PAPER

Topologically induced swarming phase transition on a 2D percolated lattice

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Topologically induced swarming phase transition on a 2d percolated lattice

David A. Quint* and Ajay Gopinathan†
Physics Department, University of California Merced
(Dated:)

The emergence of collective motion, or swarming, in groups of moving individuals who orient themselves using only information from their neighbors is a very general phenomenon that occurs at multiple spatio-temporal scales. Swarms that occur in natural environments typically have to contend with spatial disorder such as obstacles that can hinder an individual's motion or can disrupt communication with neighbors. We study swarming agents, possessing both aligning and mutually avoiding repulsive interactions, in a 2D percolated network representing a topologically disordered environment. We numerically find a phase transition from a collectively moving swarm to a disordered gas-like state above a critical value of the topological or environmental disorder. For agents that utilize only alignment interactions, we find that the swarming transition does not exist in the large system size limit, while the addition of a mutually repulsive interaction can restore the existence of the transition at a finite critical value of disorder. We find there is a finite range of topological disorder where swarming can occur and that this range can be maximized by an optimal amount of mutual repulsion.

INTRODUCTION

Collective motion of self propelled individuals is a well studied emergent phenomenon [1–20] that spans many different length and time scales. Within the literature that is aimed at studying collective motion in systems of self-propelled agents, a main underlying assumption has been that the environment is obstacle free, isotropic and ordered. In the natural world there are many examples of environments that possess physical obstacles where collective motion can exist. Examples include bats that navigate natural caverns via echolocation, schools of fish that maneuver through dark and light areas [21], microbial colonies that move about in heterogeneous soil [22], crowds of people that are evacuating a building [23] and traffic flow in major cities [24]. Environmental or *topological disorder* is manifested in the form of obstacles or regions of space that either hinder movement or disrupt the flow of communication or both. Given that natural environments can be intrinsically topologically disordered which is due to the obstacles that prevent motion, it is interesting to consider how self-propelled individuals maintain an organized state of collective motion without knowledge of a global "road map". Recent models that have included the effects of environmental disorder have found a multitude of different swarm state behaviors, which are attributed to the effects that topological disorder has at the local agent level [25–27]. In the presence of fixed obstacles in the environment, swarms can benefit from an optimal thermal noise value that allows agents to maintain a collectively moving state [27]. The optimal noise, in this context, is akin to annealing the defects in the collective caused by the heterogeneities in the environment. We compliment these previous studies, which have focused on the classical thermally induced phase transition by examining the phase transition that is associated with tuning the topological order of the environment at fixed thermal noise and how it is coupled to inter-agent repulsion. We are interested in the nature of such a topological noise driven swarming transition and its relation to the underlying topology of the background space [18]. Given that swarming cannot persist at reasonable values of obstacles [27], are there physical mechanisms that agents can control to facilitate the collective movement of agents across a topologically disordered network?

* dquint@ucmerced.edu
† agopinathan@ucmerced.edu

MODEL AND SIMULATION

In order to understand the effect of topological disorder on the swarm dynamics of locally coupled self-propelled agents the introduction and the implementation of the heterogeneities (obstacles) in the environment must be done in a more controlled and quantitative manner. To do so we implement a Monte Carlo lattice gas model [2, 28–30] that consists of N_p interacting self-propelled agents that occupy a $2d$ periodic triangular lattice with $L^2 \equiv N$ lattice sites. Idealizing obstacles in the environment utilizing ordinary bond-percolation allows for a simple and controllable method that is suited for studying disorder. Moreover, this type of study can be generalized to network topologies that aren't restricted to Euclidean embeddings [18]. The diagram contained in Fig. 1a represents a typical unit cell within the lattice with agents (red filled circles) that occupy some of the lattice sites. We introduce topological disorder by varying p, the probability that a bond exists between two lattice (grey broken bond in Fig. 1a) sites as in usual bond percolation theory [31]. Local interactions between agents have energies associated with them. The first is a velocity alignment interaction energy,

$$E_i^a = -\alpha \mathbf{u}_i \cdot \sum_{j=1}^{6} \eta_j \sum_{k=1}^{n(j)} \mathbf{v}_k \Big/ \left| \sum_{j=1}^{6} \eta_j \sum_{k=1}^{n(j)} \mathbf{v}_k \right|, \qquad (1)$$

where the magnitude is controlled by the parameter α and $\mathbf{v}_k$ is the velocity of the kth agent at nearest neighbor site j. Here $n(j)$ refers to the number of agents at a site j and this can be larger than 1. The second type of energetic interaction is a repulsive one,

$$E_i^r = \epsilon \eta_i (n(i) - n(0)), \qquad (2)$$

that is controlled by the parameter ϵ and is proportional to the difference in local density (agent number), $n(i) - n(0)$ along a lattice direction $\mathbf{u}_i$.

Bond disorder enters into the calculation of both interactions via the bond occupation parameter η_i, which is unity when a bond is present and zero when it is missing. Motion as well as information flow across a deleted bond is prohibited, hence in Fig. 1a the central agent will most likely move along the white arrow in the absence of any repulsive interaction with the neighboring agents, whereas if the bond were present the most likely direction of travel would be along the gray arrow. In this way topological disorder directly couples into the agents dynamics at the local level. Thermal effects are introduced by the parameter T that enters during the METROPOLIS Monte Carlo update. Updates were accomplished by selecting an agent at random on the lattice then preforming the Monte Carlo update to select their new possible velocity direction $\mathbf{u}_i$. The update consists of constructing a probability, $P(\mathbf{u}_i)$, for each possible direction present locally to each agent using the energies given in Eqns. 1, 2 ($P(\mathbf{u}_i) = \exp(-(E_i^r + E_i^a)/kT)$, $kT = 1$). These probabilities are then mapped to the interval $[0, 1]$ and new directions are accepted by selecting a random number $0 \leq r \leq 1$ on this interval.

After a new direction is selected the agent moves along that lattice bond to the next lattice site. In this way the only degrees of freedom that are affected by thermal noise are rotational (See Supplementary information). The data we present below was taken for system sizes no larger than $\sim 10^4$, this was due to the slow relaxation of our system for larger system sizes.

RESULTS

Results I - Topological disorder driven phase transition.

In general, the existence of intrinsic thermal noise in a system of mobile interacting agents can drive the departure from a collectively moving ordered swarm state to a completely disordered gas-like state [1–5, 8, 9, 32, 33]. Intuitively we expect that the introduction of topological

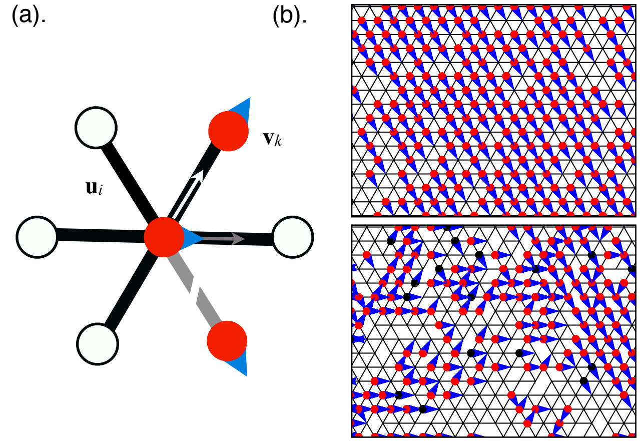

FIG. 1. (a) Diagram of a lattice unit cell with agents (red) occupying lattice sites and currently moving in the direction of the blue arrows. Here a lattice bond is labeled by $\mathbf{u}_i$ and the velocity of the nearest neighbor is denoted $\mathbf{v}_k$. The grey broken line represents a missing bond, the grey arrow represents the most probable velocity direction of the central agent, if the deleted bond were present, and the white arrow represents the most probable velocity direction of the central agent when the bond is not present. (b) Top; A snapshot of a typical simulation showing finite sized groups of agents which are collectively moving in a disorder free $(1 - p = 0)$ lattice. Bottom; A snapshot of a simulation of a system with disorder $(1 - p = 0.05)$, where missing bonds are not drawn. Black filled circles are agents that are temporarily stuck at a lattice bond defect.

disorder will effect the ability of agents to move collectively. Fig. 1b (top) shows a typical simulation snapshot of a collectively swarming group of agents moving through a disorder free lattice. In contrast, Fig. 1b (bottom) shows for the same system but with 5% disorder the effect of missing bonds in the lattice on the formation of a swarming state. Temporarily stuck agents (Fig. 1b - black filled circles) at the location of a missing bond are causing neighboring agents to move around them, and thus causing the entire group of agents to move in different directions within the lattice. To quantitatively study the transition from an ordered state to a disordered one, we propose using the average velocity order parameter,

$$\langle v \rangle = \frac{1}{N_p} \left| \sum_{i}^{N_p} \mathbf{v}_i \right|, \text{ where } |\mathbf{v}_i| = 1. \qquad (3)$$

Here $\mathbf{v}_i$ is the velocity of the ith agent and N_p is the total number of agents in the system, which has been implemented in previous studies [1–5, 8, 9, 27, 32, 33] for the disorder free case.

To isolate the effects of topological disorder on the formation of a collective swarm we fix the magnitude of thermal noise such that an ordered swarming state would naturally occur without any environmental disorder. Fig. 2a shows the effect of increasing environmental disorder on the formation of a collective swarm for a system that only utilizes alignment interactions (Eqn. 1). As we vary the density we find that swarming is not perfect ($\langle v \rangle < 1$) even for a perfectly ordered lattice at low densities of agents ($\rho \leq 0.2$), consistent with earlier studies [2, 33] (See supplementary information).

At higher densities, for low values of the lattice disorder, we find a coherent swarming state where the order parameter is near unity ($v \sim 1$). This state exhibits a phase transition to a disordered state when the fraction of missing bonds exceeds a critical value ($1 - p^*$), where we define the location of the transition by the inflection point in the order parameter versus disorder curve. This order-disorder transition is completely determined by the amount of topological disorder, which is unrelated to the standard Vicsek model transition that is induced by either thermal or vectorial noise [1, 2, 5, 9, 27, 32–34]. It should be noted that there exists a controversy as to the order of this phase transition in the standard Vicsek model driven by thermal noise with different numerical studies ascribing a different order of the phase transition depending on the parameter regimes examined, system size and how intrinsic noise is introduced into the model [5, 8, 34–36]. We will address the critical features of our phase transition later in this paper.

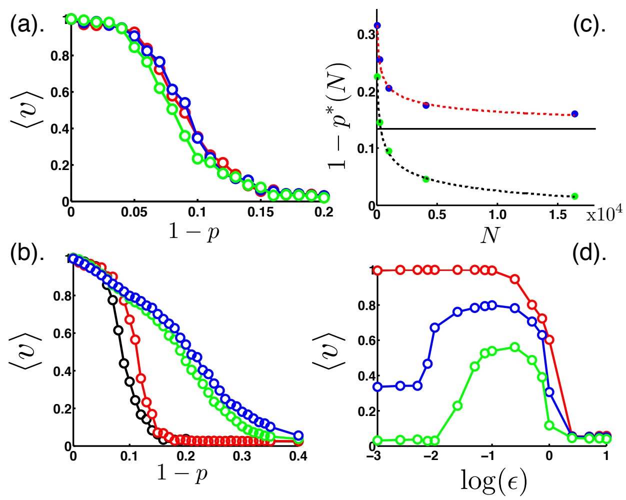

FIG. 2. (a) Global alignment, in the absence of repulsion, as measured by Eqn. 3 as a function of the disorder fraction in the lattice for different densities [$\rho = 0.8$ (red), $\rho = 1.0$ (blue), $\rho = 2.0$ (green)]. (b) Global alignment for a system of agents that interact with both a local alignment and repulsive fields for $\rho = 1.0$ and $\epsilon = [5.0 * 10^{-3}$ (black), 10^{-2} (red), $5.0 * 10^{-2}$ (green), 10^{-1} (blue)]. (c) Finite system size critical disorder fraction ($1 - p^*(N)$) for systems without/with repulsion [$\epsilon = 0$ (green filled circles)] [$\epsilon = 10^{-1}$ (blue filled circles)] with finite size scaling fits (red dashed line) and (black dashed line). (d) The order parameter for particular values of the disorder fraction [$1 - p = 0$ (red), 0.10 (blue), 0.20 (green)] over the the log of the repulsion strength. All further log data is to the base 10 unless stated otherwise. (See supplementary information for more simulation details)

To understand if our transition occurs at a critical disorder fraction, as we increase the system size $N = L^2$, we characterized the location of the critical disorder fraction ($1 - p^*(N)$) for finite systems as a function of the system size. We will use the finite size scaling ansatz ($p^*(N) - p^*(\infty) \sim N^{-\lambda}$), where we have extracted the critical disorder fraction, $1 - p^*(\infty)$ from our simulations. This type of finite size scaling has been used to characterize a thermally induced phase transition in a recent study of a continuous swarming model with spatial heterogeneities [27]. Fitting the locations of the critical disorder fractions for each of the system sizes we studied revealed that in the large system size limit, $1 - p^*(N \to \infty) \to 0$ (Fig. 2b (green filled circles, black dashed line)).

This result suggests that, with solely alignment interactions between agents, the phase transition from an ordered state to a disordered one does not exist in the large system size limit. We must point out that in other studies of swarming behavior it has been reported that there can be more complex system wide swarming patterns. Moreover, these patterns display no globally ordered state (i.e quasi long range order - moving bands, traffic jams and asters) and have been seen in both on and off lattice simulations [30, 37]. While we do see certain types of patterns in our simulations, which we plan to explore in the future, here we are only interested in the existence of global order in the presences of topological defects.

Results II - Repulsion enhances swarming in disorder

Consider the effect of adding repulsive interactions between neighboring agents (Eqn. 2). In Fig. 2b, we see that the ability of agents to form a collectively moving swarm is significantly enhanced compared to Fig. 2a even for moderately high values of the disorder fraction. Thus adding a repulsive interaction between agents appears to have shifted the location of the order-disorder transition ($1 - p^*(N)$) for finite systems. Using the finite size analysis again, we find that there exists a phase transition (Fig. 2c (blue filled circles, red dashed line)) occurring at a *non-zero* value of the topological disorder fraction. For a specific value of the repulsive interaction, ($\epsilon = 10^{-1}$), and temperature ($T = 10^{-2}$), this turns out to be $1 - p^*(N \to \infty) = 0.13(4)$(Fig. 2c (black solid line)). Finally, we find that there is an optimal repulsive magnitude, where collective motion is enhanced in the presence of disorder, that is reflected in the non-monotonic behavior in Fig. 2d. This behavior is consistent with previous studies for systems without disorder [33] and is not to be confused with optimal noise assisted swarming in the presence of obstacles [27] as here we focus on how the inter-agent repulsion can be tuned to optimally enable swarming.

Intuitively one can understand the nature of this non-monotonic behavior from the following argument. When an agent becomes temporarily stuck at a defect (i.e. a missing bond) other agents can also become stuck at that same location with little energy cost and they can pile up at a lattice defect. As the repulsive magnitude is increased agents will tend to avoid regions where the agent number density is high (Eqn. 2). At moderate values of repulsion agents can avoid sites that are temporarily occupied by another agent, causing the local number density per lattice site to decrease. If a lattice site near a missing bond is occupied the repulsive interaction between the two agents makes it less favorable for them to occupy the same lattice site. Thus, the local repulsive interaction is, in some way, mediating the local bond disorder and allowing agents to avoid lattice defects more efficiently, which can be seen in the peak in Fig. 2d. However, as the repulsive interaction is further increased individual agent motion will become more randomized because any movement toward another agent will be highly unlikely as can be seen in Fig. 2d even for the case where this is no lattice disorder ($1 - p = 0$) for $\log(\epsilon) \geq 0$. We will explore these dynamics more in a following publication.

Now we can clearly see that repulsive interactions restore the ability of the agents to swarm collectively in a finite amount of disorder, it is now interesting to speculate to which universality class our system belongs to. We probe this question by utilizing finite size scaling analysis to extract the critical exponents of the relevant thermodynamic quantities. We fix our system at $\epsilon = 10^{-1}$ and examine the behavior of the critical disorder fraction defined by $1 - p*(L)$, while varying the system size L (Fig 2c). The system size dependence of the critical point ($\xi \sim L \rightarrow (p^*(L) - p^*(\infty)) \sim L^{-1/\nu}$) provided an estimate for the correlation length exponent ν, yielding $\nu \approx 1.19 \pm 0.09$ (Fig 3a. green dashed line). Measuring the susceptibility of the order parameter to p by analyzing the fluctuations in $\langle v \rangle$ as a function of system size, $\chi_v = N\sigma_v^2(p^*) \ \sim L^{\gamma/\nu}$ [9] provided an estimate for the susceptibility critical exponent, $\gamma \approx 2.20 \pm 0.29$ (Fig. 3a, red dashed line). Using the values for both ν and γ as well as the finite size hyper-scaling (FH) relation ($2\beta + \gamma = d\nu$) gave and estimate order parameter exponent $\beta_{FH} \approx 0.098$. In order to test the universal nature of the exponents we then fixed our system at a particular system size and particle density ($N = 32^2$, $\rho = 1.0$) and examine the behavior of the order parameter critical exponent over the range of repulsive interactions where swarming occurs ($-3 \leq \log(\epsilon) \leq -1$). We measured the scaling of the order parameter near the critical point for each value of repulsive strength ($\langle v \rangle \sim (p^*(\epsilon) - p)^{\beta}$) and found an excellent agreement between both methods of estimation (Fig. 3b grey dashed line & solid data points), and the value of β was also insensitive to the variation over the range of repulsive magnitudes giving an average $\bar{\beta} = 0.094$, which compares well with the value obtained from finite size scaling $\beta_{FH} \approx 0.098$.

Results III - Universality classes

To gain perspective on which universality class best describes the order-disorder transition due to topological noise, we compared our exponents to those of two dimensional percolation [38] and those that have been associated with the *thermal noise* induced phase transition in the standard Vicsek model [1, 9] for finite systems. Our choice for this comparison is suggested by the underlying physics governing our model. On one hand we are using a variant of the standard Vicsek model as defined in Eqn. 1, while on the other hand, there is a geometric background (percolated quenched disorder) associated with ordinary percolation. We found that our system follows a universality class that is closer to that of percolation (Fig. 3b red dashed line) than the Vicsek type (Fig. 3b green dashed line). This similarity may be due to the fact that the phase transition in our system is driven by a source of noise (topological noise due to percolation effects) that is physically different from thermal noise as in the standard Vicsek model. It is interesting to note while this system is more akin to the percolation type universality class, the critical disorder fraction ($1 - p^*(\infty) = 0.13(4)$) for a repulsive strength of $\epsilon = 10^{-1}$ is vastly different than that of ordinary connectivity percolation ($1 - p_c \approx 0.66$)) for a triangular lattice [39]. This comparison suggests that swarming phenomena in such systems maybe extremely sensitive to the effects of ordinary percolation far from the percolation critical point while retaining the critical behavior of a percolation type system.

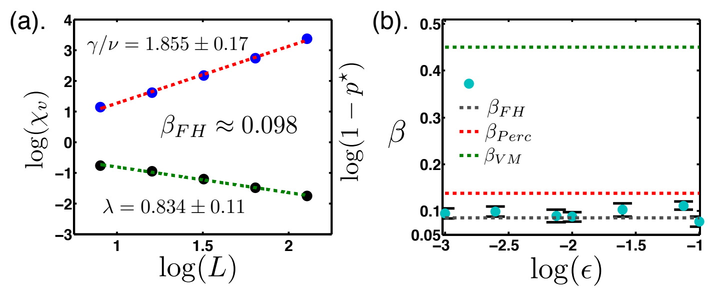

FIG. 3. (a) Finite size scaling fit (red dashed line) of the susceptibility, χ_v, (blue filled circles) and the correlation length, ξ, (green dashed line) at finite repulsion ($\epsilon = 10^{-1}$) as a function of the system size L. The critical exponents γ and $\nu = 1/\lambda$ are estimated at 2.20 ± 0.29 and 1.19 ± 0.09 respectively and $\beta_{FH} = (d\nu - \gamma)/2$. (b) Extracted critical exponents with error bars for β (light blue circles and black bars) from scaling analysis near the critical disorder fraction transition ($1 - p^*(L)$) plotted against the log of the repulsion magnitude. The grey dashed line is the predicted value of β_{FH} from the hyper-scaling relation compared with β_{VM} (green dashed line) and β_{Perc} (red dashed line) the scaling exponents for both the standard Vicsek model and 2d percolation respectively.

DISCUSSION

Systems of self propelled agents using local nearest neighbor alignment interactions can form a collectively moving swarm. However, in the presence of environmental topological disorder, we have shown that alignment alone is insufficient for agents to form a swarm. They need to possess information not only about local neighbors directions of travel but also about the surrounding environment. In summary, we showed that repulsive forces, while seemingly antagonistic to alignment, allow agents to communicate local environmental topological features to their neighbors and restores the ability of the individuals to form a swarm that collectively navigates the intrinsically disordered environment. We have shown that there exists a new type of dynamical phase transition from an ordered co-moving swarm to a disorganized collection of individuals driven by environmental disorder. It is of interest to note that this collective phase transition occurs at a critical point that is far from the underlying geometric percolation transition of the background environment, while still sharing some of it's critical features [38–40]. Furthermore, collective motion in these disordered environments can be optimized by tuning the magnitude of the repulsive interaction for a given amount of disorder. Thus, repulsion or maintaining separations in finite flocks is a simple, tunable and robust mechanism to deal with environmental disorder. In nature there may exist evolutionary pressures that select for better swarming ability within a group of individuals. For finite flocks in an infinite space our results imply that cohesive interactions required to keep the flock stable may need to be reduced, leading to a larger separation between agents, in order to successfully navigate disorder. More generally, topological disorder poses a significant challenge to systems that engage in any collective decision making process such as slime molds solving mazes [41, 42] and quorum sensing bacteria [43, 44]. Topological disorder is also intrinsic in virtual environments where collective processes are important such as social networks, the internet and scientific citation networks [45–47] as well as artificial groups in real environments such as robotic drones exploring unknown territory [48, 49]. It is interesting to speculate whether such general collective decision making systems can benefit from tempering purely consensus driven decisions with moderate amounts of antisocial behavior to overcome topological disorder.

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