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Flat tree-level inflationary potentials in the light of cosmic microwave background and large scale structure data

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Abstract. We use cosmic microwave background (CMB) and large scale structure (LSS) data to test a broad and physically well-motivated class of inflationary models: those with flat tree-level potentials (typical in supersymmetry). The non-trivial features of the potential arise from radiative corrections which give a simple logarithmic dependence on the inflaton field, making the models very predictive. We also consider a modified scenario with new physics beyond a certain high energy cut-off showing up as non-renormalizable operators (NRO) in the inflaton field. We find that both kinds of models fit CMB and LSS data remarkably well, with very few free parameters. Besides, many of these models naturally predict a reasonable number of e-folds. A robust feature of these scenarios is the smallness of tensor perturbations ($r \lesssim 10^{-3}$). The NRO case can give a sizable running of the spectral index while achieving a sufficient number of e-folds. We use Bayesian model comparison tools to assess the relative performance of the models. We believe that these scenarios can be considered as a standard physical class of inflationary models, on a similar footing to monomial potentials.

Keywords: CMBR theory, inflation, cosmology of theories beyond the SM

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1. Introduction

The WMAP 3-year data [1, 2] on the cosmic microwave background radiation (CMB) represent a milestone in the history of observational cosmology due to their great precision. The challenge for theoretical cosmology and particle physics is to understand these data within some theoretical framework and to make predictions to be tested by future observations. In this respect, inflation stands as the most successful and promising theoretical scenario. Proposed in 1981 as a solution to the flatness and horizon problems [3], it was later shown to predict an almost scale-invariant spectrum of matter perturbations [5, 5]–[8], making it a very successful idea. However, we still lack a concrete description of the mechanism producing inflation in the very early universe. The simplest implementation postulates a real scalar field, \( \phi \), whose homogeneous part drives inflation on its way towards the minimum of its potential, \( V(\phi) \). Therefore, to provide a description of the inflationary epoch of the universe we need to elucidate the functional form of the inflaton potential [9]. In this respect it is fair to say that, despite interesting developments, there is still no convincing inflationary scheme based on particle physics.

The main goal of this work is to compare a broad class of models, which are physically well motivated, with CMB and large scale structure (LSS) data. As shown below, these
models predict a scalar spectral index with non-constant slope, in contrast with the ansätze used in standard analyses. Interestingly, these models are very predictive: the number of independent parameters is comparable to those of free parameters used in more phenomenological approaches.

This paper is organized as follows. In section 2 we review the standard way of comparing inflationary potentials with cosmological data through parametrizations of the primordial scalar and tensor power spectra that involve the spectral indices and their derivatives (up to a given order). In section 3 we introduce the class of models that we are interested in, namely, models with flat tree-level potentials which are generically lifted in a controlled way by radiative corrections. This type of models is then extended by taking into account possible effects from non-renormalizable operators that appear after integrating out some heavy new physics. In both cases we discuss the physical motivation and significance of such models. Section 4 is devoted to the derivation of the power spectra for the two classes of models as a function of suitable model parameters. The data analysis procedure is discussed in section 5; we present parameter constraints within each class of models from our analysis in section 6, while section 7 is devoted to the issue of model comparison, assessing which model is in better agreement with the data. Finally, our conclusions are presented in section 8. In addition, the appendix contains some technical details relevant to the analysis of the model with non-renormalizable operators.

2. Comparing inflationary potentials with data

The usual strategy for probing inflationary potentials is the comparison of theoretical predictions for the spectrum of primordial (scalar and tensor) cosmological perturbations \( P_s(k) \) and \( P_t(k) \) respectively, where \( k \) stands for the space wavenumber) with CMB and LSS data. This is usually done in the slow-roll approximation [10]. Generically, in this situation \(|\epsilon,\eta,\xi|\ll 1\), with

\[
\epsilon \equiv \frac{1}{2} M_p^2 \left( \frac{V'}{V} \right)^2, \quad \eta \equiv M_p^2 \frac{V''}{V}, \quad \xi \equiv M_p^4 \frac{V'''V''}{V^2},
\]

(2.1)

where primes denote derivatives with respect to the inflaton \( \phi \) and \( M_p \equiv m_p / \sqrt{8\pi} \) (where \( m_p \approx 10^{19} \) GeV) is the reduced Planck mass. In this approximation

\[
P_s(k) \approx \frac{1}{24\pi^2 \epsilon} \frac{V}{M_p^4},
\]

(2.2)

\[
P_t(k) \approx \frac{3}{2\pi^2} \frac{V}{M_p^4}.
\]

(2.3)

Thus the tensor to scalar ratio simply reads

\[
r \equiv \frac{P_t}{P_s} \approx 16\epsilon.
\]

(2.4)

The relationship between the inverse distance scale \( k \) and the inflaton field \( \phi \) is given in slow-roll by

\[
\frac{d\phi}{d \ln k} \approx -M_p \sqrt{2\epsilon}.
\]

(2.5)
The number of e-folds between the beginning of inflation \( (t_*) \) and the end \( (t_e) \) is
\[
N_e = \int_{t_*}^{t_e} H \, dt \simeq \frac{1}{M_p^2} \int_{\phi_*}^{\phi_e} \frac{V V'}{V'} \, d\phi = \frac{1}{M_p} \int_{\phi_*}^{\phi_e} \frac{1}{\sqrt{2\epsilon}} \, d\phi,
\]
where we have used the slow-roll approximation to change variables from \( t \) to \( \phi \). The usual requirement for solving the horizon problem is \( N_e \simeq 50–60 \) although the precise number depends on the details of the post-inflationary epoch [11]. Of course, the total number of e-folds may be much larger, but only 50–60 e-folds are needed and actually WMAP only probes directly the first \( \sim 7 \) e-folds, which can be extended to \( \sim 10 \) by using LSS data and other observations. Thus the subscript ‘∗’ above denotes the time at 50–60 e-folds before the end of inflation. On the other hand, there might be several episodes of inflation, in order to achieve the required 50–60 e-folds. In that case, WMAP would be only sensitive to the first of these episodes after \( t_* \).

An important quantity when performing fits to the observations is the scalar spectral index, \( n \), which describes the variation of \( P_s(k) \) with \( k \):
\[
n - 1 = \frac{d \ln P_s}{d \ln k} \simeq 2\eta - 6\epsilon.
\]
(2.7)

The spectral index itself may change with \( k \):
\[
\frac{dn}{d \ln k} \simeq -2\xi + 16\epsilon\eta - 24\epsilon^2.
\]
(2.8)

The right-hand sides of (2.7) and (2.8) correspond to the slow-roll approximation. Similar equations, involving the tensor spectral index \( n_t(k) \), can be written for the variation of \( P_t(k) \) with \( k \). Summarizing, observational information about \( P_s \) and its derivatives, \( V', V'', V''' \) and so on, provides a link between theory and experiment.

**Standard parametrizations of the power spectrum**

A common approach to determining the power spectrum from CMB and LSS data is to perform a Taylor expansion of \( \ln P_s(k) \) and \( \ln P_t(k) \) in \( \ln k/k_0 \) around zero, where \( k_0 \) is a pivotal scale (from now on fixed to \( k_0 \equiv 0.002 \) Mpc\(^{-1} \)),
\[
\ln P_s(k) = \ln P_s(k_0) + [n(k_0) - 1] \ln \left( \frac{k}{k_0} \right) + \frac{1}{2} \frac{dn}{d \ln k} \ln \left( \frac{k}{k_0} \right)^2 + \cdots,
\]
(2.9)

\[
\ln P_t(k) = \ln P_t(k_0) + n_t(k_0) \ln \left( \frac{k}{k_0} \right) + \cdots.
\]
(2.10)

Note from equations (2.2) and (2.4) that it follows that, at first order in slow-roll [12],
\[
r(k_0) \equiv \frac{P_t(k_0)}{P_s(k_0)} \simeq -8n_t(k_0).
\]
(2.11)

Usually, one does not go beyond the order shown in equations (2.9) and (2.10). In particular, a running of the tensor spectral index has not been considered because at present the tensor contribution to the spectrum is only weakly constrained by the data. Therefore one typically fits four independent parameters, namely \{\ln P_s, n, dn/d\ln k, r\}_k_0,
and often the running of the spectral index, \( dn/d \ln k \bigg|_{k_0} \), and the tensor to scalar ratio, \( r(k_0) \), are set to zero. The improvement in the fit obtained when introducing the latter parameters is neither large enough to be considered as a strong indication for their presence nor small enough to be considered as irrelevant. In general, the issue of whether an extra parameter is needed or not is a difficult one and has to be addressed with care; see [13, 14] for a discussion. We return to this question in section 7.

The Taylor expansion above has been used by the WMAP collaboration. Assuming a ΛCDM universe and setting \( dn/d \ln k = r = 0 \), WMAP 3-yr data alone give [15]

\[
\begin{align*}
n(k_0) &= 0.958 \pm 0.016, \quad (68\% \text{ c.l.}) \quad (2.12) \\
\frac{dn}{d \ln k} &= -0.055^{+0.030}_{-0.031}, \quad (68\% \text{ c.l.}) \quad (2.14)
\end{align*}
\]

The absolute magnitude of \( P_s \) depends slightly on the inflationary model, but roughly one finds \( P_s(k_0) \simeq 2 \times 10^{-9} \). The WMAP collaboration used these results to probe monomial potentials \( V(\phi) \propto \phi^\alpha \) (with \( \alpha = 2, 4 \)) [2]. These models predict a negligible running of \( n \), so they are well approximated by equations (2.9) and (2.10) with \( dn/d \ln k = 0 \). The fits seem to exclude \( \alpha = 4 \) [16] and any other higher monomial power. The quadratic case, \( V = \frac{1}{2}m^2\phi^2 \), works quite well, although it requires very large values of the inflaton field, \( \phi \sim 14 M_p \). For the purposes of our discussion it must be stressed that the simple functional forms assumed previously for \( P_s(k) \) and \( P_t(k) \) may not be accurate enough for describing the actual power spectrum of other inflaton potentials which are well motivated physically. In this sense, although (2.9) and (2.10) can be useful as phenomenological approximations, it is important to be open to other parametrizations. In this paper we will show explicit examples of this. At the end of the day, the best fit together with the best physical motivation will determine the preferred functional form.

3. A broad class of models: flat tree-level potentials

We will consider models that have ‘flat tree-level potentials’\(^5\), i.e.

\[
V_{\text{tree}}(\phi) = \rho_{\text{tree}} = \text{constant}. \quad (3.1)
\]

Then, the potential derivatives \( V', V'', \ldots \) arise from the radiative corrections to \( V \). These potentials appear typically in supersymmetric (SUSY) theories: \( V_{\text{SUSY}}^{\text{tree}} \) ordinarily has plenty of accidental flat directions. A familiar example of this is the minimal supersymmetric standard model (MSSM). Such accidental flatness is broken by radiative corrections since there is no symmetry protecting it. Generically, at one-loop order,

\[
V(\phi) = \rho + \beta \ln \frac{m(\phi)}{Q}, \quad (3.2)
\]

where \( Q \) is the renormalization \( \ln \) (which might have absorbed finite pieces) and \( m(\phi) \) is the most relevant \( \phi \)-dependent mass in the spectrum. Note that \( \rho \) depends implicitly

\(^5\) For a more complete discussion of the theoretical aspects of these models see [17].
on $Q$ through its renormalization group equation (RGE) and that the $Q$-invariance of the effective potential implies

$$\beta = \frac{d\rho}{d\ln Q},$$

at one-loop order. From now on we will assume $\beta > 0$, which is the usual situation, though the opposite case is also possible and the analysis is similar.

Since these ‘almost flat’ directions are so common in SUSY scenarios, they are natural candidates for driving inflation, provided the potential stores a large enough energy density. As a matter of fact, particular examples of these approximate flat directions have been used in the literature to implement inflation, e.g. the first D-term hybrid inflation model belongs to this class [18]. The important point is that, whatever the model considered, the slope of $V(\phi)$, and thus the dynamics of inflation, is determined by radiative corrections. Since the latter have a very generic functional form (logarithmic), it is possible to make very model-independent predictions without relying on a particular model [17]. Next we work out these statements in a more detailed way.

The leading-log approximation (which amounts to summing up the leading-log contributions to all loops) is implemented in this context by simply taking $Q = m(\phi)$. This choice eliminates the potentially large (and thus dangerous) logarithms, improving the convergence of the perturbative expansion. Then

$$V(\phi) \simeq \rho[Q = m(\phi)].$$

In general, one expects $m^2(\phi) = M^2 + c^2\phi^2$, where $M$ does not depend on $\phi$, and $c$ is some coupling constant (which depends on $Q$ according to its own RGE). Normally one considers a range of $\phi$-values where either the constant $M^2$ piece (if it exists) or the $\phi$-dependent part dominates. We assume we are in the second case. Hence, we will ignore the possible presence of $M$ and take $Q = m(\phi) = c\phi$. For very small values of the inflaton field the presence of $M^2$ avoids an infrared logarithmic singularity. In practice, inflation ends before reaching this regime. Moreover, in the hybrid models, the inflaton eventually gets a value where some waterfall fields come into play and the inflaton potential changes dramatically (stopping inflation in any case). So one can assume a purely logarithmic form of the potential during all the inflationary process before reheating.

**Logarithmic regime (LOG)**

In the regime of very small coupling constants one has $d\beta/d\ln Q \ll \beta$ since the former is higher order in the couplings and has a loop suppression factor. Then we can consider $\beta$ as constant in the range of $Q \propto \phi$ of interest (which is never too wide). Now the scalar potential (3.4) can be easily written in terms of its value at $\phi_0$, the value of the inflaton at $k_0$, integrating equation (3.3):

$$V(\phi) = \rho_0 + \beta \ln \frac{\phi}{\phi_0},$$

where $\rho_0 \equiv \rho(\phi_0)$. Note that (3.5) can also be obtained from (3.2) by choosing $Q = c\phi_0$.

Since $\ln \phi = \lim_{\alpha \to 0} \alpha^{-1}(\phi^\alpha - 1)$ the logarithmic potential (3.5) can be considered in many respects as a monomial potential with $\alpha = 0$ [19]. In particular all the derivatives, which are crucial for the cosmology of the model, follow that pattern. On the other hand, as argued above, this potential is physically as well motivated as the monomial forms.
Logarithmic regime + non-renormalizable operator (LOG + NRO)

If there is a scale of new physics, $M$, higher than the scales relevant to inflation (i.e. $M^2 \gg \phi^2$), the new physics will generically show up in the effective theory at lower scales as non-renormalizable operators (NROs) of the light fields, suppressed by inverse powers of $M$. Due to the suppression factor, the impact of the NROs in the physics at low scales is usually very small. However, if the NRO has characteristics not shared by the low energy physics, its effect may be significant (as happens with higher dimension operators that mediate proton decay or give a Majorana mass to neutrinos). In our case, the new physics does not need to respect the accidental flat directions of the effective theory. Thus one expects the inflaton potential (3.5) to become

$$V(\phi) = \rho_0 + \beta \ln \frac{\phi}{\phi_0} + \phi^4 \frac{\phi^{2N}}{M^{2N}}. \quad (3.6)$$

The first two terms correspond to the generic one-loop potential in the small coupling regime while the last term is a non-renormalizable operator left in the low energy theory after integrating out some unspecified physics at the high scale $M$. Notice that this scale absorbs any possible coupling in front of the operator. Of course $V(\phi)$ may contain other NROs of different order. Here we assume that the one shown in equation (3.6) is the lowest order one, and thus the dominant one. The sign and power that we have assumed for this NRO guarantee the stability of the potential. Notice also that an even power for this operator is what one expects generically in supersymmetric theories. An explicit example of this is given in [17]. Apart from that, the potential (3.6) is completely general and therefore the analysis is quite model independent.

4. Primordial spectra in the slow-roll approximation

4.1. Logarithmic regime (LOG)

From $V(\phi)$, as given in equation (3.5), the first three slow-roll parameters (2.1) in the LOG scenario are

$$\epsilon \approx \frac{1}{2} q^2 \frac{M_p^2}{\phi^2}, \quad \eta \approx -q \frac{M_p^2}{\phi^2} \approx -2 \frac{\epsilon}{q}, \quad \xi \approx 2 \eta^2, \quad (4.1)$$

where we have introduced the quantity

$$q \equiv \frac{\beta}{\rho_0}. \quad (4.2)$$

Note that $q > 0$ since we have assumed $\beta > 0$. Satisfying the slow-roll condition $\epsilon \ll 1$ requires $q \ll 1$ for $\phi < M_p$. This implies in turn $\epsilon \ll |\eta|$. We give a more quantitative evaluation of this hierarchy in section 5.

The number of e-folds between time $t_0$ (i.e. the time when $\phi = \phi_0$) and the end of inflation, $t_e$, can be easily computed using $\epsilon$, as given by equation (4.1), in the usual expression (2.6):

$$N_e(t_0 \rightarrow t_e) \approx -\frac{1}{2} \left[ \frac{1}{\eta(\phi_0)} - \frac{1}{\eta(\phi_e)} \right] \approx -\frac{1}{2} \frac{1}{\eta(\phi_0)} \approx \frac{1}{2} \frac{\phi_0^2}{qM_p^2} \equiv N_e^0, \quad (4.3)$$
where we have used the fact that inflation comes to an end when $\eta \sim \mathcal{O}(1)$ (it could end before that time if inflation is interrupted by other mechanisms, like a waterfall field in hybrid models). The $N_0^e$ parameter defined above, besides giving an excellent approximation to $N_e(t_0 \rightarrow t_e)$, will play a relevant role when performing the fits to the data.

Now the $\phi-k$ connection, equation (2.5), can be integrated at first order in $q$, giving

$$\phi^2 = \phi_0^2 \left(1 - \frac{1}{N_0^e} \ln \frac{k}{k_0}\right), \quad (4.4)$$

where we have used (4.3). Note that increasing $\phi$ corresponds to decreasing $k$ so that the scales probed by WMAP correspond to the highest values of $\phi$ during its slow-roll towards the origin. Now we can straightforwardly evaluate $P_s(k)$ from equation (2.2). For the purpose of comparing the model with the data, it is convenient to write $P_s$ in terms of $P_0^s \equiv P_s(k_0)$ using the general expression

$$\ln P_s = \ln P_0^s + 3 \ln \frac{V(\phi)}{V(\phi_0)} - 2 \ln \frac{V'(\phi)}{V'(\phi_0)} . \quad (4.5)$$

Using equations (3.5) and (4.4) and expanding in $q$ we find, at first order,

$$\ln P_s(k) = \ln P_0^s + \left(1 + \frac{3}{2} q\right) \ln \left(1 - \frac{1}{N_0^e} \ln \frac{k}{k_0}\right), \quad (4.6)$$

where, from equation (2.2),

$$P_0^s = \frac{1}{12 \pi^2 M_p^3 \beta^2}. \quad (4.7)$$

The same result can be obtained by integrating the slow-roll equation (2.7).

Similarly, the spectrum of tensor perturbations, $P_t(k)$, can be obtained from equation (2.3):

$$P_t(k) \simeq \frac{4 q P_0^s}{N_0^e} \left[1 + \frac{q}{2} \ln \left(1 - \frac{1}{N_0^e} \ln \frac{k}{k_0}\right)\right]. \quad (4.8)$$

At the same level of approximation, the tensor to scalar ratio (2.4) reads

$$r(k) \simeq \frac{4 q}{N_0^e} \left(1 - \frac{1}{N_0^e} \ln \frac{k}{k_0}\right)^{-1}. \quad (4.9)$$

Let us now count the number of independent parameters. The power spectra, $P_s(k)$ and $P_t(k)$, contain three independent parameters, $\{P^0_s, q, N_0^e\}$, which are combinations of the initial parameters $\{\phi_0, \rho_0, \beta\}$. Incidentally note that the scalar potential (3.5) is a function of just two combinations of parameters, but a third one appears in the conversion of $\phi$ into $k$ through equation (4.4). Actually, the $q$ term in (4.6) is subdominant because $q \ll 1$. Removing it from the expression is a good approximation and eliminates one parameter. So the expression for $P_s(k)$ contains basically two parameters. This is to be compared with the three parameters (two if the running of $n$ is set to zero) of the simple standard parametrization (2.9). As a consequence this LOG scenario is highly predictive. On the other hand, the fits to WMAP data favour $P_t(k) \ll P_s(k)$, which means that $P_t$ turns out to be scarcely important in the fit, and so the number
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of relevant parameters continues to be 2. Indeed, from equation (4.9) and \( q \ll 1 \) we do
expect by construction \( P(t) \ll P_s(k) \), something that cannot be postulated from the
simple parametrizations (2.9) and (2.10), which contain the additional parameter \( P_t(k_0) \)
or, equivalently, \( r \) (unless \( r \) is set to zero by hand).

Equations (4.6) and (4.8) summarize the predictions of models with flat tree-level
potentials in the small coupling regime. It is possible to gain some intuition about the
performance of these functional forms as follows. As mentioned above, \( \epsilon \ll |\eta| \), which
means \( n - 1 \simeq 2 \eta \). Therefore (see equation (4.1)),

\[
\frac{dn}{d\ln k} \simeq -2\xi = -(n - 1)^2. \tag{4.10}
\]

As a consequence, \( dn/d\ln k \) is negative, as suggested by the data, though its value tends
to be quite small. In fact the sign of \( n - 1 \) cannot change during the inflationary process
in this class of models. Since the dependence of \( n \) on the scale is weak, and therefore
\( n \simeq \) constant, we can expect a fit similar to the one obtained by using the standard
parametrization of equation (2.9) with \( dn/d\ln k = 0 \), leading to \( n_0 \sim 0.95 \). We will see
in section 5 that this is indeed the case.

Equation (4.10) can be written in an integrated form as

\[
n = 1 - \frac{1}{N_e^0 - \ln(k/k_0)}, \tag{4.11}
\]

where we have used equation (4.3). Equation (4.11) is the prediction for the spectral
index in scenarios with flat tree-level potential in the regime of small coupling. It can be
compared with the \( n = \) constant or \( dn/d\ln k = \) constant assumptions made in standard
analyses. Note that \( N_e^0 \) is the only independent parameter in (4.11) and has a precise
physical meaning.

4.2. Logarithmic regime and non-renormalizable operator (LOG + NRO)

In this scenario the derivatives of the potential \( V(\phi) \) (equation (3.6)) with respect to \( \phi \)
read

\[
V'(\phi) = \frac{\beta}{\phi^2} + 2(N + 2)\phi^{2N} \phi^{2N}/M^{2N},
\]
\[
V''(\phi) = -\frac{\beta}{\phi^2} + 2(N + 2)(2N + 3)\phi^2 \phi^{2N}/M^{2N}, \tag{4.12}
\]
\[
V'''(\phi) = 2 \frac{\beta}{\phi^2} + 4(N + 2)(2N + 3)(N + 1)\phi \phi^{2N}/M^{2N}.
\]

The corresponding expressions for \( \epsilon, \eta \) are

\[
\epsilon = \frac{1}{2} M_p^2 \left[ \frac{\beta}{\rho_0} + 2(N + 2)\phi^4 \phi^{2N}/M^{2N} \right]^2,
\]
\[
\eta = \frac{1}{2} M_p^2 \left[ \frac{\beta}{\rho_0} + 2(N + 2)(2N + 3)\phi^4 \phi^{2N}/M^{2N} \right]. \tag{4.13}
\]

In consequence the NRO can have a significant impact on inflation when the small number
\( (\phi/M)^{2N} \) is comparable in size to \( \beta/\phi^4 \). It is also immediate from equations (4.12)
and (4.13) that, for sufficiently large φ, the higher derivatives $V'', V'''$ (and thus η, ξ) can receive a large contribution from the NRO while the contribution to $V'$ (and thus ε) is much less significant, thanks to the additional $(2N + 3)$ and $(2N + 3)(2N + 2)$ factors in $V'', V'''$. Therefore, one expects modifications of the spectral index, $n$, and its running dn/dln$k$ (which depend on η, ξ), especially at the initial scales (very small k and thus large φ), and much smaller changes in ε (and hence in $N_e$).

Let us now calculate the expression for the power spectrum $P_s(k)$ in this scenario. As in the previous subsection, we start with the general expression (4.5) where $V(φ)$ and $V'(φ)$ are given now by equations (3.6) and (4.12) respectively, and

$$P^0_s = \frac{1}{12\pi^2} \frac{[V(φ_0)]^3}{M^2_π [V'(φ_0)]^2}. \tag{4.14}$$

Next we have to convert φ into k by integrating equation (2.5). This is done in the appendix in an exact way. The corresponding formula, equation (A.1), is rather cumbersome. However, using $|q| \ll 1$, one can write the much simpler but extremely accurate expression

$$\ln \frac{k}{k_0} \simeq -φ N_0^0 2F_1 \left[ \frac{1}{N + 2}; \frac{N + 3}{N + 2}; -(4φ)N+2 \right]_{φ^2/φ_0^2}, \tag{4.15}$$

where $2F_1(a; b; c; z)$ is the Gauss hypergeometric function [20], φ is just a dummy variable and

$$A^{N+2} \equiv 2(N + 2)\frac{φ_0^4}{β} \frac{φ_0^{2N}}{M^{2N}}. \tag{4.16}$$

Note that $N_0^0$ is still defined as in equation (4.3) and gives a good estimate of the number of e-folds $N_e(t_0 \rightarrow t_0)$. It is possible to invert (4.15) numerically to get φ = φ(k). Plugging φ(k) into equation (4.5) we obtain $P_s(k)$. This is the procedure that we have followed in doing the fits. Using φ(k) we can also obtain other quantities of interest as functions of k, e.g. the spectral index $n \simeq 1 + 2η - 6ε$ or the tensor to scalar ratio $r \simeq 16ε$.

In order to get some intuition about the shapes of $P_s(k)$ and n(k) it is convenient to derive an analytical approximation to the previous numerical procedure. Actually the numerical part just comes from the φ to k conversion, i.e. the integration of equation (2.5). This equation depends on ε (on V and $V'$) but not on higher derivatives of V, which are the ones most affected by the presence of the NRO, as discussed after equation (4.12). Therefore it is sensible to use here the value of ε when the NRO is switched off, i.e. that of equation (4.1). Then the φ to k relation is still given by equation (4.4). Substituting this in the general expression for $P_s(φ)$, equation (4.5), and expanding to first order in the NRO contributions, one gets\(^6\)

$$\ln \frac{P_s(k)}{P^0_s} \simeq \left(1 + \frac{3}{2}β \right) \ln \left(1 - \frac{1}{N_0} \ln \frac{k}{k_0} \right) + \frac{γN_0^0}{N + 2} \left[1 - \left(1 - \frac{1}{N_0} \ln \frac{k}{k_0} \right)^{N+2}\right], \tag{4.17}$$

\(^6\) This approximate formula gives $P_s(k)$ with a maximum error of ≤13% in the most extreme cases although typically is much better. Anyway, we remark that in the fit we evaluate $P_s(k)$ numerically using equation (4.15).
where

$$\gamma \equiv \left\{ 2N + 3 + \left[ \frac{1}{2(N + 2)} - 3 \right] q + \frac{3}{2(N + 2)} q^2 \right\} A^{N+2} \frac{N_0^2}{N_0^e} \approx (2N + 3) A^{N+2} \frac{N_0^2}{N_0^e}. \quad (4.18)$$

The approximate equality in the last expression is justified by the smallness of $q$. Alternatively, one can use the expansion $n - 1 \simeq 2\eta - 6\epsilon$ at first order in the NRO and approximate again the $\phi$ to $k$ conversion by equation (4.4). One obtains

$$n(k) - 1 \simeq - \left( 1 + \frac{3}{2} q \right) \frac{1}{N_0^e} \left( 1 - \frac{1}{N_0^e} \ln \frac{k}{k_0} \right)^{-1} + \gamma \left( 1 - \frac{1}{N_0^e} \ln \frac{k}{k_0} \right)^{N+1}. \quad (4.19)$$

Then the direct integration of (4.19) gives back expression (4.17). Equation (4.19) corresponds to a running $n(k)$ with non-constant slope, departing from the assumption of analyses done using the standard parametrization. Unlike in the LOG scenario, in this case the running is not constrained to be very small.

It is also worth mentioning that, due to the positivity of $1 - (1/N_0^e) \ln k/k_0$, the signs of the LOG and the NRO contributions to the $n(k)$ are $\{ - , - , - \}$ and $\{ +, - , + \}$ respectively (see equation (4.19)). Since a sizable running at low $k$ requires a dominant NRO contribution, we can conclude from equation (4.19) that in that case the sign of the second derivative will be positive, although for large enough $k$ it will turn to negative as the LOG part becomes dominant.

In a similar way one can obtain expressions for $P_t(k)$ (or equivalently $r(k)$) starting with the general equations (2.3) or (2.4). In particular, the previous analytical approximation gives in this case

$$r(k) \simeq \frac{4q}{N_0^e} \left( 1 - \frac{1}{N_0^e} \ln \frac{k}{k_0} \right)^{-1} + \frac{8q A^{N+2}}{N_0^e} \left( 1 - \frac{q}{2N + 4} \right) \left( 1 - \frac{1}{N_0^e} \ln \frac{k}{k_0} \right)^{N+1}. \quad (4.20)$$

Let us finally count the number of independent parameters. From expressions (4.17) and (4.20) we see that the spectrum of primordial perturbations depends upon five parameters $\{ P_0^s, N_0^e, q, A, N \}$, which are combinations of the five parameters $\{ \rho_0, \beta, \phi_0, M, N \}$ appearing in the scalar potential (3.6). Thus the LOG + NRO has two more parameters than the LOG model. Again, as in the LOG case, the smallness of $q$ implies that $P_s(k)$ is nearly independent of $q$ and, besides, the tensor spectrum is much less important than its scalar counterpart. Hence, $q$ will be irrelevant for a broad range of values in the fit to the data. In practice the primordial spectrum depends essentially on four parameters which become just three if we consider $N$ to be a fixed integer. Again, this is to be compared with the four parameters of the simplest standard parametrizations (2.9) and (2.10). In consequence, the LOG + NRO scenario is still highly predictive.

To illustrate the shapes of the scalar power spectrum in the different scenarios discussed in this section we plot $P_s(k)$ from (4.15) in figure 1. The parameters for the models are chosen to be the best-fit values given in section 6 and discussed later in the text.

### 5. Data analysis procedure

In order to constrain the parameters of the two scenarios introduced above, by comparing their predictions with CMB and LSS data, we use a modified version of the `cosmomc`
Figure 1. Primordial power spectra in the standard parametrization (dashed lines with Taylor expansion up to second order, i.e. running of running) and as predicted by the LOG and LOG + NRO ($N = 2$) scenarios. The parameters for each case are the best-fit ones (given in tables 1–3).

As discussed in section 4.1, the LOG scenario is described by the three independent parameters $\{\rho, \beta, \phi_0\}$, appearing in the scalar potential (3.5). However, for the purpose
of comparing the model with data, it is more appropriate to use the following set:

$$\mathcal{P}_{\text{LOG}} \equiv \{ \ln P_s^0, N_e^0, q \},$$

(5.1)

which are related to the potential parameters by the relations (4.2), (4.3) and (4.7). The inverse transformations are given by

$$\phi_0 = \sqrt{2qN_e^0} M_p,$$

$$\rho_0 = \frac{6\pi^2 q P_s^0}{N_e} M_p^4,$$

$$\beta = q \frac{6\pi^2 q P_s^0}{N_e} M_p^4.$$

(5.2)

The reasons for preferring the set (5.1) over the original potential parameters are the following. First, the fit to WMAP data is very sensitive to the value of $\ln P_s^0$ at the pivotal scale, which makes it very convenient to use $\ln P_s^0$ as one of the parameters. Second, $N_e^0$ appears explicitly in the expressions for $P_s(k)$ and $P_t(k)$ (see equations (4.6) and (4.8)). In addition, $N_e^0$ has a clear physical meaning, since for small $q$ it simply expresses the number of e-folds since the time when $k_0$ crosses out of the horizon until the end of inflation. This also makes it possible to impose on it a physically motivated prior for the number of e-folds, as required to solve the homogeneity and flatness problems. Furthermore, we note from equation (4.11) that $N_e^0$ and the spectral index at $k_0$ are simply related,

$$n_0 \simeq 1 - \frac{1}{N_e^0}.$$ 

(5.3)

Finally, the third parameter, $q$, does also appear explicitly in the expressions for the spectra. As argued in section 4.1, we expect $q \ll 1$, implying that the scalar primordial spectrum depends essentially only on $\{ \ln P_s^0, N_e^0 \}$, while the tensor spectrum $P_t(k) \simeq 16\epsilon P_s(k)$ is suppressed (and much less important for the fit). Therefore it is convenient to choose $q$ as a parameter for the fit in order to single this effect out. From the above discussion, $\ln P_s^0, N_e^0$ will be well determined by the observable properties of the power spectrum, and therefore it is appropriate to impose flat priors on them, which corresponds to the assumption that they are location parameters.

The relation between $\mathcal{P}_{\text{LOG}}$ and the potential parameters is non-linear and so one expects volume effects coming from the Jacobian of the transformation that will in general make the marginalized constraints on the potential parameters sensitive to the choice of priors. Furthermore, as argued above, only two combinations of parameters of the potential are going to be well determined by the data. The constraints on these ‘principal directions’ in the potential parameter space are however essentially prior independent, as we discuss in detail in section 6.2.

Let us now focus on the physical constraints on the parameter space spanned by $\mathcal{P}_{\text{LOG}}$. The evolution equations of the classical value of the field are based on general relativity. To prevent effects of quantum gravity from becoming important, we conservatively require the energy density to satisfy

$$\rho < M_p^4.$$ 

(5.4)

Similarly, it is sensible to keep the inflaton field below the Planck scale. Note in particular that, at least in this framework, the renormalization scale $Q$ is to be identified with
the value of the inflaton, in order to keep the radiative corrections under control, and obviously the RGE are only reliable for $Q$ below the Planck scale. Thus we also require, conservatively,

$$\phi_0 < M_p.$$  \hfill (5.5)

Notice that, since the inflaton rolls towards zero, if the above condition is satisfied for $\phi_0$ it will automatically be satisfied for smaller values of $\phi$, as well. For larger values of $\phi$, imposing equation (5.5) easily guarantees that they are well below $m_p$, since there are very few e-folds before $k_0$, and they correspond to a short range of $\phi$-values.

Moreover, one must ensure that the slow-roll approximation is fulfilled, which means that we require

$$\epsilon < 1, \quad (5.6)$$

$$|\eta| < 1. \quad (5.7)$$

For simplicity we impose these conditions at $k_0$ and this automatically ensures that the slow-roll is not violated for smaller values of $k$, which means greater values of $\phi$. Therefore the slow-roll will be guaranteed at the scale $k_{\text{obs}} \equiv 10^{-4}$ Mpc$^{-1}$, which is roughly the size of the observable universe. On the other hand, larger values of $k$ are probed and the slow-roll parameters grow as $\phi$ goes to zero. Taking into account that the largest relevant multipole is about 3000, one gets a maximum $k$ around $k_{\text{max}} \equiv 0.1$ Mpc$^{-1}$ [30]. Using equation (4.4), we have checked that slow-roll for such large values of $k$ is indeed satisfied by the samples in our Markov chains. The slow-roll condition on $\eta$ is equivalent to

$$2N_{\epsilon}^0 > 1,$$  \hfill (5.8)

while the one on $\epsilon$ leads to

$$4N_{\epsilon}^0 > q.$$  \hfill (5.9)

On the other hand, the inequality (5.5) implies

$$2qN_{\epsilon}^0 < 1,$$  \hfill (5.10)

which together with equation (5.8) implies

$$q < 1,$$  \hfill (5.11)

as anticipated in section 4.1.

We found that samples in the Markov chains that fulfil the condition (5.10) automatically also satisfy conditions (5.8), (5.11) and (5.4). This can be understood as follows. As discussed in section 4.1, we expect a value for $n_0 \sim 0.95$, similar to the simplest fit (2.12). Then equation (5.3) implies (5.8). Moreover, the value of $n$, coupled with the physical prior (5.5), translates into an upper bound on $q$: since $n-1 \sim -2\eta \sim 2q(M_p/\phi_0)^2$, for $\phi_0 \leq M_p$ one gets $q \leq (1-n)/2$, and therefore condition (5.11) holds. Incidentally, this upper bound on $q$ implies that the contribution of the tensor part of the power spectrum must be necessarily small: $r = 16\epsilon \leq 2(1-n)^2$. Finally, (5.4) is granted by the smallness of $P_\epsilon^0$. Notice also that the condition (5.6) on $\epsilon$ is, in practice, irrelevant because (5.11) ensures that $\epsilon < |\eta|$, as can also be read off directly from (4.1). In consequence, condition (5.10) remains the only non-trivial constraint.
In summary, we take $\mathcal{P}_{\text{LOG}}$ (equation (5.1)) as the set of independent parameters, imposing flat priors on them and enforcing the constraint (5.10). We then compute the scalar and tensor contributions to the primordial spectrum via the expressions (4.6) and (4.8).

As mentioned above, one of the reasons for choosing $N_e^0$ as an independent parameter is its direct physical interpretation as the number of e-folds. In fact, we have a strong theoretical prejudice about its value, which should be $\sim 50$. We have taken this fact into account by performing two different analyses. The first one imposes a flat prior on $N_e^0$, therefore assuming no prejudice about its value and leaving the data to constrain it. In the second case we enforce the theoretical requirement by imposing a Gaussian prior on $N_e^0$ centred on 50 with a standard deviation of 5. The details of these two fits and the results are discussed below, in section 6.2.

5.2. Logarithmic regime and non-renormalizable operator (LOG + NRO)

Concerning the LOG + NRO scenario, for practical reasons it is convenient to work with the set of independent parameters

$$\mathcal{P}_{\text{LOG+NRO}} \equiv \{ \ln P_s^0, N_e^0, q, A, N \},$$

where $A$ was defined in equation (4.16), instead of the parameters $\{ \rho, \beta, \phi_0, M, N \}$ of the scalar potential (3.5). The relationships between $\mathcal{P}_{\text{LOG+NRO}}$ and the potential parameters are given in appendix A.

The convenience and significance of the first three parameters in $\mathcal{P}_{\text{LOG+NRO}}$ are the same as in the LOG scenario. However, the interpretation of $N_e^0$ as the number of e-folds between $k_0$ and the end of inflation is now less accurate since there are NRO corrections, although it is still a good approximation. This is also true for the connection between $N_e^0$ and the spectral index $n_0$: the relation (5.3) becomes now

$$n_0 \simeq 1 - \left( 1 + \frac{3}{2} q \right) \frac{1}{N_e^0} + \gamma \simeq 1 - \frac{1}{N_e^0} + (2N + 3) \frac{A^{N+2}}{N_e^0}. \tag{5.13}$$

This expression tells us that for not too large $N_e^0$ we should expect $A$ to be bounded from above by some number close to unity because otherwise $n_0$ can become substantially different from 1 (especially for high values of $N$), thus violating the slow-roll conditions. This is also illustrated in figure 2 which shows contour plots for the scalar spectral index and its running at $k_0$ at lowest order in slow-roll as functions of $A$ and $N_e^0$ for $N = 2$ and 10. It is worth remarking here that since we are dealing with scale-dependent quantities the appearance of these graphs would change if we made them at a different $k$. Figure 2 allows us to see that, in the context of the LOG + NRO scenario, it is possible to have simultaneously a sizable running and a reasonable number of e-folds.

Concerning the physical limits in parameter space, we must take into account the presence of the scale of new physics $M$. The role played by the Planck mass on the LOG scenario corresponds now to $M$. To retain the validity of the effective potential (3.6) the

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7 Recall however that its value could be less than 50 if there are subsequent episodes of inflation (see the discussion after equation (2.6)). On the other hand, the parameter $N_e^0$ (defined in equation (4.3)) could be larger than 50 if inflation is interrupted, e.g. by a waterfall condition in hybrid models (see the discussion after equation (4.3)).
inflaton must evolve well below that scale, which should be itself smaller than the Planck scale. Thus, we impose the following conservative limits:

\[ 2\phi_{\text{obs}} < M \leq M_p, \]
\[ \rho < M^4. \]

Notice that we set the first constraint at \( \phi_{\text{obs}} \equiv \phi(k_{\text{obs}}) \) in order to ensure it for any value of \( \phi \) in the observable range.

As in the LOG scenario, \( |\eta| \gg \epsilon \), so \( \eta \) is the relevant parameter for the breakdown of the slow-roll. However, unlike in the LOG case, due to the NRO the absolute value of \( \eta \) grows with sufficiently large \( \phi \). Therefore we must ensure the fulfillment of slow-roll not only at \( k_{\text{max}} \) but also at \( k_{\text{obs}} \). This guarantees that any point in between will satisfy the slow-roll conditions as well. So, we reject in the Monte Carlo process those points such that

\[ |\eta(k_{\text{obs}})| > \eta_{\text{lim}}, \]

or

\[ \eta(k_{\text{max}}) > \eta_{\text{lim}}, \]

\( \eta_{\text{lim}} \) being a limiting value (smaller than 1) that we set at the beginning of the run. At the end we check that \( \eta_{\text{lim}} \) was indeed well chosen to ensure the validity of the slow-roll approximation. In practice we work with \( \eta_{\text{lim}} = 0.2 \) which is a rather conservative value. We have checked that the change in the results is negligible if instead we use \( \eta_{\text{lim}} = 0.5 \).

On the other hand, we expect theoretically that the parameter \( N \) should be in the range from 1 to \( O(10) \). It is very common that flat directions in supersymmetric models are only lifted by NROs at very high order (as is the case for the MSSM \[31\]–\[33\]). This is also very common in \( D = 4 \) string compactifications due to stringy selection rules \[34,35\].
For further details see [17]. In section 6 we discuss in detail two representative cases, which reasonably encompass the range of values for $N$ ($N = 2$ and 10), and we comment on the qualitative behaviour for values of $N$ in between.

As for the LOG scenario, we can consider $N_0^0$ as a free parameter with a flat prior on it or we can alternatively constrain it to be around 50. We have performed the two types of fit.

Finally, we can anticipate theoretically the appearance of some strong bounds on the parameters of the model when performing the fits. First note that the observed power spectrum normalization $P_s^0 \sim 2 \times 10^{-9}$ implies through equation (2.2) the smallness of $\rho_0/M_p^4$. More precisely

$$\frac{\rho_0}{M_p^4} \simeq 5 \times 10^{-7} \epsilon_0,$$

where the subscript ‘0’ indicates evaluation at the pivotal scale. On the other hand, from equations (4.13) we note that the smallness of $|\eta|$ (to preserve the slow-roll) implies that the two contributions within the square brackets (i.e. the LOG and the NRO contributions) must be small separately. Otherwise one should require an unjustified fine-tuned cancellation between them. Actually, even with fine-tuning, one could arrange the parameters to produce the cancellation only at a particular $\phi$ (and thus $k$): since the $\phi$-dependences of the two terms are very different, at another (not too distant) value of $\phi$ the cancellation would not work, spoiling the slow-roll. Consequently, the smallness of $|\eta|$ implies

$$\frac{\beta}{\rho_0} \lesssim \eta_0 \left( \frac{\phi_0}{M_p} \right)^2,$$

$$\left( \frac{\phi}{M_p} \right)^{2N+2} \lesssim \frac{|\eta_0|}{2(N + 2)(2N + 3)} \frac{\rho_0}{M_p^4} \approx \frac{5 \times 10^{-7} \epsilon_0 |\eta_0|}{2(N + 2)(2N + 3)}.$$  \hspace{1cm} (5.20)

In the second equation we have used $M \leq M_p$ and equation (5.18). On the other hand, comparing the two equations (4.13), and recalling that there cannot be fine-tuned cancellations in $\eta$, it is clear that

$$\epsilon_0 \lesssim \frac{1}{2} \left( \frac{\phi_0}{M_p} \right)^2 \eta_0^2.$$  \hspace{1cm} (5.21)

Using this relation in (5.20) we get

$$\left( \frac{\phi}{M_p} \right)^{2N} \lesssim \frac{5 \times 10^{-7} |\eta_0|^3}{4(N + 2)(2N + 3)}.$$  \hspace{1cm} (5.22)

which, substituted into (5.19), gives

$$q = \frac{\beta}{\rho_0} \lesssim |\eta_0|^{1+3/N} \left[ \frac{5 \times 10^{-7}}{4(N + 2)(2N + 3)} \right]^{1/N}.$$  \hspace{1cm} (5.23)

This shows that $q$ is typically small: for $N = 2$ ($N = 10$) one obtains $q \lesssim 1.7 \times 10^{-6}$ ($q \lesssim 1.4 \times 10^{-2}$), a conservative estimate obtained by making the replacement $\eta_0 \rightarrow \eta_\text{lim} = 0.2$. In practice $\eta$ is substantially smaller since the bound $\eta \leq \eta_\text{lim}$ is to be fulfilled at all $k$, not just at the pivotal scale. Similar bounds on $q$ can be obtained using our priors together with the constraint $M \leq M_p$. We have checked numerically that the values obtained agree with the ones derived above.
The absolute value of the log-likelihood is of little interest here and in the following. For completeness, we have computed the likelihood values using the WMAP3 likelihood code version v2p2 with the default settings regarding the offset for the log-likelihood. The best-fit value for the constant tilt model is \(-2\ln L = 3614.0\).

6. Results

6.1. Standard parametrization

The results obtained using the standard parametrization for the primordial spectra (equations (2.9) and (2.10)) are summarized for easy reference in table 1 both with and without a running spectral index (i.e. including or not including the last term of equation (2.9)). For later reference, we have also considered the next term in the Taylor expansion (2.9), which has been denoted as ‘running of the running’, in the last two columns of table 1. The table shows the best-fit parameter values, the posterior values and 68% one-dimensional posterior intervals for the parameters. We also give the best-fit values for (minus twice) the log-likelihood, normalized with respect to the model with only a constant tilt included\(^\text{8}\). Let us recall that the quantities describing the primordial spectrum are defined at \(k_0 = 0.002\) Mpc\(^{-1}\). These results will be useful later on for interpreting the outcomes for the LOG and LOG + NRO models and for the comparison with them.

It is interesting to note that when using the standard parametrization up to second order a large and negative running of the running is preferred \(^[36]\), which increases the power on large scales (see the bottom panel of figure 4), even though the fit is only marginally better than the case with constant running; see table 1. This is somewhat

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<table>
<thead>
<tr>
<th>Model (-2\Delta \ln L)</th>
<th>No running (0.0)</th>
<th>With running (-3.4)</th>
<th>Running of running (-4.4)</th>
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<tr>
<td></td>
<td>1D 68%</td>
<td>Best fit</td>
<td>1D 68%</td>
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<td>Cosmological parameters</td>
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<td></td>
</tr>
<tr>
<td>(\Omega_b h^2 \times 10^2)</td>
<td>2.23 ± 0.07</td>
<td>2.25</td>
<td>2.20(^{+0.09}_{-0.08})</td>
</tr>
<tr>
<td>(\Omega_c h^2)</td>
<td>0.106 ± 0.004</td>
<td>0.107</td>
<td>0.107 ± 0.004</td>
</tr>
<tr>
<td>(\Theta_s)</td>
<td>1.043 ± 0.003</td>
<td>1.042</td>
<td>1.043 ± 0.003</td>
</tr>
<tr>
<td>(\tau)</td>
<td>0.084 ± 0.029</td>
<td>0.087</td>
<td>0.114 ± 0.035</td>
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<tr>
<td>(H_0) (km s(^{-1}) Mpc(^{-1}))</td>
<td>74.3 ± 2.1</td>
<td>73.1</td>
<td>73.1 ± 2.3</td>
</tr>
<tr>
<td>Power spectra parameters</td>
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<tr>
<td>(\ln(P_s^0 \times 10^{10}))</td>
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<td>3.15</td>
<td>3.00 ± 0.10</td>
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<tr>
<td>(n_0)</td>
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<td>0.961</td>
<td>1.141(^{+0.083}_{-0.082})</td>
</tr>
<tr>
<td>(dn/d\ln k)(_{k_0})</td>
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<td>—</td>
<td>—</td>
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<tr>
<td>(d^2n/d\ln^2 k)(_{k_0})</td>
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<td>0.003</td>
<td>&lt;0.59</td>
</tr>
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</table>

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\(^{8}\) The absolute value of the log-likelihood is of little interest here and in the following. For completeness, we have computed the likelihood values using the WMAP3 likelihood code version v2p2 with the default settings regarding the offset for the log-likelihood. The best-fit value for the constant tilt model is \(-2\ln L = 3614.0\).
surprising: since in the slow-roll approximation \( n = 1 + 2\eta - 6\epsilon \), if \( n \) departs from scale invariance too quickly then \( \eta, \epsilon \) or both grow up to \( \mathcal{O}(1) \) values, marking the end of slow-roll and the inflationary process at quite small \( k \). However, in order to solve the horizon problem we need \( \sim 50-60 \) e-folds of inflation, which corresponds to the same interval in \( \ln k \). This requires the (large and negative) \( \frac{dn}{d\ln k} \) to get suppressed at some point, suggesting a positive second derivative (unlike the result of the fit). This contradiction could only be avoided if the \( n(k) \) function changes abruptly at some point (or, maybe, if there are subsequent episodes of inflation). In any case, it seems clear that the current preference for a negative second derivative is strongly driven by the (possibly anomalous) low power of the large scale multipoles. This could easily change if observations by Planck do not confirm the lack of large scale power observed by COBE and WMAP. However, we notice that from a model selection perspective even present day data do not require a non-zero running of the running, as discussed in section 7. The previous discussion will be useful later on to interpret the outcomes for the LOG and LOG + NRO models and for the comparison with them.

### 6.2. Logarithmic regime (LOG)

We denote the choice of flat prior on \( N_0^e \) by LOG\( \mathcal{F} \), where the top-hat distribution is taken in the interval \( 2 \leq N_0^e \leq 1000 \). In the second case, denoted by LOG\( \mathcal{G} \), we impose a Gaussian prior on \( N_0^e \), with mean 50 and a standard deviation of 5 e-folds. Let us recall here that \( N_0^e \) approximates the number of e-folds since the time when the scales associated with \( k_0 \) first crossed the horizon to the end of inflation. Thus the total number of e-folds since the time when the largest observable scale, \( k_{\text{obs}} \), crossed the horizon is \( \simeq N_0^e + 3 \). This prior choice incorporates the theoretical prejudice that the total number of e-folds should be in the 50–60 range.

The results of the Monte Carlo Markov chain (MCMC) analysis for the LOG\( \mathcal{F} \) and LOG\( \mathcal{G} \) cases are given in figure 3 and are summarized in table 2. We give one-dimensional regions encompassing 68% of probability for well-determined parameters; robust upper bounds for parameters whose detailed constraints are parametrization dependent; and best-fit values. The table also gives posterior ranges and best-fit values for the corresponding expressions for the tilt, running and tensor to scalar ratio at the pivotal scale, \( n_0, \frac{dn}{d\ln k}|_0 \) and \( r_0 \), respectively. These have been obtained by using lowest order expressions in terms of the slow-roll parameters, equations (2.4), (2.7) and (2.8).

As anticipated, constraints on \( N_0^e \) and \( \ln P_0^s \) are quite tight, and we have checked that they are almost independent of the choice of prior by performing a run with priors flat in \( \{\ln N_0^e, \ln q, \ln P_0^s\} \) instead. It is interesting to notice that in the LOG\( \mathcal{F} \) case the 1D 68% (95%) posterior region (two tails) is approximately \( 21 < N_0^e < 46 \) (16 < \( N_0^e < 81 \)), even though the mean is somewhat lower, at around 33 e-folds, and a best fit around 26 e-folds. This result is close to the theoretical prejudice \( N_0^e \sim 50 \), required to solve the horizon problem. Thus assuming a flat tree-level potential for the inflaton, the observed shape of the power spectrum appears to automatically point to model parameters giving a very sensible number of e-folds, in particular given the heavy tail of the probability distribution function (pdf) for large \( N_0^e \). This is not trivial at all: in principle any value for \( N_0^e \) could have emerged from the analysis.

The values of the spectral index and its running at the pivotal scale are easily derived from the parameters of the fit and are also given in table 2. Note that \( \frac{dn}{d\ln k}|_0 \) is very
Flat tree-level inflationary potentials in the light of CMB and LSS data

Figure 3. 1D marginalized probability distributions for the well-constrained parameters in the LOG scenario (compare table 2). Black curves are for the case with a flat prior on $N_0^e$, while red is for the case where a Gaussian prior around $N_0^e = 50$ has been enforced.

small (of order $\sim 10^{-3}$), as expected from the relation (4.10). So the LOG scenario is indeed close to the $n = constant$ limit. The value of $n$ at the pivotal scale, $n_0$, is directly related to the value of $N_0^0$ by equation (5.3), leading to the values of $n_0$ quoted in the table. The LOGF best-fit value, $n_0 = 0.961$ (corresponding to $N_0^e = 25.6$), coincides with the value obtained assuming $n = constant$ and negligible running (table 1). It is interesting to note that although the LOGF and $n = constant$ fits are very similar, they are not identical, and indeed LOGF gives a slightly better fit, as can be checked by comparing the best-fit likelihood values (also compare figure 4). Furthermore, if future CMB and LSS data favour a value of $n_0$ closer to $\sim 0.98$, the value of $N_0^e$ will come out even closer to the theoretically preferred value, $N_0^e \sim 50$. The upper bound on $q$ is a consequence of the measured tilt and of the physical boundaries imposed on the potential parameters (Recall from the discussion after equation (5.11) that we expect $q \leq (1 - n)/2$. For the reasons explained below, the pdf for $q$ depends on the prior chosen, and therefore we do
not show it in figure 3. However the upper bound is robust with respect to a change of priors, and therefore we chose to report only this value. The tensor contribution remains negligible, below the level $\sim 10^{-3}$, since from (4.9) the value of the tensor to scalar ratio at the pivotal scale is

$$r_0 \simeq 4 \frac{q}{N_e^0}.$$  \hfill (6.1)

Consequently the upper bound on $q$ corresponds to an order of magnitude smaller bound on $r_0$.  

Figure 4. CMB temperature power spectrum for the best-fit model parameters for the standard parametrization, the LOG and the LOG + NRO scenarios. The bottom panel shows the details of the large scale region.
Table 2. Marginalized 68% regions and best-fit values for the class of models LOG (small coupling regime) for quantities that are well determined and essentially prior/parametrization independent. For $q$ and the tensor to scalar ratio at the pivot scale, $r_0$, we give absolute upper limits that are a consequence of the spectral tilt and of physical priors on the potential parameter space, equation (5.5). These bounds have no confidence level attached as the precise numerical value would depend on the prior/parametrization choice, a consequence of the PCA component $\varepsilon_3$ being an unconstrained, degenerate direction in parameter space (see the text for details).

<table>
<thead>
<tr>
<th>Model $-2\Delta \ln L$</th>
<th>LOG $G$</th>
<th>LOG $F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D 68%</td>
<td>Best fit</td>
<td>1D 68%</td>
</tr>
<tr>
<td><strong>Cosmological parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Omega_b h^2 \times 10^2$</td>
<td>$2.30 \pm 0.04$</td>
<td>$2.30$</td>
</tr>
<tr>
<td>$\Omega_c h^2$</td>
<td>$0.107 \pm 0.004$</td>
<td>$0.1058$</td>
</tr>
<tr>
<td>$\Theta_*$</td>
<td>$1.044 \pm 0.003$</td>
<td>$1.044$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$0.103 \pm 0.026$</td>
<td>$0.107$</td>
</tr>
<tr>
<td>$H_0$ (km s$^{-1}$ Mpc$^{-1}$)</td>
<td>$74.5 \pm 1.5$</td>
<td>$74.9$</td>
</tr>
<tr>
<td><strong>Power spectra parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ln(P_s^0 \times 10^{10})$</td>
<td>$3.14 \pm 0.05$</td>
<td>$3.14$</td>
</tr>
<tr>
<td>$N_e^0$</td>
<td>$48.3 \pm 5.1$</td>
<td>$48.5$</td>
</tr>
<tr>
<td>$q$</td>
<td>$&lt;0.04$ for any parametrization</td>
<td></td>
</tr>
<tr>
<td><strong>Derived power spectra parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n_0$</td>
<td>$0.979 \pm 0.002$</td>
<td>$0.979$</td>
</tr>
<tr>
<td>$dn/d \ln k</td>
<td>_{k_0} \times 10^3$</td>
<td>$-0.45 \pm 0.09$</td>
</tr>
<tr>
<td>$r_0$</td>
<td>$&lt;4 \times 10^{-3}$ for any parametrization</td>
<td></td>
</tr>
<tr>
<td><strong>Potential parameters and PCA components</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho/M_p^4$</td>
<td>$&lt;4 \times 10^{-10}$ for any parametrization</td>
<td></td>
</tr>
<tr>
<td>$\beta/M_p^4$</td>
<td>$&lt;1 \times 10^{-12}$ for any parametrization</td>
<td></td>
</tr>
<tr>
<td>$\phi_0/M_p$</td>
<td>$&lt;1$ (from prior, equation (5.5))</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_1$</td>
<td>$0.00 \pm 0.02$</td>
<td>$0.01$</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>$0.52 \pm 0.18$</td>
<td>$0.67$</td>
</tr>
<tr>
<td>$\varepsilon_3$</td>
<td>Essentially unconstrained</td>
<td></td>
</tr>
</tbody>
</table>

The fact that we can extract two measured quantities in this scenario (the tilt and normalization) from three model parameters (either (5.1) or (5.2)) means that we expect a strongly degenerate direction in the primordial power spectrum parameter space. In fact, the constraints coming from the data define a region shaped as a long solid cylinder in the 3D subspace spanned by (5.2). Since this cylinder is not aligned with the potential parameters direction, if one tries to convert limits on (5.1) into limits on the potential parameters (5.2) one unavoidably picks up the degenerate direction, i.e. along the axis of the cylinder. This means that while in the set (5.1) the constraints on $N_e^0, \ln P_s^0$ are robust with respect to a change in the parametrization of the problem (since all the
parametrization dependence is dumped into \( q \), it is impossible to translate these into parametrization-independent results for the potential parameters (5.2).

However, one can still define well-constrained (and parametrization-independent) directions in the subspace spanned by (5.2) by performing a principal component analysis (PCA), i.e. by rotating into a new coordinate system aligned with the degenerate direction. We therefore consider the covariance matrix \( C \) in the subspace spanned by the reduced variables \( \zeta = (\ln \rho_0, \ln \dot{q}, \ln \dot{\phi}_0) \), where hats indicate that the variables have been shifted by their posterior mean and normalized to their posterior standard deviation. Then the PCA vector \( \varepsilon \) is given by

\[
\varepsilon = U\zeta,
\]

(6.2)

where \( U \) is the 3D rotation matrix that diagonalizes \( C \):

\[
\zeta^t C \zeta = \varepsilon^t \Lambda \varepsilon,
\]

(6.3)

and \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \) is the matrix of eigenvalues, whose square roots give the error along the directions defined by \( \varepsilon \). The matrix \( U \) is numerically given by

\[
U = \begin{pmatrix}
0.46 & -0.68 & 0.57 \\
0.80 & 0.05 & -0.60 \\
0.38 & 0.73 & 0.56 \\
\end{pmatrix},
\]

(6.4)

and \( \sqrt{\lambda_1} = 0.02, \sqrt{\lambda_2} = 0.48 \) while \( \sqrt{\lambda_3} \gg 1 \), showing that \( \varepsilon_3 \) is indeed the degenerate direction. We have checked that the constraints on \( (\varepsilon_1, \varepsilon_2) \) are largely independent of the chosen parametrization.

Turning now to the \( \text{LOG} G \) case, which imposes a theoretically motivated prior on the number of e-folds, we notice that the prior enforces \( N_e^0 \sim 50 \). This means that the model essentially loses one further parameter, and therefore the best-fit log-likelihood is slightly worse (see table 2). In fact, the \( \text{LOG} G \) fit has basically just one free parameter for the power spectrum (namely the normalization \( \ln P_s^0 \)), since \( \dot{q} \) is almost irrelevant. Still, it gives an excellent best fit to observational data! Also, enforcing 50 e-folds results in a very strong prediction for the tilt to be \( n_0 \sim 0.98 \) (compare figure 3 and the tightness of the posterior probability for \( n_0 \), red curve), while both the tensor contribution and the running are predicted to be very small.

In figure 4 we plot the CMB temperature power spectrum for the best-fit models discussed here along with the compilation of the data used.

6.3. Logarithmic regime and non-renormalizable operator (LOG + NRO)

In this scenario the five parameters that we use to describe the primordial spectrum are

\[
P_{\text{LOG}+\text{NRO}} \equiv \{ \ln P_s^0, N_e^0, q, \ A, N \}.
\]

(6.5)
As discussed in section 5.2, the meanings of the first three are similar to those for the LOG case, and we impose flat priors on ln $P^0_s$ and $q$. In analogy, we consider two types of fits: LOG + NRO$^F$ (with a flat prior on $N_e^0$) and LOG + NRO$^G$ (with a Gaussian prior centred at $N_e^0 = 50$ with the standard deviation of 5). Since $A$ is expected to be of order unity or less according to the discussion above, it is appropriate to use a flat prior on $A$ between 0 and 1.

Let us recall that $N$ determines the order of the NRO, equation (3.6). As discussed in section 5.2, we expect it to be within 1 to $O(10)$. One could imagine treating $N$ as a free parameter and trying to derive a posterior bound on it from the data. However it is technically difficult to ensure that the MCMC study is correctly performed across disjoint regions of the parameter space (since $N$ is an integer, using it as a free parameter effectively gives $N$ separated patches across which it is very difficult to sample). Furthermore, NROs with different values of $N$ are best considered as different models, since the underlying physics is likely to be different. Therefore distinguishing between values of $N$ can be regarded as a model selection task, rather than a parameter constraint exercise. For this reason it is more instructive to consider two separate cases which are representative of the general behaviour at low ($N = 2$) and large ($N = 10$) values of $N$. Parameter constraints from CMB and LSS data, which are discussed next, are summarized in table 3 for the $N = 2$ case and in table 4 for the $N = 10$ case.

Starting from the $N = 2$ case, we find a strong upper bound on $q$, which reflects the theoretical considerations exposed above and is a consequence of the physically motivated prior (5.14). As a consequence, the tensor contribution is always negligible. The number of e-folds for the LOG + NRO$^F$ case ($N = 2$) is $N_e^0 = 14.5 \pm 3.5$ at 68%, becoming $10.1 \leq N_e^0 \leq 25.8$ at 95%, which is too small to solve the horizon problem. Meanwhile, the parameter $A$ is rather tightly constrained, $A = 0.60^{+0.08}_{-0.09}$. These results can be intuitively understood in the following way. As discussed in section 4.2, the presence of the NRO increases $dn/d\ln k$. This effect is maximal at low $k$. The lower $N$, the more gradual the decrease of $dn/d\ln k$ with $k$. In the $N = 2$ case the value of the running of the spectral index is fairly constant in the region of $k$ accessible to observations. Thus the model (for not too small $A$, which would lead back to the LOG scenario) approximately resembles the $dn/d\ln k = \text{constant}$ standard parametrization. We empirically know from the WMAP analyses [1,2] (and our own analysis in this paper) that for this standard parametrization the value of $n$ at $k = k_0$ cannot be very far from $n = 1$. This implies from equations (4.19), (4.18) and the smallness of $q$ that $A$ cannot be far from $A \sim (2N + 3)^{-1/(N+2)}$, which explains the value $A \sim 0.6$. The running $dn/d\ln k$ is then determined by $N_e^0$ (see equation (4.19)). Not surprisingly, the preferred value for the running turns out to be consistent with the one from the standard parametrization (compare tables 1 and 3), which corresponds to the value of $N_e^0$ quoted above. This is also consistent with our discussion of the (too small) number of e-folds in the standard parametrization. Some of these features are illustrated in figure 5 (left panel), which shows the interplay of $N_e^0$ and $A$ and their impact on the spectral index; and figure 6, which shows the best fit $n(k)$ (left panel) and the curve corresponding to the posterior mean, alongside the favoured 95% posterior region of $n(k)$ for $N = 2$. The corresponding $P_s(k)$ is shown in the right panel.

Note that the numbers of free parameters are essentially the same for the LOG + NRO case and the constant running parametrization: 4 for the latter (3 if $r$ is set to zero) and 5...
for LOG + NRO (among which \( q \) is almost irrelevant and \( N \) has been fixed), and their best-fit log-likelihoods are similar. We comment further on this in section 7.

Further enforcing a sufficient number of e-folds by imposing a Gaussian prior on \( N_e^0 \) (the LOG + NRO\( G \) case in table 3) results in a worsening of the quality of fit (an increase of minus twice the best-fit log-likelihood by 2.1 with respect to the standard power-law case). This is because a larger \( N_e^0 \) for a given \( A \) implies a spectrum closer to scale invariance, which is rather strongly disfavoured by data (for example, [14] reports evidence of 17:1 against a scale-invariant spectrum).

As observed in the LOG case, in the LOG + NRO case too the strong degeneracy among the potential parameters makes it impossible to robustly translate the constraints on \( P_{\text{LOG+NRO}} \) into prior-independent constraints for the potential parameters, (3.5). As we

### Table 3. As in table 1, but for the class of models referred to in the text as LOG + NRO, for \( N = 2 \). Upper or lower bounds at the specified confidence level are understood to be one-tail limits.

<table>
<thead>
<tr>
<th>Model</th>
<th>LOG + ( \text{NRO}_G )</th>
<th>LOG + ( \text{NRO}_F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -2\Delta \ln \mathcal{L} )</td>
<td>2.4</td>
<td>2.7</td>
</tr>
<tr>
<td>1D 68%</td>
<td>Best fit</td>
<td>1D 68% Best fit</td>
</tr>
<tr>
<td>Cosmological parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Omega_b h^2 \times 10^2 )</td>
<td>2.32 ± 0.05</td>
<td>2.18 ± 0.07</td>
</tr>
<tr>
<td>( \Omega_c h^2 )</td>
<td>0.107 ± 0.004</td>
<td>0.108 ± 0.004</td>
</tr>
<tr>
<td>( \Theta_* )</td>
<td>1.044 ± 0.003</td>
<td>1.044 ± 0.003</td>
</tr>
<tr>
<td>( \tau )</td>
<td>0.112 ± 0.026</td>
<td>0.010 ± 0.030</td>
</tr>
<tr>
<td>( H_0 , (\text{km s}^{-1} \text{Mpc}^{-1}) )</td>
<td>74.7 ± 1.5</td>
<td>75.0</td>
</tr>
<tr>
<td>Power spectra parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \ln(P_s^0 \times 10^{10}) )</td>
<td>3.14 ± 0.05</td>
<td>3.15 ± 0.05</td>
</tr>
<tr>
<td>( N_e^0 )</td>
<td>47.0 ± 5.1</td>
<td>14.5 ± 3.5</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.46 ± 0.11</td>
<td>0.27</td>
</tr>
<tr>
<td>( q )</td>
<td>&lt;5 \times 10^{-6}</td>
<td>for any parametrization</td>
</tr>
<tr>
<td>Derived power spectra parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n_0 )</td>
<td>0.987 ± 0.007</td>
<td>0.981</td>
</tr>
<tr>
<td>( \frac{dn}{d \ln k}</td>
<td>k_m^0 \times 10^2 )</td>
<td>-0.11 ± 0.11</td>
</tr>
<tr>
<td>( \frac{d^2n}{d \ln^2 k}</td>
<td>k_m^0 \times 10^3 )</td>
<td>0.02 ± 0.02</td>
</tr>
<tr>
<td>( r_0 )</td>
<td>&lt;3 \times 10^{-8}</td>
<td>for any parametrization</td>
</tr>
<tr>
<td>Potential parameters and PCA components</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho/M_p^4 )</td>
<td>&lt;1 \times 10^{-10}</td>
<td>for any parametrization</td>
</tr>
<tr>
<td>( \beta/M_p^4 )</td>
<td>&lt;7 \times 10^{-13}</td>
<td>for any parametrization</td>
</tr>
<tr>
<td>( \phi_0/M_p )</td>
<td>&lt;1 \times 10^{-3}</td>
<td>for any parametrization</td>
</tr>
<tr>
<td>( M/M_p )</td>
<td>&lt;1 (from prior, equation (5.14))</td>
<td></td>
</tr>
<tr>
<td>( \varepsilon_1 )</td>
<td>0.04 ± 0.02</td>
<td>0.06</td>
</tr>
<tr>
<td>( \varepsilon_2 )</td>
<td>0.28 ± 0.13</td>
<td>0.05</td>
</tr>
<tr>
<td>( \varepsilon_3 )</td>
<td>-2.3 ± 0.4</td>
<td>-3.1</td>
</tr>
<tr>
<td>( \varepsilon_4 )</td>
<td>Essentially unconstrained</td>
<td></td>
</tr>
</tbody>
</table>
have done above, we can still define well-constrained directions in the subspace spanned by the parameters $\zeta = (\ln \hat{\rho}_0, \ln \hat{q}, \ln \hat{\phi}_0, \ln \hat{M})$ (hats indicate that the variables have been shifted by their posterior mean and normalized to their posterior standard deviation). The eigenvalues of the three well-constrained directions are now $\sqrt{\lambda_1} = 0.02, \sqrt{\lambda_2} = 0.08, \sqrt{\lambda_3} = 0.72$ while $\sqrt{\lambda_4} \gg 1$, and the corresponding rotation matrix is

$$U = \begin{pmatrix} 0.49 & -0.56 & 0.43 & 0.51 \\ -0.84 & -0.05 & 0.23 & 0.53 \\ 0.31 & 0.79 & 0.04 & 0.54 \\ 0.04 & -0.26 & -0.87 & 0.42 \end{pmatrix}. \tag{6.6}$$

Figure 7 shows marginalized one-dimensional posterior distributions for some of the well-constrained parameters in the LOG + NRO ($N = 2$) scenario. Since the values of the tilt, running and running of the running at the pivotal scale are not really representative of the functional form of $n(k)$ in this case, we do not show pdfs for those quantities (even though their constraints are reported for completeness in table 3). A more faithful representation is actually given in figure 6.

Table 4. As in table 3 but for $N = 10$ in the LOG + NRO scenario. We did not perform in this case a principal component analysis as for $N = 2$.

<table>
<thead>
<tr>
<th>Model</th>
<th>LOG + NRO</th>
<th>LOG + NRO</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2\Delta \ln \mathcal{L}$</td>
<td>LOG + NRO</td>
<td>LOG + NRO</td>
</tr>
<tr>
<td>$\Omega_b h^2 \times 10^2$</td>
<td>$2.31 \pm 0.05$</td>
<td>$2.30$</td>
</tr>
<tr>
<td>$\Omega_c h^2$</td>
<td>$0.107 \pm 0.004$</td>
<td>$0.107 \pm 0.004$</td>
</tr>
<tr>
<td>$\Theta_*$</td>
<td>$1.044^{+0.003}_{-0.002}$</td>
<td>$1.044$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$0.105 \pm 0.026$</td>
<td>$0.104$</td>
</tr>
<tr>
<td>$H_0$ ($\text{km s}^{-1} \text{Mpc}^{-1}$)</td>
<td>$74.6 \pm 1.6$</td>
<td>$74.5$</td>
</tr>
</tbody>
</table>

Cosmological parameters

| $\ln(P_s^0 \times 10^{10})$ | $3.14 \pm 0.05$ | $3.14$ |
| $N_0^0$ | $48.2 \pm 5.1$ | $48.7$ |
| $A$ | $0.52^{+0.19}_{-0.16}$ | $0.54$ |
| $q$ | $<2 \times 10^{-3}$ for any parametrization |

Power spectra parameters

<table>
<thead>
<tr>
<th>Derived power spectra parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_0$</td>
</tr>
<tr>
<td>$\frac{dn}{d\ln k}</td>
</tr>
<tr>
<td>$\frac{d^2n}{d\ln k^2}</td>
</tr>
<tr>
<td>$q_0$</td>
</tr>
</tbody>
</table>

Potential parameters

| $\rho/M_p^4$ | $<1 \times 10^{-11}$ for any parametrization |
| $\beta/M_p^4$ | $<1 \times 10^{-14}$ for any parametrization |
| $\phi_0/M_p$ | $<0.2$ for any parametrization |
| $M/M_p$ | $<1$ (from prior, equation (5.14)) |
Figure 5. Dependence of $n(k)$ on the parameters $N^0_e$ (blue, from thin to thick $N^0_e = 10, 20, 30, 50$, for fixed $A = 0.6$ in the $N = 2$ case, left, and for fixed $A = 0.77$ in the $N = 10$ case, right) and on $A$ (red, from thin to thick $A = 0.2, 0.4, 0.6, 0.8$, for fixed $N^0_e = 15$ in the two panels) in the LOG + NRO scenario with $N$ as indicated.

Let us now turn to the $N = 10$ case. The main difference from the previous $N = 2$ case is that the effect of NRO is much more pronounced on large scales. In particular the running $dn/d\ln k$ can now be quite large at very low $k$ and decrease very quickly, converging to the LOG scenario. Thus, the scenario is qualitatively different from the constant running standard parametrization. As for $N = 2$, the value of $q$ is small and below the theoretically expected bound given by equation (5.23), which translates again into a negligible tensor contribution ($r_0 < 3 \times 10^{-4}$). The preferred value of $A$ is still approximately determined by the empirical condition $n_0 \sim 1$. From equations (4.19) and (4.18), this translates into $A \sim (2N + 3)^{-1/(N+2)} \sim 0.77$. The fact that the model rapidly converges to the LOG scenario allows us to increase the number of e-folds to values similar to those of the LOG case. More precisely, in the LOG + NRO case, the probability distribution for the number of e-folds has a heavier tail for large values of $N^0_e$, and the two-tails posterior 68% (95%) region is given by $18.8 < N^0_e < 37.1$ (15.8 < $N^0_e < 69.5$), which appears to be solving the horizon problem within $2\sigma$.

Some of these features are illustrated figure 8. Note in particular that the running of running for the best fit (yellow line in figure 8) is positive and sizable. The quality of the fit is similar to that for the constant running case for the standard parametrization (though the actual shape of the spectrum is quite different!), although slightly worse. This is not surprising. From section 6.1 we know that data favour a negative second derivative, which cannot be achieved in the LOG + NRO scenario at low $k$ (see the discussion after equation (4.19)). However, as explained there, if the spectral index really runs, a positive second derivative can be much more satisfactory from the physical point of view, in particular for producing a reasonable number of e-folds. This is precisely the case here.
Figure 6. Preferred shape of the spectral index $n(k)$ (left) and the corresponding power spectrum (right) from CMB and LSS data (at 95%, red curves) in the LOG + NRO scenario for $N = 2$. The yellow line shows the best-fit value, the green line the posterior mean while the cyan line is the best fit further imposing a Gaussian prior on the number of e-folds (LOG + NROG scenario). The dotted blue lines represent for reference the best-fit power spectra in the standard parametrization with tilt only, with running and with running of the running (from top to bottom on the right-hand side of the $n(k)$ panel, from top to bottom on the left-hand side of the $P_s(k)$ panel; compare table 1).

If one enforces a Gaussian prior around $N^0_\epsilon = 50$ (LOG + NROG case), then the best-fit spectrum becomes again featureless (light blue, solid line in figure 8) but with a smaller tilt than for the standard parametrization ($n_0 = 0.980$ for the $N = 10$ LOG + NROG case), which in turn means that the goodness of fit becomes worse than for the standard case (see table 4). Actually the model becomes in this case quite similar to the simpler LOGG scenario.

Finally, for values of $N$ between 2 and 10 we have found that the behaviour is intermediate between the cases discussed in the text.

The above discussion shows that there are cases where the standard Taylor expansion of $n(k)$ fails to capture the physics of the models. Generally, the LOG + NRO scenario predicts a running of $n$ which is stronger on large scales. This can only be recovered with several terms in the Taylor expansion, which results in a higher number of free parameters in the fit. On the other hand, the functional form of $n(k)$ in the LOG + NRO scenario implies the positiveness of the second derivative in most of the parameter space, unlike the standard fit. These facts make it impossible to use the results of the standard fits to constrain the LOG + NRO scenario: a direct comparison of the model with the data (as we have given here) becomes necessary. This situation could easily apply to other theoretical models, as well, and therefore great caution is necessary when interpreting generic constraints on the coefficient of the standard Taylor expansion in terms of specific physical models.
7. Model comparison

In the previous section we have presented parameter constraints for each class of model, namely the phenomenological model in the standard parametrization and the more physically motivated LOG and LOG + NRO scenarios. Assessing the relative performance of the three models is a model comparison question, to which we now turn our attention.

In the traditional frequentist approach to statistics, model comparison is tackled in terms of hypothesis testing: for example, we might ask whether the improvement in the best-fit likelihood in terms of the effective $\Delta \chi^2 = -2\Delta \ln L$ when adding a running to the tilt is ‘significant’ enough to warrant the inclusion of a non-zero running. There are however several reasons why answering this question is far from trivial. A technical reason is that the usual rule of thumb of ‘$\Delta \chi^2$ per extra degree of freedom’ can only be applied if certain regularity conditions are met, and in particular only if the extra parameter for
Figure 8. As in figure 6 but for the LOG + NRO scenario with $N = 10$.

the nested model does not lie on the boundary of the parameter space (see [37] for an astrophysical example and references therein). So for example, the $\Delta \chi^2$ criterion could not be applied to compare the quality of fit of the LOG model with that of LOG + NRO, since the former is obtained from the latter by setting $A = 0$, and $A < 0$ is not allowed.

Another, more fundamental aspect has to do with the meaning and interpretation of frequentist hypothesis testing. As discussed in detail in [13], frequentist likelihood ratio tests assume that the hypothesis $H$ is true and give the probability of observing data $d$ as extreme or more extreme than what has actually been measured. This is a statement on the probability of the data assuming a hypothesis $H$ to be true (which in Bayesian terms amounts to the choice of a model, $\mathcal{M}$), i.e. frequentist hypothesis testing gives $P(d|H)$. But this is not the quantity that one is usually interested in, which is actually $P(H|d)$, the probability of the model $\mathcal{M}$ given the observations, which can only be obtained by using the Bayes theorem to invert the order of conditioning. For this reason, model selection is an inherently Bayesian question [14].

Bayesian model selection is based on the computation of the model likelihood $P(d|\mathcal{M}) \equiv \mathcal{E}(\mathcal{M})$ (also called ‘evidence’), which is the normalization constant in the denominator of the Bayes theorem (see [14, 38] for details) obtained by averaging the likelihood $P(d|\theta, \mathcal{M})$ over the prior $P(\theta|\mathcal{M})$ in the parameter space $\theta$ of the model

$$\mathcal{E}(\mathcal{M}) = \int P(d|\theta, \mathcal{M})P(\theta|\mathcal{M})\,d\theta. \tag{7.1}$$

From the model likelihood one obtains the model probability given the data by using once more the Bayes theorem, $P(\mathcal{M}|d) \propto P(\mathcal{M})\mathcal{E}(\mathcal{M})$, where $P(\mathcal{M})$ is the prior probability assigned to the model (usually taken to be non-committal and equal to $1/N_m$ if one considers $N_m$ different models). When comparing two models one usually computes the Bayes factor $B_{12}$, given by the ratio of the evidences for the two models:

$$\ln B_{12} = \ln \mathcal{E}(\mathcal{M}_1) - \ln \mathcal{E}(\mathcal{M}_2). \tag{7.2}$$

The Bayes factor thus gives the factor by which the relative odds among two models have changed after the arrival of the data. As a simple calculation shows for the case of Gaussian
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likelihood and prior, equation (7.1) contains both a likelihood ratio term which rewards better fitting, and an ‘Occam’s razor’ term that disfavours unnecessary model complexity, defined in terms of useless parameters (see [39] for a discussion of model complexity). The ‘best’ model is one that combines good fitting with model predictivity. Bayes factors are usually interpreted against the Jeffreys’ scale for the strength of evidence, which we qualify as follows: ‘weak evidence’ for $|\ln B| < 2.5$, ‘moderate evidence’ for $2.5 < |\ln B| < 5.0$ and ‘strong evidence’ for $|\ln B| > 5.0$. The computation of the model likelihood is in general numerically a difficult task, as it involves a multi-dimensional integration over the whole of parameter space. Furthermore, a prior dependence is (correctly) built into the method, as the Occam’s razor term depends on the ratio of the prior to posterior volume, which gives the amount of ‘wasted’ parameter space of the model. Therefore it is problematic to evaluate the Bayes factor unless one has a physically motivated way of setting the prior volume.

The difficulty of using a fully Bayesian approach to the comparison of the standard Taylor series parametrization with either the LOG or the LOG + NRO scenario is that the former represents a purely phenomenological fit to the data, while the LOG and LOG + NRO models are physically motivated. In particular, setting a prior on the potential parameters of the LOG and LOG + NRO models is not comparable to setting a strictly phenomenological prior on the quantities of direct relevance for the fit, i.e. the spectral tilt, the running, etc, in the standard parametrization. The Occam’s razor effect which rewards highly predictive models does not work properly if we do not compare like with like, i.e. if we are unable to set priors on the parameter space of the phenomenological parametrizations used for the fit. Since the standard parametrization is by construction phenomenological, it cannot be directly compared using the Bayesian evidence to the LOG and LOG + NRO scenarios.

However, we can still draw some interesting, partial conclusions from a Bayesian approach. In [13], a method was presented for deriving upper bounds on the Bayesian evidence for nested models, called ‘Bayesian calibrated p-values’, that is useful in cases such as this where there is only a very loose physical basis for assigning priors to phenomenological quantities in the fit (here, the various terms in the expansion of the potential). This allows us to assess whether extra parameters are unnecessary within the framework of nested models, as it gives the Bayes factor which (under mild assumptions) maximizes the evidence in favour of the more complex model (i.e., with more terms in the Taylor expansion). If this turns out to be not very strong, then one can confidently conclude that the extra parameters are not needed.

Table 5 summarizes some relevant model comparison statistics. Focusing first on the ‘standard parametrization’ section, we have employed the method of [13] to derive a prior-independent upper bound on the Bayesian evidence in favour of extra terms in the Taylor expansion. The maximum Bayesian evidence in favour of a running is only $\ln \hat{B}_{21} = 0.7$ (compared to a model with just a spectral tilt), which falls short of even the ‘weak evidence’ threshold. The maximum evidence in favour of a third term in the Taylor expansion is even weaker. We can therefore conclude that, for the standard parametrization, present data do not require any terms of higher order than a spectral tilt (for which reference [13] found a maximum evidence of $\ln \hat{B} = 2.9$ compared to a scale-invariant spectrum). Notice that in this phenomenological approach the number of e-folds has to be added in by hand as an extra parameter of the model (although it would be derivable given a
Table 5. Summary of model comparison statistics. Wherever the Bayes factor is given, the notation $\ln B_{ij}$ indicated the Bayes factor between model $i$ and model $j$ (with $\ln B_{ij} > 0$ favouring model $i$). An overbar indicates a prior-independent upper limit obtained using the Bayesian calibrated $p$-values method. The quantity $n$ gives the number of effective parameters in the model.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Delta \chi^2$</th>
<th>$n$</th>
<th>&gt;50 e-folds?</th>
<th>Bayes factor</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Standard parametrization</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No running</td>
<td>0.0</td>
<td>3(+1)</td>
<td>Ad hoc</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>With running</td>
<td>-3.4</td>
<td>4(+1)</td>
<td>Ad hoc</td>
<td>$\ln B_{21} = 0.7$</td>
<td>No evidence</td>
</tr>
<tr>
<td>Running of running</td>
<td>-4.4</td>
<td>5(+1)</td>
<td>Ad hoc</td>
<td>$\ln B_{32} = 0.0$</td>
<td>No evidence</td>
</tr>
<tr>
<td><strong>LOG models</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LOG$^F$</td>
<td>-0.4</td>
<td>~2</td>
<td>Yes (at 2$\sigma$)</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>LOG$^G$</td>
<td>+2.1</td>
<td>~1</td>
<td>Yes</td>
<td>$\ln B_{45} = 0.4$</td>
<td></td>
</tr>
<tr>
<td><strong>LOG+NRO models</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LOG + NRO$^F$, $N = 2$</td>
<td>-2.7</td>
<td>~3</td>
<td>No</td>
<td>—</td>
<td>Excluded</td>
</tr>
<tr>
<td>LOG + NRO$^G$, $N = 2$</td>
<td>+2.4</td>
<td>~2</td>
<td>Yes</td>
<td>—</td>
<td>Disfavoured</td>
</tr>
<tr>
<td>LOG + NRO$^F$, $N = 10$</td>
<td>-1.8</td>
<td>~3</td>
<td>Yes (at 2$\sigma$)</td>
<td>$\ln B_{84} \approx 0.0$</td>
<td>Au par LOG$^F$</td>
</tr>
<tr>
<td>LOG + NRO$^G$, $N = 10$</td>
<td>+2.1</td>
<td>~2</td>
<td>Yes</td>
<td>$\ln B_{95} &lt; -2.3$</td>
<td>Disfavoured</td>
</tr>
</tbody>
</table>

specific enough model for the inflaton potential), indicated as (+1) in the column giving an approximate value of the effective number of parameters in the model.

Regarding the LOG class of models, the goodness of fit of the LOG$^F$ case is similar to the one for the simple tilt model. Although the number of free parameters of the LOG scenario is 3, the parameter $q$ is irrelevant to the fit and therefore the effective number of parameters is closer to 2. A more precise counting of the effective parameters could be achieved using the notion of Bayesian complexity [39], but this is not required in the context of the present discussion. It is interesting to notice that the LOG scenario also solves the horizon problem (within 2$\sigma$ of the posterior mean) with an extreme economy of free parameters. The LOG$^G$ case dispenses with one further parameter (as $N_e^0$ becomes almost fixed to $N_e^0 \approx 50$) and the upper bound on the Bayesian evidence in favour of LOG$^F$ indicates that the difference of $\Delta \chi^2 = 2.5$ between the flat and Gaussian prior on $N_e^0$ is not strongly significant.

The LOG model can be considered nested within the LOG + NRO class of models, with the former formally obtained from the latter by setting $A = 0$. For $N = 2$, the LOG + NRO$^F$ model falls short of achieving the necessary number of e-folds, and for this reason it must be excluded, even though its quality of fit is comparable to the standard case with constant running. The LOG + NRO$^G$ case has one extra parameter (for fixed $N$) than LOG$^G$, and a best-fit value which is actually slightly worse, a consequence of the Gaussian prior forcing the posterior distribution around a value of $N_e^0$ which is not strongly favoured by the data. Hence we can conclude that LOG + NRO$^G$ ($N = 2$) is disfavoured with respect to LOG$^G$ and LOG$^F$ since it is unable to achieve a better fit even with one extra parameter.
LOG + NRO$^F$ with $N = 10$ has a better fit than the LOG scenario and it achieves a sufficient number of e-folds within $2\sigma$. The method of Bayesian calibrated $p$-values cannot be used to compare the two models because the LOG model (obtained by setting $A = 0$ in the LOG + NRO model) lies at the boundary of parameter space. However, we can still roughly estimate the Bayes factor between LOG$^F$ and LOG + NRO$^F$ by taking a prior width on the extra parameter $A$ of order unity (as motivated by the theoretical expectations presented in section 5.2) and using that, for nested models, the Bayes factor in favour of the simpler model is approximately (see equation (9) in [14])

$$\ln B \sim I - \lambda^2/2,$$

(7.3)

where $I$ is the logarithm of the ratio of the prior to posterior volume (the information gain) for the extra parameter and $\lambda$ is the number of sigmas of discrepancy between the likelihood peak and the value of the extra parameter under the nested model (here, $A = 0$). Using the values in table 4 one obtains $\lambda \sim 2.8$ and $I \sim 1.6$ and thus $\ln B \sim -2.3$, which would weakly favour the LOG + NRO$^F$ model. However, one has to bear in mind that the parameter $N$ has been fixed to a value picked among a range of order 10 possible values—hence one has to factor in an extra Occam’s razor effect coming from the fact that $N = 10$ is one of about 10 possible choices for $N$. Hence $\ln B$ has to be increased by about a factor $\ln 10 = 2.3$, which brings the final odds between LOG + NRO$^F$ and LOG$^F$ to unity (i.e., $\ln B \sim 0$). Finally, the LOG + NRO$^G$ case has the same quality of fit as the LOG$^G$ case and one extra parameter. The Occam’s razor term from the choice of $N$ alone would disfavour LOG + NRO$^G$ by a factor $\ln B = 2.3$ with respect to LOG$^G$, so even without computing the precise Bayes factor we can conclude that this scenario is disfavoured.

In conclusion, a model comparison approach singles out the LOG scenario and the LOG + NRO$^F$ ($N = 10$) model as the most viable cases in the light of the present data. This kind of consideration could be extended to compare this class of models with other inflationary scenarios, once they have been suitably parametrized in terms of fundamental variables. However, a direct comparison with a phenomenological approach such as the standard Taylor expansion of the spectrum is not feasible due to the lack of predictivity of the latter. The Bayesian evidence still leads to the conclusion that no term of higher order than the tilt is currently required in the series.

Finally, we emphasize that the LOG and LOG + NRO models predict tensor contributions that are generally very small and will be largely undetectable. The most optimistic case is that of the LOG, where the upper bound is of order $r_0 \sim 10^{-3}$, which might be just within reach of future $B$-mode observations. Conversely, a detection of tensor modes above $\sim 10^{-3}$ would disprove the scenario of flat tree-level inflationary potentials.

8. Conclusions

In this paper we have compared a broad and physically well-motivated class of inflationary models with CMB and LSS observational data. Namely, we have considered models with flat tree-level potentials, which typically appear in supersymmetric theories, where $V_{\text{SUSY}}^{\text{tree}}$ ordinarily has plenty of accidental flat directions. These models, beside being very well motivated from the physical point of view (on a similar footing to monomial potentials),
lead to very model-independent cosmological predictions. The reason is that the potential derivatives $V', V'', \ldots$ arise from the radiative corrections to $V$, which has a characteristic logarithmic dependence on the inflaton field. This scenario has been labelled ‘LOG’ throughout the paper. In addition, we have considered the possible presence of new physics beyond a certain high energy cut-off. This physics does not need to respect the flat directions of the ‘low energy’ theory, and thus it will show up as non-renormalizable operators (NRO) in the inflaton field, which will be dominated by the lowest order one. This modified scenario (labelled ‘LOG + NRO’) is also very well motivated and still quite model independent.

We have studied the performance of these scenarios when compared with CMB and LSS. We have made first a detailed study of the features of these models, working out both numerically accurate results and approximate analytical expressions for $P_s(k), P_t(k)$ and other relevant quantities, such as the spectral index, $n(k)$, as a function of suitably defined model parameters. We also discussed the number of independent parameters and the theoretical and phenomenological constraints on them (to be imposed a priori). As a matter of fact, one (combination) of the parameters is almost irrelevant, which makes these models even more predictive. Another parameter is essentially the number of e-folds, $N^0_e$, since the time when the largest observable scales crossed out of the horizon to the end of inflation. This allows us to perform the fits in a twofold way: either leaving $N^0_e$ free, and letting the data determine its value, or imposing a prior on its value according to the usual theoretical prejudice ($N^0_e = 50–60$). The two approaches (labelled $\mathcal{F}$ and $\mathcal{G}$ respectively) are interesting and complementary.

In the analysis we also study the performance of standard parametrizations of the power spectrum, based on Taylor expansion of $\ln P_s(k)$ and $\ln P_t(k)$ around an (arbitrary) pivotal scale, $\ln k_0$. At first (second) order these parametrizations correspond to a constant (constantly running) spectral index, $n(k) = \text{constant}$ ($dn/d\ln k = \text{constant}$). They have been used in reference analyses, in particular by the WMAP collaboration. It is important to keep in mind that, although useful, the standard parametrizations are not inspired by any particularly well-motivated inflationary physics. For example, the results of the fit with the $dn/d\ln k = \text{constant}$ assumption are not consistent with a number of e-folds in the required range. Still we have also studied them (going one order beyond the WMAP analysis) to facilitate the discussion of the performance of the LOG and LOG + NRO scenarios. As a general comment, care must be taken to test inflationary models which predict a non-negligible scale dependence. In many cases the standard Taylor series parametrization of equations (2.9) and (2.10) cannot be accurately used in such a situation unless a high number of terms is taken in the expansion.

Our main results are the following:

- Both the LOG and LOG + NRO scenarios predict small tensor perturbations: $r_0 \leq O(10^{-3})$.
- The LOG scenario has essentially two parameters, $P_s(k_0)$ and $N^0_e$, and implies a nearly constant $n(k)$.
- Leaving $N_e$ as a free parameter (LOG-$\mathcal{F}$ fit), one gets $24 < N^0_e < 49$ (16 < $N^0_e$ < 84) at 68% (95%) c.l. while the corresponding spectral index is close to $n_0 = 0.96$. This result is consistent with the theoretical prejudice $N^0_e \sim 50–60$, required to solve the horizon problem, which is remarkable. In the LOG-$\mathcal{G}$ fit (i.e. imposing $N^0_e = 50–60$)
one gets $n_0 \approx 0.98$. Note that this fit has only one parameter and still works very well.

- The LOG + NRO scenario has two more parameters, which arise from the order and the suppression scale of the NRO. We have fixed the order of the NRO (denoted as $4 + 2N$ throughout the paper) to two representative values ($N = 2, 10$) that reasonably encompass the sensible physical range. Thus in practice we are playing with just one additional parameter. This scenario can produce a sizable running of the spectral index, and still be consistent with the data and a reasonable number of e-folds, especially if $N$ is not very small. The impact of the NRO (driving a running of the spectral index) is relevant at small $k$, corresponding to the first stages of the inflationary period, and then it quickly converges (especially for not too small $N$) to the LOG scenario.

- The model comparison is delicate for several technical and fundamental reasons. In the paper we give a fully Bayesian discussion of the relative quality of the various scenarios considered. Qualitatively, it can be said that the goodness of the LOG ($\text{NRO} + \text{LOG} \mathcal{F}$) fits is similar to that of the standard $n = \text{constant}$ ($\frac{dn}{d \ln k} = \text{constant}$) parametrization. The improvement in the goodness of the fit obtained by the inclusion of an extra parameter (as the LOG + NRO scenario implies) is not enough (with the present data) to justify such modification, but still it remains an interesting theoretical possibility. On the other hand, a rigorous comparison between the evidences for LOG, LOG + NRO scenarios and for the standard parametrizations (which are phenomenological by construction) is not feasible. However, the prior-independent method of using the Bayesian calibrated $p$-values still indicates that no term of higher order than a tilt is required in the standard Taylor expansion.

As a final conclusion, the LOG and LOG + NRO scenarios analysed in this paper (based on flat tree-level potentials without or with the presence of extra physics) are not only very well motivated from the physical point of view, but they also fit the CMB and LSS data remarkably well, with very few parameters (the predictions are quite model independent). In addition they are naturally consistent with a reasonable number of e-folds. Therefore, they can be considered as a standard physical class of inflationary models, on a similar footing to monomial potentials.

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Appendix. Some complementary formulae

In this appendix we provide some important expressions for the case of the potential with NRO. Applying the following two identities of hypergeometric functions:

\[ 2 F_1(a, b; c; z) = 2 F_1(b, a; c; z), \]
\[ 3 F_2(a, b, c; a + 1, b + 1; z) = \frac{1}{b - a} \left[ b_2 F_1(a, c; a + 1; z) - a_2 F_1(b, c; b + 1; z) \right], \]

one can see, integrating equation (2.5), that

\[
\ln \frac{k}{k_0} \simeq -q N_0 e^{\Phi} \left\{ \frac{1}{N + 2} + \left( \frac{2}{q} - \frac{1}{N + 2} + \ln \phi \right) 2 F_1 \left( \frac{1}{N + 2}; \frac{N + 3}{N + 2}; -(A\Phi)^{N+2} \right) \right. \\
\left. + \frac{N + 2}{(N + 1)^2} \left[ (A\Phi)^{-N-2} \left( 1 - (A\Phi)^{N+2} \right)^{-1/(N+2)} + 1 - (A\Phi)^{N+2} \right] \right\} \bigg|_{\phi = 1},
\]  

(A.1)

the slow-roll approximation being the only reason for the symbol of approximate equality. The expression (4.15) that we use in the fits is directly obtained from (A.1) neglecting the irrelevant addends, i.e. all but the one that is not proportional to the small parameter \( q \).

The first-order slow-roll parameters in this scenario are

\[
\epsilon = -\frac{q N_0^2}{2 e^\Phi} \left[ \frac{1}{q} + \left( \frac{1}{2} \right)^M (A\Phi)^M + \left( \frac{1}{2} \right) \ln \Phi \right] \right)^2, \quad (A.2)
\]
\[
\eta = \frac{1}{2 q N_0^2 e^\Phi} \left[ \frac{(2M - 1)(A\Phi)^M - 1}{1/q + (1/2)M (A\Phi)^M + (1/2) \ln \Phi} \right], \quad (A.3)
\]

where we have defined

\[
\Phi \equiv \left( \frac{\phi}{\phi_0} \right)^2, \quad (A.4)
\]

to simplify the notation. Notice that in the limit of the NRO going to zero we recover the formulae (4.1).

The relations between \( P_{LOG+NRO} \) and the physical parameters of the potential are given by

\[
\phi_0/M_p = \sqrt{2q N_0^2}, \quad (A.5)
\]
\[
\rho/M_p^4 = 48\pi^2 q(N + 2)^3 \frac{P_0}{N_0^2} \frac{(1 + A^{N+2})^2}{2(N + 2) + q A^{N+2}}, \quad (A.6)
\]
\[
\beta/M_p^2 = 48\pi^2 q^2(N + 2)^3 P_s^0 \frac{P_0}{N_e^0} \frac{(1 + A^{N+2})^2}{[2(N + 2) + qA^{N+2}]^3}, \quad (A.7)\\
M/M_p = \sqrt{2qN_e^0} \left\{ 6\pi^2(N + 2)^2 P_s^0 \frac{A^{N+2}}{N_e^0^3} \frac{(1 + A^{N+2})^2}{[2(N + 2) + qA^{N+2}]^3} \right\}^{-1/2N}. \quad (A.8)
\]

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