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TM, TE and 'TEM' beam modes: exact solutions and their problems

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Abstract

The simplest of the recently published exact solutions of Maxwell's equations which represent TM, TE and 'TEM' beam modes is shown to be not physically realizable, because the total electromagnetic energy in any transverse section of the beam would have to be infinite. This problem is already apparent in the scalar (particle) beam case. A related exact solution is investigated, and is shown to have the required convergence properties. In the latter case the ratio of the integrals of (i) c times the momentum density, and (ii) the energy density, over any transverse section of the beam, tends to unity for broad beams, but can be very different from unity for tightly focused beams.

Keywords: Laser beams, nonparaxial TM modes, particle beams

1. Introduction

Laser and microwave beams are usually described by solutions of Maxwell's equations based on the paraxial approximation, to be defined below. The Helmholtz equation

$$\left(\nabla^2 + k^2\right)\psi = 0 \tag{1}$$

follows from Maxwell's equations for monochromatic waves of angular frequency $\omega = ck$, on expressing the magnetic and electric fields in terms of the vector and scalar potentials $\boldsymbol{A}$ and Φ

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \tag{2}$$

and choosing the Lorentz gauge

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0. \tag{3}$$

In free space, Φ and all three components of $\boldsymbol{A}$ then satisfy (1), as is well known (see e.g. [1], pp 218 ff).

For beams propagating primarily in the z direction (and converging or spreading in the x and y directions) it is convenient to use cylindrical coordinates (ρ, ϕ, z), $\rho^2 = x^2 + y^2$, in terms of which (1) reads

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0. \tag{4}$$

Approximate solutions of (4) are obtained on the assumption that the dominant rapid variation of ψ is contained in the factor $\exp(\mathrm{i}kz)$. Thus one sets $\psi = \mathrm{e}^{\mathrm{i}kz}G$, and then for the axially symmetric modes G satisfies

$$\frac{\partial^2 G}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial\rho} + 2\mathrm{i}k\frac{\partial G}{\partial z} + \frac{\partial^2 G}{\partial z^2} = 0. \tag{5}$$

The paraxial approximation [2–5] consists in neglecting the term $\partial^2 G/\partial z^2$; the fundamental mode thus obtained is

$$\psi_{\mathrm{G}} = \frac{b}{\sqrt{b^2+z^2}}\exp\left[\frac{-kb\rho^2}{2\left(b^2+z^2\right)}\right] \times \exp\mathrm{i}\left\{kz - \operatorname{atn}\left(\frac{z}{b}\right) + \frac{kz\rho^2}{2\left(b^2+z^2\right)}\right\}. \tag{6}$$

The length b is the Rayleigh or diffraction length [6, 7]. The beam waist size (at $z = 0$) is $w_0 = (2b/k)^{1/2}$, since there the modulus of ψ_{G} is $\exp\left(-\rho^2/w_0^2\right)$.

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The errors involved in the paraxial approximation can be seen by considering $\psi_G^{-1}\nabla^2\psi_G$: instead of $-k^2$ we get $-k^2$ times

$$1+\frac{2}{k^2(b+\mathrm{i}z)^2}-\frac{2\rho^2}{k(b+\mathrm{i}z)^3}+\frac{\rho^4}{4(b+\mathrm{i}z)^4}. \quad (7)$$

The errors made are thus negligible in the regions where both

$$k^2\left(b^2+z^2\right)\gg 1 \qquad \text{and} \qquad b^2+z^2\gg\rho^2 \quad (8)$$

hold true (compare [8, 9]). Thus when kb is of order unity or smaller, the paraxial approximation will fail in the beam waist region ($|z|\lesssim b$).

Exact solutions of the Helmholtz equation which represent beams are known; of these we consider those based on complex source and sink points. The vector potential due to a current source localized near the origin behaves as an outgoing spherical wave $\mathrm{e}^{\mathrm{i}kr}/r$ with an angle-dependent coefficient (see e.g. section 9.1 of [1]). The Helmholtz equation is translationally invariant, and Deschamps [10] noted that

$$\psi=\frac{\mathrm{e}^{\mathrm{i}kR}}{R}, \qquad R^2=x^2+y^2+(z-\mathrm{i}b)^2=\rho^2+(z-\mathrm{i}b)^2 \quad (9)$$

is an exact solution of (1), and has some of the properties of the Gaussian beam (6) (see also [11–13] for developments of this beam wavefunction). A fundamental problem is that R is zero on the circle $\rho=b$ in the focal plane $z=0$, and so the function $\mathrm{e}^{\mathrm{i}kR}/R$ diverges on this circle. Sheppard and Saghafi [14–17] replaced the complex-source outgoing spherical wave by a non-singular superposition of incoming and outgoing spherical waves:

$$\psi=\frac{\sin kR}{kR}=j_0(kR). \quad (10)$$

Ulanowski and Ludlow [18] considered the same function in oblate spheroidal coordinates (ξ,η,ϕ), in terms of which

$$\rho=\left(1+\xi^2\right)^{1/2}\left(1-\eta^2\right)^{1/2}b, \qquad z=\xi\eta b, \\ R=(\xi-\mathrm{i}\eta)b. \quad (11)$$

They also point out that (cf [12]) the zeroth spherical Bessel function of (10) generalizes to the set of solutions

$$\psi_{\ell m}=j_\ell(kR)P_{\ell m}\left(\frac{z-\mathrm{i}b}{R}\right)\mathrm{e}^{\pm\mathrm{i}m\phi} \quad (12)$$

where the $P_{\ell m}$ are the associated Legendre polynomials.

Here we calculate the momentum and energy densities associated with the electromagnetic field corresponding to (for example) the TM mode, in which the vector potential is [19]

$$\boldsymbol{A}=(0,0,A_0\psi). \quad (13)$$

We show that though the momentum density integrated over a beam section is finite, the integral over the energy density diverges logarithmically. The same kind of problem arises in the scalar case, considered in the next section.

2. Particle beams: the probability density and current density

Before considering electromagnetic beams, we show that the beam wavefunction (10) gives problems even in the scalar case, where ψ represents of a beam of (non-interacting) particles. In that case $|\psi|^2$ gives the probability density, and $\boldsymbol{J}=\frac{\hbar}{m}\mathrm{Im}\,(\psi^*\nabla\psi)$ is the probability density current (here m is the particle mass). If we write ψ in terms of its modulus M and phase P,

$$\psi(\rho,z)=M(\rho,z)\exp\left[\mathrm{i}P(\rho,z)\right] \quad (14)$$

we have $\boldsymbol{J}=\frac{\hbar}{m}M^2\nabla P$. For a beam propagating in the z direction with wavenumber k we might expect each particle to have a speed of approximately $\hbar k/m$ in the z direction, on average, through any cross section of the beam. Thus we expect $\int_0^\infty \mathrm{d}\rho\,\rho J_z$ to be independent of z (conservation of flux), and to have a value of approximately $\frac{\hbar k}{m}\int_0^\infty \mathrm{d}\rho\,\rho M^2$, so that

$$\int_0^\infty \mathrm{d}\rho\,\rho M^2\frac{\partial P}{\partial z}\approx k\int_0^\infty \mathrm{d}\rho\,\rho M^2. \quad (15)$$

This relation does hold for the approximate Gaussian wavefunction ψ_G of (6): the left side is $b/2-1/4k$, the right side $b/2$, for any z. (Note $kb\gg 1$ is a necessary condition for the validity of the paraxial approximation: see (8).)

For the exact wavefunction (10) we will use the oblate spheroidal coordinates (ξ,η), which are convenient because of their proportionality to the real and imaginary parts of R, as is made explicit in the last equality of (11). The modulus and phase of $\psi=j_0(kR)$ are (with $\beta=kb$)

$$M=\frac{\left(\sin^2\beta\xi+\sinh^2\beta\eta\right)^{1/2}}{\beta\left(\xi^2+\eta^2\right)^{1/2}} \quad (16)$$

$$P=\mathrm{atn}\,(\eta/\xi)-\mathrm{atn}\left(\frac{\tanh\beta\eta}{\tan\beta\xi}\right) \\ =\mathrm{atn}\left(\frac{\tan\beta\xi}{\tanh\beta\eta}\right)-\mathrm{atn}\left(\frac{\xi}{\eta}\right) \\ =\beta\xi+\mathrm{atn}\left\{\frac{\sin 2\beta\xi}{\mathrm{e}^{2\beta\eta}-\cos 2\beta\xi}\right\}-\mathrm{atn}\left(\frac{\xi}{\eta}\right). \quad (17)$$

To calculate $\partial P/\partial z$ we use

$$b^2\left(\xi^2-\eta^2\right)=\rho^2+z^2-b^2, \qquad b\xi\eta=z \quad (18)$$

to evaluate the partial derivative with respect to z:

$$\frac{\partial}{\partial z}=\frac{\partial\xi}{\partial z}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial z}\frac{\partial}{\partial\eta} \\ =\frac{1}{b\left(\xi^2+\eta^2\right)}\left\{\eta(1+\xi^2)\frac{\partial}{\partial\xi}+\xi(1-\eta^2)\frac{\partial}{\partial\eta}\right\}. \quad (19)$$

For future reference we also give the partial derivative with respect to ρ:

$$\frac{\partial}{\partial\rho}=\frac{\partial\xi}{\partial\rho}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial\rho}\frac{\partial}{\partial\eta} \\ =\frac{\left(1+\xi^2\right)^{1/2}\left(1-\eta^2\right)^{1/2}}{b\left(\xi^2+\eta^2\right)}\left\{\xi\frac{\partial}{\partial\xi}-\eta\frac{\partial}{\partial\eta}\right\}. \quad (20)$$

We shall now evaluate the left and right sides of (15) in the focal plane $z = 0$. For $\rho \leqslant b$ we have $\xi = 0$, and η varies from unity at $\rho = 0$ to zero at $\rho = b$, with

$$M^2 = \left(\frac{\sinh\beta\eta}{\beta\eta}\right)^2, \qquad \left(\frac{\partial P}{\partial z}\right)_{z=0} = \frac{k}{\eta\tanh\beta\eta} - \frac{1}{b\eta^2}. \tag{21}$$

For $\rho \geqslant b$ we have $\eta = 0$, and ξ varies from zero to infinity as ρ increases from b to infinity, with

$$M^2 = \left(\frac{\sin\beta\xi}{\beta\xi}\right)^2, \qquad \left(\frac{\partial P}{\partial z}\right)_{z=0} = \frac{1}{b\xi^2} - \frac{k}{\xi\tan\beta\xi}. \tag{22}$$

We find, on using (21) and (22),

$$\int_0^\infty \mathrm{d}\rho\,\rho M^2\left(\frac{\partial P}{\partial z}\right)_{z=0} = \left(\frac{\sinh\beta}{\beta}\right)^2\frac{b}{2}. \tag{23}$$

If ψ had been normalized to unity at the origin (as ψ_G is) this result would have been $b/2$, in close correspondence with the Gaussian result $b/2 - 1/4k$ given above. It can be verified that the value given in (23) is obtained for any z: the total flow of probability in the beam does not depend on position along the beam axis.

The integral over probability density on the right side of (15) is however logarithmically divergent:

$$\int_b^\infty \mathrm{d}\rho\,\rho M^2 = k^{-2}\int_0^\infty \mathrm{d}\xi\,\xi^{-1}\sin^2\beta\xi = \infty. \tag{24}$$

Thus the particle beam represented by ψ cannot be physically realized: it is not normalizable in any finite section. We shall reach similar conclusions for the electromagnetic beams based on ψ.

3. Energy and momentum of the electromagnetic TM beam

The energy density u and the momentum density $\boldsymbol{p}$ of an electromagnetic field in vacuum are given by (see e.g. [1], section 6.8)

$$u = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right), \qquad \boldsymbol{p} = \frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B} \tag{25}$$

(these are given in Gaussian units; the SI expressions are $u = \frac{1}{2}\varepsilon_0\left(\boldsymbol{E}^2 + c^2\boldsymbol{B}^2\right)$, $\boldsymbol{p} = \varepsilon_0\boldsymbol{E}\times\boldsymbol{B}$). These expressions are bilinear in the real fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$. The complex fields $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r})$ are given in terms of the complex vector potential $\boldsymbol{A}(\boldsymbol{r})$ by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\mathrm{i}}{k}\left[\nabla\left(\nabla\cdot\boldsymbol{A}(\boldsymbol{r})\right) + k^2\boldsymbol{A}(\boldsymbol{r})\right], \qquad \boldsymbol{B}(\boldsymbol{r}) = \nabla\times\boldsymbol{A}(\boldsymbol{r}) \tag{26}$$

and the real fields are, for example,

$$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left\{\boldsymbol{E}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t}\right\} = \tfrac{1}{2}\left[\boldsymbol{E}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t} + \boldsymbol{E}^*(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\omega t}\right] \tag{27}$$

(with $\omega = ck$). Thus $\boldsymbol{E}^2(\boldsymbol{r},t)$ will consist of $\frac{1}{2}\boldsymbol{E}(\boldsymbol{r})\cdot\boldsymbol{E}^*(\boldsymbol{r})$ plus oscillatory terms which have zero average over one cycle, and the time-averaged energy density is

$$\bar{u} = \frac{1}{16\pi}\left[\boldsymbol{E}(\boldsymbol{r})\cdot\boldsymbol{E}^*(\boldsymbol{r}) + \boldsymbol{B}(\boldsymbol{r})\cdot\boldsymbol{B}^*(\boldsymbol{r})\right]. \tag{28}$$

The time-averaged momentum density is likewise

$$\bar{\boldsymbol{p}} = \frac{1}{16\pi c}\left[\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{B}^*(\boldsymbol{r}) + \boldsymbol{E}^*(\boldsymbol{r})\times\boldsymbol{B}(\boldsymbol{r})\right]. \tag{29}$$

If we think of the beam as consisting of photons carrying energy $\hbar ck$ and momentum $\hbar k$ in the z direction, we expect on integrating over any section z = constant to find that

$$c\int_0^\infty \mathrm{d}\rho\,\rho\bar{p}_z \approx \int_0^\infty \mathrm{d}\rho\,\rho\bar{u}. \tag{30}$$

We have written an *approximate* equality, since not all of the electromagnetic energy is necessarily associated with propagation in the direction of positive z.

Let us evaluate both sides of (30) in the case of the TM beam, for which [19] $\boldsymbol{A} = (0, 0, A_0\psi)$ where A_0 is a real amplitude, and ψ will later be taken to be $\sin kR/kR$, but at the moment is a general function of (x, y, z) or (ρ, z, ϕ) or (ξ, η, ϕ). From (26) we have

$$\begin{aligned}\boldsymbol{B} &= A_0\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right),\\ \boldsymbol{E} &= \frac{\mathrm{i}A_0}{k}\left(\frac{\partial^2\psi}{\partial x\partial z}, \frac{\partial^2\psi}{\partial y\partial z}, \frac{\partial^2\psi}{\partial z^2} + k^2\psi\right).\end{aligned} \tag{31}$$

We see that $\boldsymbol{B}$ is purely transverse. Note that setting $\psi = \mathrm{e}^{\mathrm{i}kz}$ gives identically zero $\boldsymbol{B}$ and $\boldsymbol{E}$: there is no plane-wave limit for this class of beams.

For ψ independent of the azimuthal angle ϕ, as will be the case for the $m = 0$ subset of the $\psi_{\ell m}$ given in (12),

$$\boldsymbol{B} = A_0\frac{\partial\psi}{\partial\rho}(\sin\phi, -\cos\phi, 0) \tag{32}$$

i.e. the magnetic field lines are circles, with centres on the beam axis. The electric field in the $m = 0$ case is

$$\boldsymbol{E} = \frac{\mathrm{i}A_0}{k}\left(\cos\phi\frac{\partial^2\psi}{\partial\rho\partial z}, \sin\phi\frac{\partial^2\psi}{\partial\rho\partial z}, \frac{\partial^2\psi}{\partial z^2} + k^2\psi\right) \tag{33}$$

i.e. $\boldsymbol{E}$ is the sum of a radial vector in the xy plane plus a component along the beam axis.

With these results we find the time-averaged energy and momentum densities for axially symmetric fields:

$$\bar{u} = \frac{A_0^2}{16\pi k^2}\left\{\left|\frac{\partial^2\psi}{\partial\rho\partial z}\right|^2 + \left|\frac{\partial^2\psi}{\partial z^2} + k^2\psi\right|^2 + k^2\left|\frac{\partial\psi}{\partial\rho}\right|^2\right\} \tag{34}$$

$$\bar{p}_z = \frac{A_0^2}{8\pi ck}\mathrm{Im}\left(\frac{\partial\psi^*}{\partial\rho}\frac{\partial^2\psi}{\partial\rho\partial z}\right). \tag{35}$$

In terms of the modulus and phase, $\psi = M\mathrm{e}^{\mathrm{i}P}$, we have

$$\left|\frac{\partial\psi}{\partial\rho}\right|^2 = \left(\frac{\partial M}{\partial\rho}\right)^2 + M^2\left(\frac{\partial P}{\partial\rho}\right)^2$$

$$\left|\frac{\partial^2\psi}{\partial\rho\partial z}\right|^2 = \left(\frac{\partial^2 M}{\partial\rho\partial z} - M\frac{\partial P}{\partial\rho}\frac{\partial P}{\partial z}\right)^2 + \left(M\frac{\partial^2 P}{\partial\rho\partial z} + \frac{\partial M}{\partial\rho}\frac{\partial P}{\partial z} + \frac{\partial M}{\partial z}\frac{\partial P}{\partial\rho}\right)^2$$

$$\left|\frac{\partial^2\psi}{\partial z^2} + k^2\psi\right|^2 = \left[\frac{\partial^2 M}{\partial z^2} + k^2 M - M\left(\frac{\partial P}{\partial z}\right)^2\right]^2 + \left(2\frac{\partial M}{\partial z}\frac{\partial P}{\partial z} + M\frac{\partial^2 P}{\partial z^2}\right)^2 \tag{36}$$

$$\mathrm{Im}\left(\frac{\partial\psi^*}{\partial\rho}\frac{\partial^2\psi}{\partial\rho\partial z}\right) = \frac{\partial M}{\partial\rho}\frac{\partial M}{\partial z}\frac{\partial P}{\partial\rho} + \left(\frac{\partial M}{\partial\rho}\right)^2\frac{\partial P}{\partial z} + M\left[\frac{\partial M}{\partial\rho}\frac{\partial^2 P}{\partial\rho\partial z} - \frac{\partial^2 M}{\partial\rho\partial z}\frac{\partial P}{\partial\rho}\right] + M^2\left(\frac{\partial P}{\partial\rho}\right)^2\frac{\partial P}{\partial z}.$$

An equivalent way of deriving these results is to work entirely with real fields $\boldsymbol{E}$ and $\boldsymbol{B}$, though starting with a complex vector potential $\boldsymbol{A}$. From (26), these real fields are

$$\boldsymbol{B} = \mathrm{Re}\left\{\nabla\times\boldsymbol{A}\mathrm{e}^{-\mathrm{i}\omega t}\right\},$$
$$\boldsymbol{E} = -\frac{1}{k}\mathrm{Im}\left\{\left[\nabla(\nabla\cdot\boldsymbol{A}) + k^2\boldsymbol{A}\right]\mathrm{e}^{-\mathrm{i}\omega t}\right\} \tag{37}$$

and with $\boldsymbol{A} = \left(0, 0, A_0 M\mathrm{e}^{\mathrm{i}P}\right)$, A_0 assumed real,

$$\boldsymbol{B} = A_0\nabla\times(0, 0, M\cos(P-\omega t))$$
$$\boldsymbol{E} = -A_0\left\{\nabla\frac{\partial}{\partial z}\left[M\sin(P-\omega t)\right] + k^2\left(0, 0, M\sin(P-\omega t)\right)\right\}. \tag{38}$$

We will now evaluate $\bar{p}_z$ and $\bar{u}$ and then both sides of (30), starting with the paraxially approximated ψ_{G} given in (6). For $\psi = \psi_{\mathrm{G}}$ we find both sides to be independent of z, with values

$$c\int_0^\infty \mathrm{d}\rho\,\rho\bar{p}_z = \frac{A_0^2}{16\pi}\left(1-\beta^{-1}\right)$$
$$\int_0^\infty \mathrm{d}\rho\,\rho\bar{u} = \frac{A_0^2}{16\pi}\left(1 - \frac{3}{4}\beta^{-2} + \frac{3}{4}\beta^{-3}\right). \tag{39}$$

Thus (30) is satisfied by the paraxial Gaussian ψ_{G} within its range of validity (the first inequality in (8) requires $\beta = kb \gg 1$ for errors to be negligible near the focal plane).

A much longer calculation, with $\psi = (\beta/\sinh\beta)\sin kR/kR$, gives (see the appendix)

$$c\int_0^\infty \mathrm{d}\rho\,\rho\bar{p}_z = \frac{A_0^2}{16\pi}\frac{\left[1-\frac{1}{2}\beta^{-1}+\mathrm{e}^{-2\beta}\right]\left[1-(2\beta+1)\,\mathrm{e}^{-2\beta}\right]}{\left[1-\mathrm{e}^{-\beta}\right]^2} = \frac{A_0^2}{16\pi}\left\{1 - \frac{1}{2}\beta^{-1} + \mathrm{O}\left(\mathrm{e}^{-\beta}\right)\right\} \tag{40}$$

in close correspondence with the paraxial expression in (39). However $\int_0^\infty \mathrm{d}\rho\,\rho\bar{u}$ is *logarithmically divergent*, as was $\int_0^\infty \mathrm{d}\rho\,\rho M^2$ in the scalar case in section 2. The implication is that a TM beam with the wavefunction $\psi = j_0(kR)$ is not physically realizable, because an infinite energy would be required to construct any finite length of such a beam.

4. TE and 'TEM' beams

Davis and Patsakos [19] have pointed out that TE modes can be obtained from TM modes (and vice versa) by means of the duality transformations $\boldsymbol{E}\to\boldsymbol{B}$, $\boldsymbol{B}\to-\boldsymbol{E}$ (for the general transformation, see section 6.12 of [1]). The vector potential $\boldsymbol{A}_{\mathrm{TE}}$ which gives the dual of the TM field arising from $\boldsymbol{A}_{\mathrm{TM}} = (0, 0, A_0\psi)$ is $\boldsymbol{A}_{\mathrm{TE}} = (\mathrm{i}k)^{-1}\nabla\times\boldsymbol{A}_{\mathrm{TM}} = (\mathrm{i}k)^{-1}\boldsymbol{B}_{\mathrm{TM}}$, i.e.

$$\boldsymbol{A}_{\mathrm{TE}} = \frac{A_0}{\mathrm{i}k}\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right). \tag{41}$$

The corresponding fields are

$$\boldsymbol{E} = A_0\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right),$$
$$\boldsymbol{B} = \frac{A_0}{\mathrm{i}k}\left(\frac{\partial^2\psi}{\partial x\partial z}, \frac{\partial^2\xi}{\partial y\partial z}, \frac{\partial^2\psi}{\partial z^2} + k^2\psi\right). \tag{42}$$

In this case the electric field is transverse, and for axially symmetric ψ the electric field lines are circles concentric with the z axis. The results for $\bar{\boldsymbol{p}}$ and $\bar{u}$ are the same as for the TM mode.

Clearly combinations of $\boldsymbol{A}_{\mathrm{TM}}$ and $\boldsymbol{A}_{\mathrm{TE}}$ will also give exact solutions of Maxwell's equations. Consider

$$\boldsymbol{A} = \boldsymbol{A}_{\mathrm{TM}} + \mathrm{i}\boldsymbol{A}_{\mathrm{TE}} = A_0\left(\frac{1}{k}\frac{\partial\psi}{\partial y}, -\frac{1}{k}\frac{\partial\psi}{\partial x}, \psi\right). \tag{43}$$

The magnetic field is

$$\boldsymbol{B} = \frac{A_0}{k}\left(\frac{\partial^2\psi}{\partial x\partial z} + k\frac{\partial\psi}{\partial y}, \frac{\partial^2\psi}{\partial y\partial z} - k\frac{\partial\psi}{\partial x}, \frac{\partial^2\psi}{\partial z^2} + k^2\psi\right) \tag{44}$$

and the electric field is $\boldsymbol{E} = \mathrm{i}\boldsymbol{B}$ ($\boldsymbol{A}_{\mathrm{TM}} - \mathrm{i}\boldsymbol{A}_{\mathrm{TE}}$ gives $\boldsymbol{E} = -\mathrm{i}\boldsymbol{B}$). Thus the electric and magnetic fields are in phase quadrature. Neither field is purely transverse: hence our notation 'TEM'.

The combinations TM $\pm$ iTE have the special property that the momentum and energy densities are *time independent*: all time dependence is eliminated through application of $\cos^2(P-\omega t) + \sin^2(P-\omega t) = 1$. This is in contrast to the usual situation where $\boldsymbol{p}$ and u oscillate (at angular frequency 2ω) about their average values $\bar{\boldsymbol{p}}$ and $\bar{u}$.

Again we find that for $\psi = (\beta/\sinh\beta)\sin kR/kR$ the integral over p_z takes the value given in (40), and that the integral over u is logarithmically divergent.

5. Related wavefunctions

The wavefunction $j_0(kR)$, $R^2 = \rho^2 + (z-\mathrm{i}b)^2$ has been shown to lead to physically unrealizable particle or electromagnetic beams. Here we discuss the generalized set mentioned in section 1,

$$\psi_{\ell m} = j_\ell(kR)P_{\ell m}\left(\frac{z-\mathrm{i}b}{R}\right)\mathrm{e}^{\pm\mathrm{i}m\phi}. \tag{45}$$

The corresponding functions with $b = 0$ are eigenfunctions of the angular momentum operator $\boldsymbol{L} = -\mathrm{i}\hbar\boldsymbol{r} \times \nabla$, with the eigenvalues of $\boldsymbol{L}^2$ and L_z being $\hbar^2\ell(\ell+1)$ and $\hbar m$. Since the functions (45) are obtainable by replacing z by $z - \mathrm{i}b$ in the spherical components of a plane wave, it is plausible (and can be verified) that the $\psi_{\ell m}$ of (45) are eigenfunctions of $\tilde{\boldsymbol{L}}^2$ and $\tilde{L}_z$, where

$$\tilde{\boldsymbol{L}} = -\mathrm{i}\hbar\,(\boldsymbol{r} - \boldsymbol{r}_b) \times \nabla, \qquad \boldsymbol{r}_b = (0, 0, \mathrm{i}b). \tag{46}$$

This is the angular momentum operator relative to the imaginary point $\boldsymbol{r}_b$, rather than relative to the origin. (For a discussion of the orbital angular momentum of light, see the recent review [21].)

Let us consider the $\ell = 1, m = 0$ function. In oblate spheroidal coordinates we have, with $\beta = kb$ as before,

$$\psi_{10} = j_1[\beta(\xi - \mathrm{i}\eta)]\frac{\xi\eta - \mathrm{i}}{\xi - \mathrm{i}\eta}. \tag{47}$$

Consider the focal plane variation of ψ_{10}:

$$\psi_{10} = \begin{cases} -\mathrm{i}j_1(\beta\xi)/\xi \\ \left[\rho \geqslant b, \eta = 0, \xi = k\sqrt{\rho^2 - b^2}, 0 \leqslant \xi < \infty\right] \\ j_1(-\mathrm{i}\beta\eta)/\eta \\ \left[0 \leqslant \rho \leqslant b, \xi = 0, \eta = k\sqrt{b^2 - \rho^2}, 0 \leqslant \eta \leqslant 1\right]. \end{cases} \tag{48}$$

The Legendre function contributes the factor $-\mathrm{i}\xi^{-1}$ for $\rho \geqslant b$, resulting in a finite normalization integral for scalar beams:

$$\begin{aligned} N(0) &\equiv b^{-2}\int_0^\infty \mathrm{d}\rho\,\rho M^2(z = 0) \\ &= \frac{2\beta\sinh(2\beta) - \cosh(2\beta) - 2\beta^2 + 1}{8\beta^4}. \end{aligned} \tag{49}$$

At arbitrary z the normalization integral becomes

$$\begin{aligned} N(z) = {}& (2\beta\sinh(2\beta) - \cosh(2\beta) + 1 - (2\beta b/z)\sin(2kz) \\ &+ (b/z)^2[1 - \cos(2kz)])\{8\beta^4\}^{-1}. \end{aligned} \tag{50}$$

The corresponding current density integral is is independent of z, as expected:

$$\begin{aligned} K &\equiv b^{-2}\int_0^\infty \mathrm{d}\rho\,\rho M^2\frac{\partial P}{\partial z} \\ &= \frac{2\beta^2\cosh(2\beta) - 2\beta\sinh(2\beta) + \cosh(2\beta) - 1}{8b\beta^4}. \end{aligned} \tag{51}$$

As discussed in section 2, we expect the ratio of K to kN to be approximately unity, and it is for $\beta \gg 1$ (for all z):

$$\frac{K}{kN} = 1 - \frac{\beta - 1}{\beta(2\beta - 1)} + \mathrm{O}\big(\mathrm{e}^{-2\beta}\big). \tag{52}$$

Finally, we investigate the momentum and energy density integrals for the electromagnetic TM wave. The integral over a plane of fixed z of the time-averaged momentum density $\bar{p}_z$ (as given in (35)) can be evaluated. It is independent of z, as expected from the conservation of momentum:

$$\begin{aligned} c\int_0^\infty \mathrm{d}\rho\,\rho\bar{p}_z = {}& \left(\frac{A_0^2}{8\pi}\right)\frac{1}{8\beta^5}\big\{2\beta^3\sinh(2\beta) - 5\beta^2\cosh(2\beta) \\ &+ 6\beta\sin(2\beta) - 3\cosh(2\beta) - \beta^2 + 3\big\}. \end{aligned} \tag{53}$$

The integral over the energy density is finite (in contrast to that for the j_0 wavefunction): from (34) we find

$$\begin{aligned} \int_0^\infty \mathrm{d}\rho\,\rho\bar{u} = {}& \left(\frac{A_0^2}{16\pi}\right)\frac{1}{8\beta^6}\bigg\{\beta^2\big[4\beta^2\cosh(2\beta) - 6\beta\sinh(2\beta) \\ &+ 3\cosh(2\beta) + 2\beta^2 - 3\big] + 2\beta^4\left(\frac{b}{z}\right)^2 \\ &+ 8\beta^3\left(\frac{b}{z}\right)^3\sin(2kz) + 6\beta^2\left(\frac{b}{z}\right)^4[1 + 4\cos(2kz)] \\ &- 30\beta\left(\frac{b}{z}\right)^5\sin(2kz) + 15\left(\frac{b}{z}\right)^6[1 - \cos(2kz)]\bigg\}. \end{aligned} \tag{54}$$

This expression is even in z, as we expect; at $z = 0$ it takes the value

$$\begin{aligned} &\left(\frac{A_0^2}{16\pi}\right)\frac{1}{8\beta^4}\big\{4\beta^2\cosh(2\beta) \\ &- 6\beta\sinh(2\beta) + 3\cosh(2\beta) - \tfrac{4}{3}\beta^4 + 2\beta^2 - 3\big\}. \end{aligned} \tag{55}$$

We note that for large β the ratio of c times the momentum integral to the energy integral tends to unity for all z:

$$\frac{c\int_0^\infty \mathrm{d}\rho\,\rho\bar{p}_z}{\int_0^\infty \mathrm{d}\rho\,\rho\bar{u}} = 1 - \frac{4\beta^2 - 9\beta + 6}{\beta\left(4\beta^2 - 6\beta + 3\right)} + \mathrm{O}\big(\mathrm{e}^{-2\beta}\big). \tag{56}$$

6. Discussion

We have seen that the wavefunction $j_0(kR)$, which has been proposed as an exact solution of the Helmholtz equation representing particle or electromagnetic beams, gives a divergent normalization integral in the particle case, and a divergent energy density integral in the electromagnetic TM, TE and 'TEM' cases. Thus such beams, in the exact form given, cannot exist.

On the other hand, the wavefunction $j_1(kR)P_{10}\left(\frac{z-\mathrm{i}b}{R}\right)$ does lead to finite integrals, with the physically desirable properties that (i) the integral over the momentum density is independent of z (conservation of z-component of the momentum), and (ii) that for large beam waist compared to wavelength the ratio of the $c\bar{p}_z$ to the $\bar{u}$ integrals tends to unity (energy $= c\times$ momentum, on average, as expected for a broad beam). For more tightly focused beams, there is significant z-variation of the energy integral, especially near the focal plane.

We note in conclusion that the j_1P_{10} wavefunction is not an isolated case: the same convergence factor is provided by all of the Legendre functions which have odd $\ell - m$ (see e.g. [20], p 1325). Thus

$$j_2P_{21},\ j_3P_{30},\ j_3P_{32}\ldots \tag{57}$$

are all solutions of the Helmholtz equation which are expected to have convergent energy integrals, but these have not yet been investigated.

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Appendix. Energy and momentum integrals in oblate spheroidal coordinates

The expressions (34) and (35) give $\bar{u}$ and $\bar{p}_z$ in terms of the partial derivatives $\partial\psi/\partial\rho$, $\partial^2\psi/\partial\rho\partial z$ and $\partial^2\psi/\partial z^2$. To find these in terms of ξ and η we proceed as in the derivation of (19) and (20); these already give us

$$\frac{\partial\psi}{\partial\rho} = \frac{(1+\xi^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}{b(\xi^2+\eta^2)}\left\{\xi\frac{\partial\psi}{\partial\xi} - \eta\frac{\partial\psi}{\partial\eta}\right\} \tag{A.1}$$

$$\frac{\partial\psi}{\partial z} = \frac{1}{b(\xi^2+\eta^2)}\left\{\eta(1+\xi^2)\frac{\partial\psi}{\partial\xi} + \xi(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right\}. \tag{A.2}$$

The others (we list all the partial derivatives of second order for completeness) are

$$\begin{aligned}\frac{\partial^2\psi}{\partial\rho^2} = \frac{1}{b^2(\xi^2+\eta^2)^3}\Bigg\{&(\xi^2+\eta^2)(1+\xi^2)(1-\eta^2)\\ &\times\left[\xi^2\frac{\partial^2\psi}{\partial\xi^2} - 2\xi\eta\frac{\partial^2\psi}{\partial\xi\partial\eta} + \eta^2\frac{\partial^2\psi}{\partial\eta^2}\right]\\ &+\xi\left[(1-\xi^2\eta^2)(3\eta^2-\xi^2) + 2\eta^2(3\xi^2-\eta^2)\right]\frac{\partial\psi}{\partial\xi}\\ &+\eta\left[(1-\xi^2\eta^2)(3\xi^2-\eta^2) - 2\xi^2(3\eta^2-\xi^2)\right]\frac{\partial\psi}{\partial\eta}\Bigg\}\end{aligned} \tag{A.3}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial\rho\partial z} = \frac{(1+\xi^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}{b^2(\xi^2+\eta^2)^3}\Bigg\{&(\xi^2+\eta^2)\Bigg[\xi\eta(1+\xi^2)\frac{\partial^2\psi}{\partial\xi^2}\\ &-(2\xi^2\eta^2-\xi^2+\eta^2)\frac{\partial^2\psi}{\partial\xi\partial\eta} - \xi\eta(1-\eta^2)\frac{\partial^2\psi}{\partial\eta^2}\Bigg]\\ &+\eta\left[\xi^2(3\eta^2-\xi^2) - (3\xi^2-\eta^2)\right]\frac{\partial\psi}{\partial\xi}\\ &+\xi\left[\eta^2(3\xi^2-\eta^2) + 3\eta^2 - \xi^2\right]\frac{\partial\psi}{\partial\eta}\Bigg\}\end{aligned} \tag{A.4}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial z^2} = \frac{1}{b^2(\xi^2+\eta^2)^3}\Bigg\{&(\xi^2+\eta^2)\Bigg[\eta^2(1+\xi^2)^2\frac{\partial^2\psi}{\partial\xi^2}\\ &+2\xi\eta(1+\xi^2)(1-\eta^2)\frac{\partial^2\psi}{\partial\xi\partial\eta} + \xi^2(1-\eta^2)^2\frac{\partial^2\psi}{\partial\eta^2}\Bigg]\\ &-(1+\xi^2)(1-\eta^2)\Bigg[\xi(3\eta^2-\xi^2)\frac{\partial\psi}{\partial\xi}\\ &+\eta(3\xi^2-\eta^2)\frac{\partial\psi}{\partial\eta}\Bigg]\Bigg\}.\end{aligned} \tag{A.5}$$

As a check we calculate the well known Laplacian (we omit the $\rho^{-2}\partial^2\psi/\partial\phi^2$ term)

$$\begin{aligned}\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{b^2(\xi^2+\eta^2)}\Bigg\{&\frac{\partial}{\partial\xi}(1+\xi^2)\frac{\partial\psi}{\partial\xi}\\ &+\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial\psi}{\partial\eta}\Bigg\}\end{aligned} \tag{A.6}$$

(see e.g. pp 1292 ff of [20]).

With these expressions we can calculate $\bar{p}_z$ as given by (35) and (36). The integral we need is $\int_0^\infty \mathrm{d}\rho\,\rho\,\bar{p}_z$, for an arbitrary fixed z. We set $\xi = z/b\eta$; since $\rho^2 = (1+\xi^2)(1-\eta^2)\,b^2$ we now have, for integration over a plane of fixed z

$$\int_0^\infty \mathrm{d}\rho\,\rho\,\bar{p}_z = b^2\int_0^1 \mathrm{d}\eta\,\eta^{-1}[\eta^2 + (\zeta/\eta)^2]\,\bar{p}_z \tag{A.7}$$

where we have set $z = b\zeta$. The integrand is proportional to $(1-\eta^2)(\eta^2+\zeta^2)(\eta^4+\zeta^2)^{-4}$ times

$$\begin{aligned}&\beta^3(\eta^4+\zeta^2)^2\left[(\eta^2+\zeta^2)\,CS + \zeta(1-\eta^2)\,cs\right]\\ &\quad-\beta^2\eta(\eta^4+\zeta^2)\left\{2\left[3\eta^2\zeta^2 - \zeta^2 + 2\eta^4\right]C^2\right.\\ &\quad\left.+2\left[3\eta^2\zeta^2 - 2\zeta^2 + \eta^4\right]c^2 - 3\left[2\eta^2\zeta^2 + \zeta^2 - \eta^4\right]\right\}\\ &\quad+6\beta\eta^2(\eta^4 + 2\eta^2\zeta^2 - \zeta^2)(\eta^2 CS + \zeta cs)\\ &\quad-3\eta^3(C^2 - c^2)(\eta^4 + 2\eta^2\zeta^2 - \zeta^2)\end{aligned} \tag{A.8}$$

(in the case of the wavefunction $\psi = j_0[\beta(\xi - \mathrm{i}\eta)]$), where we have put $C = \cosh(\beta\eta)$, $S = \sinh(\beta\eta)$, $c = \cos(\beta\zeta/\eta)$ and $s = \sin(\beta\zeta/\eta)$. As expected from momentum conservation, the integral is independent of $\zeta = z/b$, and gives the result (40) after multiplication by $(\beta/\sinh\beta)^2$.

The corresponding integral over the energy density is logarithmically divergent for any z, as we found before on integrating over the focal plane $z = 0$.

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