms, which is also reflected in the dimensional analysis which gives LT^{-2} and $\text{LA}^2 \text{T}^{-2}$ for the latter two terms, whereas a_c has the dimension LT^{-2} . This difference can be resolved by deriving the complete expression for the acceleration from basic principles.

Consider the uniform circular motion of a mass described by its position in cylindrical coordinates $\mathbf{r}(r, \phi)$. For motion in the $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ plane, we have

$$\mathbf{r}(r, \phi) = r \cos(\phi) \hat{\mathbf{i}} + r \sin(\phi) \hat{\mathbf{j}}, \quad (70)$$

where $\phi = \omega t$. The centripetal acceleration is

$$\mathbf{a}_c = \frac{d^2}{dt^2} \mathbf{r}(r, \phi) = \omega^2 \frac{d^2}{d\phi^2} \mathbf{r}(r, \phi). \quad (71)$$

From equations (25), (29) and (32), we have

$$\frac{d^2}{d\phi^2} \cos(\phi) = -\left(\frac{2\pi}{\Theta}\right)^2 \cos(\phi), \quad (72)$$

$$\frac{d^2}{d\phi^2} \sin(\phi) = -\left(\frac{2\pi}{\Theta}\right)^2 \sin(\phi), \quad (73)$$

so that

$$\mathbf{a}_c = -\left(\frac{2\pi\omega}{\Theta}\right)^2 \mathbf{r}(r, \phi), \quad (74)$$

or

$$a_c = r \left(\frac{2\pi\omega}{\Theta}\right)^2. \quad (75)$$

This has proper dimension of LT^{-2} , which resolves the disagreement in equation (69).

For the radian as the angle unit, we have $\Theta = 2\pi$ rad and

$$a_c = r\omega_{\text{rad}}^2, \quad (76)$$

where $\omega_{\text{rad}} \equiv \omega/(1 \text{ rad})$. This means that $a_c = r\omega^2$ is not a complete equation, although $a_c = v^2/r$ and equation (75) are, which resolves the mismatch [61].

5.2. Volume integration

If we do a transformation of a Cartesian volume element to a volume element in spherical coordinates, we have

$$\langle dx dy dz \rangle = \text{L}^3, \quad (77)$$

$$\langle r^2 dr \sin \theta d\theta d\phi \rangle = \text{L}^3 \text{A}^2. \quad (78)$$

How can this be made consistent?

The transformed coordinates are

$$x = r \sin(\theta) \cos(\phi), \quad (79)$$

$$y = r \sin(\theta) \sin(\phi), \quad (80)$$

$$z = r \cos(\theta). \quad (81)$$

The transformation is given by the Jacobian determinant

$$\begin{aligned} dx dy dz &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} dr d\theta d\phi \\ &= \left(\frac{2\pi}{\Theta}\right)^2 \begin{vmatrix} \sin(\theta) \cos(\phi) & r \cos(\theta) \cos(\phi) & -r \sin(\theta) \sin(\phi) \\ \sin(\theta) \sin(\phi) & r \cos(\theta) \sin(\phi) & r \sin(\theta) \cos(\phi) \\ \cos(\theta) & -r \sin(\theta) & 0 \end{vmatrix} dr d\theta d\phi \\ &= \left(\frac{2\pi}{\Theta}\right)^2 r^2 dr \sin(\theta) d\theta d\phi, \end{aligned} \quad (82)$$

which has the dimension L^3 , as required. Thus, for an integral, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) dx dy dz \\ &= \left(\frac{2\pi}{\Theta} \right)^2 \int_0^{\Theta} \int_{-\Theta/2}^{\Theta/2} \int_0^{\infty} f(r \sin(\theta) \cos(\phi), r \sin(\theta) \\ & \quad \times \sin(\phi), r \cos(\theta)) r^2 dr \sin(\theta) d\theta d\phi. \end{aligned} \quad (83)$$

If the unit for angles is the radian, then $\Theta = 2\pi$ rad, and

$$\sin(\theta) = \sin_{\text{rad}}(\{\theta\}_{\text{rad}}) \equiv \sin_{\text{rad}}(\theta_{\text{rad}}) \quad \text{etc.} \quad (84)$$

$$\frac{2\pi}{\Theta} d\theta = d\{\theta\}_{\text{rad}} \equiv d\theta_{\text{rad}} \quad \text{etc.} \quad (85)$$

This gives

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) dx dy dz \\ &= \int_0^{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} f(r \sin_{\text{rad}}(\theta_{\text{rad}}) \cos_{\text{rad}}(\phi_{\text{rad}}), \\ & \quad r \sin_{\text{rad}}(\theta_{\text{rad}}) \sin_{\text{rad}}(\phi_{\text{rad}}), r \cos_{\text{rad}}(\theta_{\text{rad}})) \\ & \quad \times r^2 dr \sin_{\text{rad}}(\theta_{\text{rad}}) d\theta_{\text{rad}} d\phi_{\text{rad}}, \end{aligned} \quad (86)$$

which reduces to the conventional result if the radian label is dropped:

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) dx dy dz \\ &= \int_0^{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} f(r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), \\ & \quad r \cos(\theta)) r^2 dr \sin(\theta) d\theta d\phi. \end{aligned} \quad (87)$$

5.3. Water waves

The problem of the phase velocity of shallow water waves provides an interesting case about unit compatibility.

The explicit conventional expression for the phase velocity is [62]

$$c_p = \frac{\omega}{k} = \frac{\sqrt{gk \tanh(kh)}}{k} = \sqrt{gh} \sqrt{\frac{\tanh(kh)}{kh}}, \quad (88)$$

corresponding to

$$\omega^2 = gk \tanh(kh), \quad (89)$$

where ω is the frequency of the wave, $k = 1/\lambda$ is the wavenumber, g is the acceleration of gravity, and h is the depth of the water. The corresponding dimensions are $\langle \omega^2 \rangle = A^2 T^{-2}$, $\langle k \rangle = A L^{-1}$, $\langle g \rangle = L T^{-2}$, and $\langle h \rangle = L$.

The problem is that the left-hand side of equation (89) has the dimension $A^2 T^{-2}$ and the right-hand side has dimension $A T^{-2}$, so that one angle dimension appears to have been lost in the derivation. This contradiction can be fixed by repeating the derivation with angles properly taken into

account. Equation (89) follows from solving the equation for the velocity potential given by [62]

$$\frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z} = 0, \quad (90)$$

at $z = 0$, with the boundary condition

$$\frac{\partial \varphi}{\partial z} \Big|_{z=-h} = 0. \quad (91)$$

The solution is

$$\varphi = \frac{\cosh k(z+h)}{k \sinh kh} \omega a \sin(\mathbf{k} \cdot \mathbf{x} - \omega t), \quad (92)$$

where a is a normalization constant and $k_z = 0$. Differentiation gives

$$\frac{\partial^2 \varphi}{\partial t^2} = - \left(\frac{2\pi\omega}{\Theta} \right)^2 \varphi, \quad (93)$$

so that

$$\omega^2 = \frac{g}{\varphi} \left(\frac{\Theta}{2\pi} \right)^2 \frac{\partial \varphi}{\partial z} \Big|_{z=0} = gk \frac{\Theta}{2\pi} \tanh kh. \quad (94)$$

The right-hand side has the correct dimension of $A T^{-1}$.

5.4. Units for the cyclotron resonance frequency

The relevant conventional equation is

$$\omega = \frac{qB_c}{m}, \quad (95)$$

where ω is the frequency, q/m is the charge to mass ratio of the particle, and B_c is the classical magnetic field. The dimensions are $\langle \omega \rangle = A T^{-1}$, $\langle q/m \rangle = Q M^{-1}$, and $\langle B_c \rangle = M T^{-1} Q^{-1}$.

The left-hand side of equation (95) has dimension $A T^{-1}$ and the right-hand side works out to T^{-1} , so there is a mismatch of the angle dimension and the equation is not complete. Moreover, because the right-hand side has dimension T^{-1} , the current SI prescribes that the unit is s^{-1} , or Hz, which is not correct (see equation (105)) [63]. As a result, *Mathematica* [60], which provides calculations with units, gets regular bug reports about this. To resolve this problem, we work out the complete version of this equation.

For motion of a particle of mass m and charge q in the \hat{i}, \hat{j} plane along a circular path of radius r , centered at $\mathbf{x} = 0$, we have

$$\mathbf{x} = r \cos(\omega t) \hat{i} + r \sin(\omega t) \hat{j} \quad (96)$$

$$\mathbf{v} = \frac{d}{dt} \mathbf{x} = \frac{d}{dt} [r \cos(\omega t) \hat{i} + r \sin(\omega t) \hat{j}] \quad (97)$$

$$= \frac{2\pi\omega}{\Theta} [-r \sin(\omega t) \hat{i} + r \cos(\omega t) \hat{j}] \quad (98)$$

$$\mathbf{a} = \frac{d}{dt} \mathbf{v} = - \left(\frac{2\pi\omega}{\Theta} \right)^2 [r \cos(\omega t) \hat{i} + r \sin(\omega t) \hat{j}] \quad (99)$$

$$= - \left(\frac{2\pi\omega}{\Theta} \right)^2 \mathbf{x}. \quad (100)$$

For a magnetic field given by $\mathbf{B}_c = -B_c \hat{\mathbf{k}}$, the classical force is

$$\mathbf{F}_c = q\mathbf{v} \times \mathbf{B}_c = -qB_c \frac{2\pi\omega}{\Theta} \left[r \sin(\omega t) \hat{\mathbf{j}} + r \cos(\omega t) \hat{\mathbf{i}} \right] \quad (101)$$

$$= -qB_c \frac{2\pi\omega}{\Theta} \mathbf{x}. \quad (102)$$

Thus from Newton's law, $\mathbf{F}_c = m\mathbf{a}$, we have

$$qB_c = m \frac{2\pi\omega}{\Theta} \quad (103)$$

or

$$\omega = \frac{qB_c}{m} \frac{\Theta}{2\pi}. \quad (104)$$

This is the complete equation, and the dimensions, \mathbf{AT}^{-1} , match.

In SI units, the frequency is

$$\omega = \left\{ \frac{qB_c}{m} \right\} \times \left\{ \frac{\text{rad s}^{-1}}{\frac{\text{cycles s}^{-1}}{2\pi}} = \frac{\text{Hz}}{2\pi} \right\}. \quad (105)$$

5.5. Classical pendulum

The conventional expression for the small-oscillation frequency ω of a classical pendulum is

$$\omega = \sqrt{\frac{g}{L}}, \quad (106)$$

where g is the acceleration of gravity and L is the length of the pendulum arm. The current SI unit for $\sqrt{g/L}$ is s^{-1} , which suggests the frequency is in Hz, but, as for the case of the cyclotron frequency above, this is not correct, as shown by considering the complete equation calculation.

The equation of motion is

$$m \frac{d^2 s(t)}{dt^2} = -mg \sin(\theta(t)), \quad (107)$$

where θ and s are the angle and arc length from the lowest position of the pendulum, as indicated in figure 2. The mass cancels out of the equation. We consider small displacements from the lowest point, so that

$$\sin(\theta(t)) = \frac{2\pi\theta(t)}{\Theta} + \dots, \quad (108)$$

according to equation (38). From equations (26) and (29), we have

$$\frac{2\pi}{\Theta} \int d\theta = \frac{1}{L} \int ds, \quad (109)$$

which provides the relation

$$s(t) = \frac{2\pi L}{\Theta} \theta(t). \quad (110)$$

We thus have

$$\frac{d^2 \theta(t)}{dt^2} + \frac{g}{L} \theta(t) = 0. \quad (111)$$

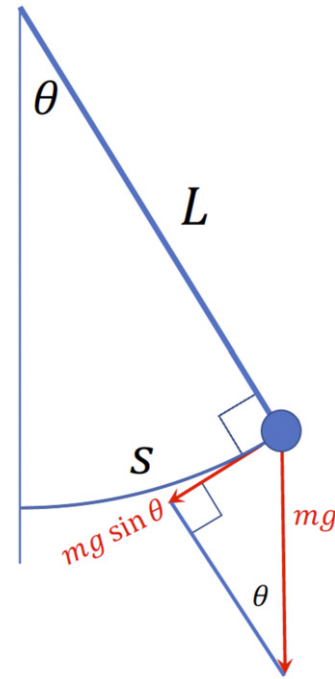


Figure 2. Classical pendulum.

The general solution is

$$\theta(t) = a \sin(\omega t) + b \cos(\omega t), \quad (112)$$

and so the complete second derivative

$$\frac{d^2 \theta(t)}{dt^2} = -\left(\frac{2\pi\omega}{\Theta} \right)^2 \theta(t) \quad (113)$$

gives

$$\omega = \sqrt{\frac{g}{L}} \frac{\Theta}{2\pi}, \quad (114)$$

which has the dimension \mathbf{AT}^{-1} . Finally, the proper SI expression for ω is

$$\omega = \sqrt{\left\{ \frac{g}{L} \right\}} \text{ rad s}^{-1}. \quad (115)$$

5.6. Jacobi elliptic functions

The Jacobi elliptic functions $\text{sn}(u, k)$ and $\text{cn}(u, k)$ may be viewed as generalizations of the sine and cosine functions because [64]

$$\text{sn}(u, 0) = \sin u, \quad (116)$$

$$\text{cn}(u, 0) = \cos u. \quad (117)$$

It is natural to seek the generalization of the differentiation formula in equation (25).

The functions, together with $\text{dn}(u, k)$, are defined by

$$u = \int_0^\phi \frac{d\phi'}{\sqrt{1 - k^2 \sin^2 \phi'}}, \quad (118)$$

$$\text{sn}(u, k) = \sin \phi, \quad (119)$$

$$\operatorname{cn}(u, k) = \cos \phi, \quad (120)$$

$$\operatorname{dn}(u, k) = \sqrt{1 - k^2 \sin^2 \phi}. \quad (121)$$

From equation (118), we have

$$\frac{du}{d\phi} = \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad (122)$$

and so

$$\begin{aligned} \frac{d}{du} \operatorname{sn}(u, k) &= \frac{d}{du} \sin \phi = \left(\frac{d}{d\phi} \sin \phi \right) \frac{d\phi}{du} \\ &= \frac{2\pi}{\Theta} \cos \phi \sqrt{1 - k^2 \sin^2 \phi} \\ &= \frac{2\pi}{\Theta} \operatorname{cn}(u, k) \operatorname{dn}(u, k), \end{aligned} \quad (123)$$

which reduces to the conventional result for the derivative of $\operatorname{sn}(u, k)$ if rad is replaced by 1 and agrees with equation (25) if $k = 0$.

6. Conclusion

Here, we review some key points relevant to the treatment of angles and their units in equations of physics. In this context, the term angle is used to mean both plane angle and phase angle.

A guiding principle is that the use and properties of units (e.g. their dimensions) is a matter of choice. Units are not a property of nature, but rather they provide a rational method for quantitatively describing natural phenomena with mathematical equations. Of course, the choice is not completely arbitrary, because the units must fit into a logically consistent framework.

We may choose to assign to angles an independent dimension with associated units, or not. The current practice is generally to treat angles simply as numbers and view them as dimensionless. However, as should be evident from the forgoing sections, this point of view makes unstated assumptions. In particular, it assumes that the number representing an angle is the numerical factor of the angle expressed in the radian unit. This view also assumes that the radian unit is not actually a unit but is just the number one. An advantage of this approach is that it represents the status quo and as such requires no further thought or analysis. However, this leads to ambiguities and errors as described in the forgoing examples. Because the radian is a coherent unit in the SI, there is no numerical factor when the radian is restored, as illustrated by equations (105) and (115). On the other hand, Hz is not a coherent unit as seen in equation (105) where there is an additional factor of $1/2\pi$.

Such ambiguities and errors can be avoided by assigning an independent dimension to angles and treating this dimension in a consistent way, insuring the correct dimension for the equations of physics, as well as insuring the proper units among the various units that might be used for angles. The use of complete functions and complete equations is a powerful tool to insure a consistent treatment of dimension and units. An important advantage is that it eliminates errors of a factor

of 2π as shown in the examples. The assignment of a dimension to angles also raises awareness that there are built-in hidden assumptions in the widely accepted way in which angles and trigonometric functions of angles are treated. Moreover, the analysis of the consequences of assigning a dimension to angles provides a generalization of the relevant equations, so that they become complete equations. As already noted, complete equations do not assume that the quantities appearing in them are expressed in particular units.

To elaborate on this point, we consider the example, often used by theoretical physicists, of setting the speed of light c to 1 [65]. When this is done, c disappears from equations that would otherwise show it. Then we have $E = m$ instead of $E = mc^2$, which introduces an ambiguity between energy and mass. Restoration of c at the end of a calculation requires judgment by the practitioner. This is completely analogous to the practice of setting $C = 2\pi/\Theta$, as defined in the preceding sections, equal to 1, which is the default for the ‘radian assumption’. If C is restored, it shows up as an additional parameter in some equations, just as c shows up when it is not 1.

The cost of this restoration of $C = 2\pi/\Theta$ is the task of determining where it belongs in equations in which it has been replaced by 1, by default. In this paper, the generalization has been worked out for transcendental functions and applied in various examples.

References

- [1] Quincey P, Mohr P J and Phillips W D 2019 Angles are inherently neither length ratios nor dimensionless *Metrologia* **56** 043001
- [2] Fourier J-B-J 1822 *Théorie analytique de la chaleur* (Paris: Chez Firmin Didot, Père et fils) see section 160 et seq
- [3] Maxwell J C 1870 Mathematics and physics *Report of the 40th Meeting of the British Association for the Advancement of Science* ed John Murray (London, Liverpool) pp 1–9
- [4] Maxwell J C 1873 *A Treatise On Electricity and Magnetism* vol 1 (London: Macmillan)
- [5] Muir T 1910 The term ‘radian’ in trigonometry *Nature* **83** 459–60
- [6] Buckingham E 1914 On physically similar systems; illustrations of the use of dimensional equations *Phys. Rev.* **4** 345–76
- [7] Brinsmade J B 1936 Plane and solid angles their pedagogic value when introduced explicitly *Am. J. Phys.* **4** 175–9
- [8] Page C H 1961 Physical entities and mathematical representation *J. Res. Natl. Bur. Stand. B* **65B** 227–35
- [9] Romain J E 1962 Angle as a fourth fundamental quantity *J. Res. Natl. Bur. Stand. B* **66B** 97–100
- [10] Page C H 1978 Classes of units in the SI *Am. J. Phys.* **46** 78–9
- [11] Stiehler D 1978 Getting the right angle *ASEE* vol 5 p 3
- [12] de Boer J 1979 Group properties of quantities and units *Am. J. Phys.* **47** 818–9
- [13] Page C 1979 Rebuttal to de Boer’s ‘Group properties of quantities and units’ *Am. J. Phys.* **47** 820
- [14] Eder W E 1980 Rotary motion and SI *Eur. J. Eng. Educ.* **4** 319–27
- [15] Eder W E 1982 A viewpoint on the quantity ‘plane angle’ *Metrologia* **18** 1–12
- [16] Scott B L 1985 Letter to the editor *Am. J. Phys.* **53** 520

- [17] Freeman G R 1986 SI units of frequency, angular velocity, Planck's constant and \hbar *Metrologia* **23** 221–2
- [18] Torrens A B 1986 On angles and angular quantities *Metrologia* **22** 1–7
- [19] Thor A J 1986 On angles and angular quantities *Metrologia* **23** 55
- [20] Torrens A B 1986 On angles and angular quantities' Torrens' reply to Thor's comments *Metrologia* **23** 57–8
- [21] Wittmann H 1988 A new approach to the plane angle *Metrologia* **25** 193–203
- [22] Brownstein K R 1991 Concept of angle as a dimensional quantity *AAPT* **21** 92–3
- [23] Oberhofer E S 1992 What happens to the 'radians' *Phys. Teach.* **30** 170–1
- [24] French A P 1992 What happens to the 'radians'? *Phys. Teach.* **30** 260–1
- [25] Cooper M 1992 Who named the radian? *Math. Gaz.* **76** 100–1
- [26] Aubrecht G J II, French A P, Iona M and Welch D W 1993 The radian-that troublesome unit *Phys. Teach.* **31** 84–7
- [27] Scott B L 1993 It's obvious-now *Phys. Teach.* **31** 262
- [28] Brownstein K R 1997 Angles-let's treat them squarely *Am. J. Phys.* **65** 605–14
- [29] Lévy-Leblond J-M 1998 Dimensional angles and universal constants *Am. J. Phys.* **66** 814–5
- [30] Mills I M, Taylor B N and Thor A J 2001 Definitions of the units radian, neper, bel and decibel *Metrologia* **38** 353–61
- [31] Emerson W H 2002 A reply to 'Definitions of the units radian, neper, bel and decibel' by I M Mills *et al Metrologia* **39** 105
- [32] Emerson W H 2005 On the concept of dimension *Metrologia* **42** L21–2
- [33] Emerson W H 2005 Differing angles on angle *Metrologia* **42** L23–6
- [34] Karshenboim S G 2005 Fundamental physical constants: looking from different angles *Can. J. Phys.* **83** 767–811
- [35] Foster M P 2010 The next 50 years of the SI: a review of the opportunities for the e-science age *Metrologia* **47** R41–51
- [36] Mohr P J and Phillips W D 2015 Dimensionless units in the SI *Metrologia* **52** 40–7
- [37] Leonard B P 2015 Comment on 'dimensionless units in the SI' *Metrologia* **52** 613–6
- [38] Mohr P J and Phillips W D 2015 Reply to comments on 'dimensionless units in the SI' *Metrologia* **52** 617–8
- [39] Krystek M P 2015 The term 'dimension' in the international system of units *Metrologia* **52** 297–300
- [40] Quincey P 2016 The range of options for handling plane angle and solid angle within a system of units *Metrologia* **53** 840–5
- [41] Mills I 2016 On the units radian and cycle for the quantity plane angle *Metrologia* **53** 991–7
- [42] Leonard B P 2016 Comment on 'on the units radian and cycle for the quantity plane angle' *Metrologia* **53** 1281–5
- [43] Mills I M 2016 Reply to comment on 'on the units radian and cycle for the quantity plane angle' *Metrologia* **53** 1286
- [44] Quincey P and Brown R J C 2016 Implications of adopting plane angle as a base quantity in the SI *Metrologia* **53** 998–1002
- [45] Quincey P 2016 Natural units in physics, and the curious case of the radian *Phys. Educ.* **51** 065012
- [46] Nature Editorials 2017 Lost dimension *Nature* **548** 135
- [47] Flater D 2017 Redressing grievances with the treatment of dimensionless quantities in SI *Measurement* **109** 105–10
- [48] Bich W 2019 The third-millennium international system of units *Riv. Nuovo Cimento* **42** 49–102
- [49] Kalinin M I 2019 On the status of plane and solid angles in the international system of units (SI) *Metrologia* **56** 065009
- [50] Quincey P and Burrows K 2019 The role of unit systems in expressing and testing the laws of nature *Metrologia* **56** 065001
- [51] Bunker P R, Mills I M and Jensen P 2019 The Planck constant and its units *J. Quant. Spectrosc. Radiat. Transfer* **237** 106594
- [52] Lovatt I 2019 The physics of $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ *Phys. Teach.* **57** 558–9
- [53] Bunker P R and Jensen P 2020 The Planck constant of action h_a *J. Quant. Spectrosc. Radiat. Transfer* **243** 106835
- [54] Quincey P 2020 Angles in the SI: treating the radian as an independent, unhidden unit does not require the redefinition of the term 'frequency' or the unit hertz *Metrologia* **57** 053001
- [55] Leonard B P 2021 Proposal for the dimensionally consistent treatment of angle and solid angle by the international system of units (SI) *Metrologia* **58** 052001
- [56] Quincey P 2021 Angles in the SI: a detailed proposal for solving the problem *Metrologia* **58** 053002
- [57] Leonard B P 2022 Comment on 'Angles in the SI: a detailed proposal for solving the problem' *Metrologia* **59** 038001
- [58] Quincey P 2022 Reply to comment on 'Angles in the SI: a detailed proposal for solving the problem' *Metrologia* **59** 038002
- [59] Grötschel M, Hanche-Olsen H, Holden H and Krystek M P 2022 On angular measures in axiomatic Euclidean planar geometry *Meas. Sci. Rev.* **22** 152–9
- [60] Mention of specific commercial products does not imply endorsement by NIST.
- [61] The 'rule of thumb' unit for centripetal force given in table II of Mohr and Phillips [36] is not consistent with this result.
- [62] Phillips O M 1978 *The Dynamics of the Upper Ocean* 2nd edn (Cambridge: Cambridge University Press)
- [63] BIPM 2019 *Le Système international d'unités (SI)* 9th edn (Sèvres, France: Bureau International des Poids et Mesures)
- [64] Weisstein E W 2022 *Jacobi Elliptic Functions* <https://mathworld.wolfram.com/JacobiEllipticFunctions.html>
- [65] Jackson J D 1998 *Classical Electrodynamics* 3rd edn (New York: Wiley)