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Thermo Field Dynamics on BTZ spacetime

W. A. Rojas C.[1,2,*] and J. R. Arenas S.[2,†]

[1] *Universidad Distrital Francisco José de Caldas, Bogotá, Colombia*

[2] *Observatorio Astronómico Nacional, Universidad Nacional de Colombia, Bogotá, Colombia*

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A thin dust shell contracting from infinity to near its gravitational radius r_+, in a spacetime AdS_3 is analyzed; its equation of motion is determined and the solution $R(t)$ as seen by a FIDO observer is estimated. It is concluded that this Shell's exterior looks like a BTZ black hole with similar properties.

Based on the Thermo Field Dynamics technique, a scalar field Φ in the proximity of a non-rotating BTZ $(2+1)$ black hole is studied. From the corresponding Killing-Boulware $|0\rangle_{KB*}$ and Hartle-Hawking $|0\rangle_{HH*}$ vacuum states, the associated Wightman function $W(x,x')_{HH^*} - W(x,x')_{KB^*}$ is determined and based on it, the time component of the momentum-energy tensor of the system $\partial_0\partial_{0'}\left(W_{HH^*} - W_{KB^*}\right)(x,x') \approx \langle T_{00}(x,x')\rangle = \sigma(r)$ is calculated. Which allows establishing the origin and location of the degrees of freedom responsible for the entropy that describes a source for the Bekenstein-Hawking S_{BH} entropy.

The thermal environment described by this model manifests itself with a well-defined and concentrated energy density near the event horizon, according to a FIDO observer.

I. INTRODUCTION

In the framework of general relativity, the presence of black holes arises as solutions to Einstein's field equations, describing regions of spacetime where the curvature is so intense that not even light can escape. Black hole thermodynamics, as formulated by Hawking and Bekenstein [1–3], posits that black holes have an entropy proportional to the area of their event horizon, defying the classical notion that these objects can only destroy information

The origin of S_{BH} may be attributed to various factors, one of which is the concept of entanglement entropy, S_{Ent}, linked to quantum modes and correlations concealed from an external observer near a horizon.

Assuming a black hole exists in an unknown pure quantum state, correlations arise between internal and external modes relative to the horizon. Thus, S_{Ent} can be inferred by tallying the modes beyond the horizon. Seminal works by Bombelli [4], Frolov and Novikov [5], Srednicki [6], Terashima [7], among others, propose: $S_{\text{Ent}} \propto A$, where A signifies the area of the partition wall, a characteristic not exclusive to black holes but extends to other scenarios [6, 7]. Srednicki estimated the ground state density matrix for a scalar field, yielding an entropy proportional to area, $S = KM^2A$, where K depends on M [6].

Similarly, Terashima interpreted S_{BH} as the entanglement entropy between the exterior and a thin region inside the horizon, where the thickness of both regions is approximately the Planck length l_P. Thus, Terashima derived:

$$S \approx C\frac{A}{a^2}, \tag{1}$$

with a representing horizon fluctuations and C as a constant [7].

This entanglement interpretation aligns closely with 't Hooft's wall model [8], subsequently modified by Mukohyama and Israel [9], especially when applied to the modeling of a black shell compressing towards its gravitational radius [10–13].

In a broader context, the thermal entanglement interpretation of Bekenstein-Hawking entropy (S_{BH}) reflects vacuum properties in strong gravitational fields, wherein vacuum fluctuations persist even in the absence of excitations. This vacuum appears as a thermal atmosphere to a Fiducial Observer (FIDO) near the horizon [14, 15], resulting in:

$$S = \beta^2\frac{\partial F(\beta)}{\partial\beta} \approx \frac{1}{360\pi\epsilon^2}A, \tag{2}$$

where $\beta = \frac{1}{T_\infty}$, ϵ represents a horizon cutoff [16, 17], and A denotes the horizon area. This entropy, associated with vacuum properties, arises due to FIDO's perception of the vacuum as a mixed state. Consequently, the entanglement entropy coincides with the entropy of the thermal atmosphere around the horizon [13].

Further insight into S_{BH} comes from symmetries-based research, particularly in the context of BTZ spacetime, which aids in understanding AdS/CFT duality [18]. This approach highlights the duality between thermal entropy

* warojasc@unal.edu.co
† jrarenass@unal.edu.co

of conformal field theory and the entropy of a BTZ black hole [19].

In this latter approach, the interpretation of the entropy of entanglement for black holes was developed by Shinsei Ryu and Tadashi Takayanagi. For the case of a BTZ black hole, in the high temperature limit, they succeeded in obtaining the Bekenstein-Hawking entropy, which is a thermal entropy of entanglement [19, 20]

$$S_A = \frac{\gamma_A}{4G_N^{d+2}}, \quad (3)$$

where γ_A is the minimum surface area in d dimensions in AdS_{d+2}, whose boundary is given by ∂A. It is important to note that γ_A plays the role of a holographic screen for an external observer. In the case of BTZ, S_A and

$$S_A(\beta) = \frac{c}{3}\ln\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right] \quad (4)$$

encode a topological entanglement entropy[19].

In this sense, there are several works that aim to explain S_{Ent} from an analytical and/or numerical point of view. For example, Dharm Veer Singh and Sanjay Siwach calculated S_{Ent} for a massless scalar field in BTZ numerically, of the form [21]

$$S_{\text{Ent}} = C_s\frac{r_+}{a}, \;\; C_s = 0.294. \quad (5)$$

which is proportional to the area of the horizon $2\pi r_+$, with a being the UV cut-off used to discretize the system.

This article seeks to extend the black shell entanglement entropy model to BTZ spacetime, with the aim of relating classical symmetries and vacuum properties in strong gravitational fields. The understanding of Bekenstein-Hawking entropy in this comprehensive context offers insights into the real localization of degrees of freedom within quantum statistical mechanics systems, bridging external and internal characterizations according to the Mukohyama-Israel complementary principle.

This study is structured as follows: In Section II, we provide a brief overview of the BTZ spacetime structure. In Section III, we conduct an analysis of a dust shell contracting from infinity to near its gravitational radius r_+. This analysis includes estimating the differential equation of motion and proposing a possible solution for a Fiducial Observer (FIDO).

In Section IV, we introduce the thermodynamics of a scalar field near the horizon. Sections V and VI focus on the application of Thermofield dynamics to the scalar field, facilitating the estimation and calculation of the $\langle T_{00}\rangle$ component of the energy-momentum tensor. This analysis provides insights into the field distribution near the horizon and the precise localization of the degrees of freedom responsible for the entanglement entropy S_{Ent} and other thermal properties of the scalar field.

In Section VII, we delve into the statistical analysis of a scalar field in BTZ spacetime.

Finally, in Section VIII, we present the discussions and conclusions of the present study.

II. BTZ BLACK HOLE STRUCTURE

The BTZ solution is an exact solution of Einstein's general theory of relativity in three dimensions.

$$ds^2 = -\left(N^\perp\right)^2 dt^2 + \frac{1}{f(r)^2}dr^2 + r^2\left(d\phi + N^\phi dt\right)^2 \quad (6)$$

where

$$N^\perp = f(r) = \sqrt{-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}}, \quad (7)$$

$$N^\phi = -\frac{J}{2r^2}, \;\; |J| \leq Ml, \quad (8)$$

The BTZ solution describes a three-dimensional spacetime with a constant and negative curvature, denoted as $R < 0$, known as anti-de Sitter (AdS) space. This solution exhibits rotational symmetry and describes a black hole with angular momentum J and mass M in spacetime. Similar to its counterpart in (3+1) dimensions, the Kerr black hole, it possesses an ergosphere. In the case of BTZ, the ergosphere is located a

$$r_{erg} = \sqrt{r_+ - r_-} = l\sqrt{M}, \quad (9)$$

$$r_\pm^2 = \frac{Ml^2}{2}\left[1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\,\right]. \quad (10)$$

The BTZ black hole is interesting because it has provided insights into fundamental issues between string theory and quantum gravity within the framework of the AdS/CFT conjecture. If we consider $J = 0$, Equation (6) leads to a static black hole, analogous to the Schwarzschild black hole. Both types of black holes share similar characteristics, including a bifurcate horizon and a central singularity. The presence of a horizon implies a causal disconnection between the interior and the exterior of the black hole. Therefore, it is not possible to extract information from the interior once an object has crossed the horizon. These similarities allow for the construction of the Penrose diagram for BTZ, which visualizes the spacetime structure near the event horizon

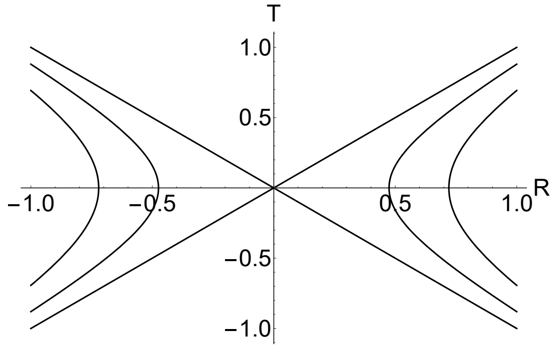

FIG. 1. Kruskal diagram for the BTZ black hole.

Thus, for the BTZ solution, accelerated observers move along a hyperbolic trajectory that is perpendicular to the black hole's event horizon

$$\frac{T^2}{1/2} - \frac{R^2}{1/2} = 1. \tag{11}$$

For more detailed information, please consult Appendix A.

III. COMPACT OBJECTS THAT MIMIC BLACK HOLES

In the framework of General Relativity in 2+1 dimensions, two different solutions of the Einstein field equations are considered at a point in the manifold. This means that the junction conditions require the metrics and their derivatives to be continuous across the interface between two regions. Additionally, the conservation of energy and momentum is required at the interface.

A. KINEMATICS OF HYPERSURFACES

According to the Darmois-Israel formalism [22–24], consider a manifold $\mathcal{M}$ of dimension $(2+1)$ and within it a hypersurface Σ, with the condition that $\Sigma \subset \mathcal{M}$ and can be time-like, space-like or null. A specific hypersurface Σ can be chosen when the coordinates x^α of variety $\mathcal{M}$ are constrained of the form $\Phi(x^\alpha) = 0$. Thus, Σ can be specified with a constraint on the coordinates whose parametric equations are of the form $x^\alpha = x^\alpha(y^a)$, $x^\alpha \in \mathcal{M}$, $y^a \in \Sigma$ y $\Sigma \subset \mathcal{M}$, where y^a corresponds to intrinsic coordinates in Σ. Also, the hypersurface Σ is determined by its normal vector n_α, such that the unit normal vector is -1, if Σ is space-like and $+1$, if Σ is time-like.

The induced metric on Σ is obtained when the displacements are limited to such a hypersurface of the form $ds^2_\Sigma = h_{ab}dy^a dy^b$. Where h_{ab} is known as the induced metric $h_{ab} = g_{\alpha\beta}e^\alpha_a e^\beta_b$ and the tangent vectors to the integral curves contained in Σ.

The extrinsic curvature or Second Fundamental Form K_{ab}, defines how the hypersurface Σ is curved with respect to $\mathcal{M}$ in which it is embedded, and it is $K_{ab} = \frac{1}{2}\left[\mathcal{L}_n g_{\mu\nu}\right] e^\alpha_a e^\beta_b$.

Now, in the Darmois-Israel formalism [22, 23], let hypersurface Σ divide spacetime into two regions: M^+ and M^-, such that $g^+_{\alpha\beta} \in M^+$ and $g^-_{\alpha\beta} \in M^-$ and the junction conditions for BTZ spacetime, we have a shell (dust ring), in $(2+1)$ dimensions. Such that it contracts from infinity to near its gravitational radius $r = r_+ + \epsilon$. To an external observer, the shell looks like a BTZ black hole. In a first approximation with $J = 0$, let (A1)

$$ds^2_+ = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\phi^2, \tag{12}$$

where

$$f(r) = \left(-M + \frac{r^2}{l^2}\right) \tag{13}$$

for the outer shell. And for the inner shell, there is a Minkowski spacetime

$$ds^2_- = -dt^2 + dr^2 + r^2 d\phi^2. \tag{14}$$

The coordinates for the inner solution of the shell (14) are $t = \bar{T}(\tau)$, $r = R(\tau)$. From the foregoing, (14) simplifies to

$$ds^2_- = -\left[\dot{\bar{T}}^2 + \dot{R}^2\right] d\tau^2 + R^2 d\phi^2. \tag{15}$$

The extrinsic coordinates defined on $\Sigma \subset M^-$, $y^a_- = (\tau, \phi) \in \Sigma$ and the intrinsic coordinates $x^\alpha_- \in M$ are $x^\alpha_- = (\bar{T}(\tau), R(\tau), \phi)$. From the foregoing, the relationship between the coordinates between Σ and M^- is $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$, then

$$e^\alpha_\tau = u^\alpha_- = \left[\dot{\bar{T}}, \dot{R}, 0\right], \tag{16}$$

where u^α, defines the 3-velocity for an observer falling radially with the shell (FFO). Also,

$$e^\alpha_\phi = u^\beta_- = [0, 0, 0]. \tag{17}$$

The normal vector $n^-_\alpha \perp \Sigma$ is defined as

$$n^-_\alpha = [n_\tau, n_r, n_\phi]. \tag{18}$$

The orthonormality condition $n^\pm_\alpha u^\alpha_\pm = 0$ and the normalization condition $n^\pm_\alpha n^\alpha_\pm = 1$. The foregoing makes it possible to conclude that n^-_α is a time-like vector and that Σ corresponds to a space-like hypersurface.

$$n^-_\alpha u^\alpha_- = n^-_\tau \dot{\bar{T}} + n^-_r \dot{R} = 0, \tag{19}$$

therefore, obtaining that $n^-_\phi = 0$. From the condition of normalization , $g^{\alpha\alpha} n^-_\alpha n^-_\alpha = 1$. Which allows determining the system of equations

$$n^-_\tau \dot{\bar{T}} + n^-_r \dot{R} = 0, \; -\left[n^-_t\right]^2 + \left[n^-_r\right]^2 = 1. \tag{20}$$

Once (20) has been solved, the normal vector $n^-_\alpha = \left[-\dot{R}, \dot{T}, 0\right]$ and $\dot{T}^2 - \dot{R}^2 = 1$ [23].

The outer solution of the Shell is defined by (12) and (13). Let $F(R) = f(r)$, then (12) transforms into

$$ds^2_+ = -F(R)dt^2 + \frac{1}{F(R)}dr^2 + r^2 d\phi^2. \tag{21}$$

Such that $\Sigma \in M^+$ is obtained $x^\alpha_+ = (t, r, \phi) \in M^+$ and $y^a_+ = (\tau, \phi) \in \Sigma$. It is useful to express $t = T(\tau)$, $r = R(\tau)$ and $F = F(R) = M + \frac{R(\tau)^2}{l^2}$. The metric (21) is simplified to

$$ds^2_+ = -\left[F\dot{T} - \frac{\dot{T}}{F}\right]d\tau^2 + \dot{R}^2 d\phi^2. \tag{22}$$

The intrinsic coordinates $x^\alpha_+ \in M^+$. It is possible to obtain $u^\alpha_+ = \left[\dot{T}, \dot{R}, 0\right]$, where u^α_+ is the 3-velocity measured from M^+. The normal vector n^+_α is determined as $n^+_\alpha = \left[n^+_\alpha, n^+_r, n^+_\phi\right]$.

From the orthogonality and normalization conditions, we find the radial component n^+_r is obtained, which is

$$n^+_r = \frac{\dot{T}}{\sqrt{F\dot{T}^2 - \frac{\dot{R}^2}{F}}}, \tag{23}$$

and the time component, n^+_t is

$$n^+_t = \frac{-\dot{R}}{\sqrt{F\dot{T}^2 - \frac{\dot{R}^2}{F}}}. \tag{24}$$

This leads to

$$n^+_\alpha = \frac{1}{\sqrt{F\dot{T}^2 - \frac{\dot{R}^2}{F}}}\left[-\dot{R}, -\dot{T}, 0\right] = \left[-\dot{R}, -\dot{T}, 0\right] \tag{25}$$

where $\sqrt{F\dot{T}^2 - \frac{\dot{R}^2}{F}} = 1$ [23].

It is possible to determine the extrinsic curvature for BTZ spacetime as $K^\pm_{\alpha\beta} = n^\pm_{\alpha;\beta}$. So

$$K^{+0}_0 = \frac{\dot{\beta}_+}{2\dot{R}}, \;\; K^{+1}_1 = \frac{\beta_+}{2} \;\text{ and }\; K^{+2}_2 = \frac{\beta_+}{R} \tag{26}$$

where

$$\beta_+ = \dot{T}F, \;\; \dot{\beta}_+ = F'\dot{R}\dot{T}. \tag{27}$$

And

$$K^{-\phi}_\phi = \frac{\sqrt{1+\dot{R}^2}}{R} = \frac{\beta_-}{R}, \tag{28}$$

where β_- is defined

$$\beta_- = \dot{T} = \sqrt{1+\dot{R}^2}. \tag{29}$$

The equation of motion for the shell is obtained from

$$S_{ab} = -\frac{\epsilon}{8\pi}\left([K_{ab}] - [K]\,h_{ab}\right), \tag{30}$$

where $h_{ab}|_\Sigma$, is the metric induced on the hypersurface Σ. Also,

$$[K_{ab}] = K^+_{ab}|_\Sigma - K^-_{ab}|_\Sigma, \tag{31}$$

$$[K] = K^+|_\Sigma - K^-|_\Sigma. \tag{32}$$

Finally, the equation of motion for the shell in the BTZ spacetime is

$$\frac{dR}{d\tau} = \sqrt{\left[\frac{4M^2F}{(F-M)^2} - 1\right]F}. \tag{33}$$

where

$$F = F(R) = M + \frac{R(\tau)^2}{l^2}. \tag{34}$$

A very important observation is that (33) corresponds to the shell motion equation contracting in spacetime BTZ [10, 16, 22, 25], whose motion is measured by a shell-comoving observer (FFO) with proper time τ , according to Figure 1.

Thus, it is possible to rewrite Equation (33) in terms of the coordinate time, t as follows

$$\frac{dF}{dt} = \alpha\sqrt{F^2\left[F - \vartheta(F-M)\right]}. \tag{35}$$

and

$$\alpha = \frac{4M}{aR_0}, \;\; \vartheta = \left[\frac{aR_0}{2Ml}\right]^2. \tag{36}$$

The solution to which can be written in the form

$$R(t) = \sqrt{\frac{l^2\vartheta M}{1-\vartheta}\left\{\left[\frac{\mathbf{A}e^{-\alpha\sqrt{\vartheta M}(t-t_0)} - 1}{\mathbf{A}e^{-\alpha\sqrt{\vartheta M}(t-t_0)} + 1}\right]^2 + 1\right\}} \tag{37}$$

where

$$\mathbf{A} = \frac{\sqrt{\vartheta M} + \sqrt{F_0 + \vartheta M - F_0\vartheta}}{\sqrt{\vartheta M} - \sqrt{F_0 + \vartheta M - F_0\vartheta}}. \tag{38}$$

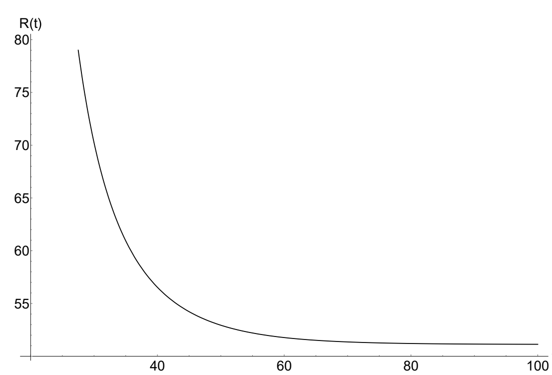

FIG. 2. Representation of $R(t)$ given by (37).

The Figure 2 represents the trajectory and behavior of a dust shell that contracts from infinity to near its gravitational radius as observed by a FIDO [16, 17]. This result is interesting because a dust shell can mimic a black hole, making it difficult for a FIDO to distinguish between an Exotic Compact Object (ECO) and a black hole [26]. To access specific details, kindly turn to Appendix B.

IV. THERMODYNAMICS NEAR THE HORIZON

The BTZ spacetime can be written in the following form

$$ds^2 = \frac{l^4}{r_+^2}\frac{(r+r_+)^2}{l^2} dU dV + r^2 d\phi. \tag{39}$$

For simplicity, consider a scalar field $\Phi(t, \underline{x})$, where $\underline{x} = r, \phi$, described by the Lagrangian density:

$$\mathcal{L}_M = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m^2\Phi^2\right], \tag{40}$$

Such that the variation of the action, δS, leads to the Klein-Gordon equation (KGE)

$$\left[\Box - m^2\right]\Phi = 0, \;\; \Box = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\right]. \tag{41}$$

By considering a WKB-type solution for Equation (41), given by $\varphi_\Omega(r) = e^{-i\phi(r)}$, it is possible to estimate the modes under the Hartle-Hawking (HH^*) scheme and the Killing-Boulware (KB^*) modes [9, 10, 25] for the scalar field over the BTZ spacetime. The relationship between the HH^* modes and the KB^* modes is defined through a Bogolubov transformation

$$\Psi_\Omega^{(\epsilon)}(\underline{x}) = \Phi_\Omega^{(\epsilon)}(\underline{x})\cosh(x) + \Phi_\Omega^{(-\epsilon)}(\underline{x})\sinh(x). \tag{42}$$

For in-depth specifics, please turn to Appendix C.

From the above, it is possible to consider a quantum field theory on a curved spacetime with respect to a Killing time t. By considering the KB^* modes, the field $\Phi_\Omega(t, \underline{x})$ is promoted to a field operator as follows:

$$\Phi_\Omega(t, \underline{x}) = \sum_\Omega \left[a_\Omega F_\Omega + b_\Omega^\dagger F_\Omega^*\right]. \tag{43}$$

Thus, the Hamiltonian of the field can be obtained as follows

$$H = \int d^2x \left\{\frac{\sqrt{-g}}{2}\left[g^{00}\dot{\Phi}^\dagger\dot{\Phi} + g^{ij}\partial_i\Phi^\dagger\partial_j\Phi\right] + \left[m^2\Phi^\dagger\Phi\right]\right\}. \tag{44}$$

Consequently, under the framework of second quantization, it is obtained that the Hamiltonian operator of the field is

$$H = \sum_\Omega \frac{1}{\sqrt{\mathbf{T}}}\omega\left[a_\Omega^\dagger a_\Omega + b_\Omega^\dagger b_\Omega\right] + Z.P.E., \tag{45}$$

where $Z.P.E.$ corresponds to the zero-point energy.

V. THERMO FIELD DYNAMICS FOR BTZ SPACETIME

This section discusses the implications of Thermo Field Dynamics (TFD) in the proximity of a BTZ black hole [7, 10, 12, 25, 27–31]. TFD is a technique used to describe systems in equilibrium, as it explains how the thermodynamic properties of quantum observables arise from quantum correlations of local operators as perceived by an observer. In the case of BTZ, an entangled state is established for the field modes in the KB^* scheme between the R and L regions of the maximally extended BTZ for $J = 0$ [32].

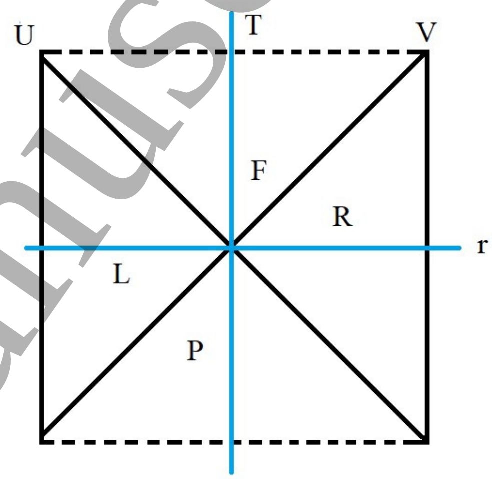

FIG. 3. Carter-Penrose diagram for a BTZ black hole.

The Hamiltonian operator for the regions R and L is

$$\begin{aligned} H &= H^{(+)} - H^{(+)} \\ &= \sum_\Omega \left[N_\Omega^{(+)} - N_\Omega^{(-)} + \bar{N}_\Omega^{(+)} - \bar{N}_\Omega^{(-)}\right]\omega, \end{aligned} \tag{46}$$

where $(+) \in R$ y $(-) \in L$, $N_\Omega^\pm = a_\Omega^{(\pm)\dagger}a_\Omega^{(\pm)}$ and $\bar{N}_\Omega^\pm = b_\Omega^{(\pm)\dagger}b_\Omega^{(\pm)}$.

The vacuum states of the field are given by

$$a_\Omega^{(+)}a_\Omega^{(-)}\left|0^{(+)}, 0^{(-)}\right\rangle_B^+ = 0, \tag{47}$$

$$b_\Omega^{(+)}b_\Omega^{(-)}\left|0^{(+)}, 0^{(-)}\right\rangle_B^- = 0. \tag{48}$$

By considering the thermal vacuum state $\left|0(\beta)_B^\pm\right\rangle$ and the normalization condition of such states ${}_B^\pm\langle 0(\beta)|0(\beta)\rangle_B^\pm = 1$, it is possible to obtain the partition function.

$$Z^\pm(\beta) = \frac{1}{1 - e^{-\beta|\omega|}}. \tag{49}$$

And the thermal vacuum state is defined as follows

$$|0(\beta)\rangle_B^+ = \sqrt{1-e^{-\beta|\omega|}}\exp\left[e^{-\frac{\beta|\omega|}{2}}a^{\dagger(+)}a^{\dagger(-)}\right]\left|0^{(+)},0^{(-)}\right\rangle, \tag{50}$$

$$|0(\beta)\rangle_B^- = \sqrt{1-e^{-\beta|\omega|}}\exp\left[e^{-\frac{\beta|\omega|}{2}}b^{\dagger(+)}b^{\dagger(-)}\right]\left|0^{(+)},0^{(-)}\right\rangle. \tag{51}$$

The occupation number is given by

$$\langle N\rangle^+ = \langle\bar{N}\rangle^- = \frac{1}{e^{\beta|\omega|}-1}. \tag{52}$$

For a detailed breakdown, please refer to Appendix E.

VI. MOMENTUM-ENERGY TENSOR FOR A SCALAR FIELD IN BTZ SPACETIME

Using the usual recipe, where the Lagrangian density of the matter fields $\mathcal{L}_M$ for BTZ (40) is known, we can obtain the energy-momentum tensor $T\mu\nu$ as follows

$$T_{\mu\nu} = \left[\partial(_\mu\partial_{\nu'}) - \frac{g^{\mu\nu}}{2}\left(\partial^\beta\partial_{\beta'} - m^2\right)\right]\Phi^2. \tag{53}$$

By considering the Wightman function for $T_{\mu\nu}$, we can obtain:

$$\langle T_{\mu\mu}(x,x')\rangle = \mathcal{D}_{\mu\nu'}W(x,x'). \tag{54}$$

Thus, we can write the difference between the Wightman functions HH^* and KB^* as

$$\left(W_{HH^*} - W_{KB^*}\right)(x,x') = \sum_\Omega \frac{1}{e^{\beta\omega}-1}F_\Omega^{*(+)}F_\Omega^{(+)}. \tag{55}$$

By considering the temporal component of $T_{\mu\nu}$, we obtain

$$\left\langle T_0^0(x,x')\right\rangle = \sigma(r) = -\int_0^\infty \frac{E}{e^{E/T(r)}-1}\frac{2\pi p\,dp}{h^2}. \tag{56}$$

Supposing an ideal gas model and $E = pv$

$$\begin{aligned} P &= \frac{1}{2}\sigma(r) \\ &= \frac{1}{2}\int_0^\infty \frac{pv}{e^{E/T(r)}-1}\frac{2\pi p\,dp}{h^2}. \end{aligned} \tag{57}$$

VII. THERMODYNAMIC ANALYSIS OF A SCALAR FIELD IN BTZ

Let partition function of a scalar field in BTZ per mode, as

$$Z_\Omega = \sum_{n=0}^\infty e^{-\beta n\omega} = \frac{1}{1-e^{\beta\omega}}, \tag{58}$$

where $\Omega = \omega, \mathfrak{m}$ and also

$$\underline{n} = \{n_\omega \forall\Omega, \omega > 0\}. \tag{59}$$

Thus, the energy of the field is $E_{\underline{n}} = \sum_\Omega n_\Omega\omega$. From the foregoing, the partition function for all possible modes near the gravitational radius r_+

$$Z = \prod_{\Omega,\omega>0}\frac{1}{1-e^{\beta\omega}}. \tag{60}$$

On the other hand, the entropy is determined as $S = \beta\langle H\rangle + \ln|Z|$ [33], where $\langle H\rangle = -\frac{\partial}{\partial\beta}\ln|Z|$. Then

$$\sum_\Omega f(\omega) = \int_0^\infty N(\omega)f(\omega)d\omega. \tag{61}$$

Ergo, for the partition function,

$$\ln|Z| = \int_0^\infty N(\omega)f(\omega)d\omega \tag{62}$$

where

$$f(\omega) = \ln\left|\frac{1}{1-e^{-\beta\omega}}\right|, \tag{63}$$

also, $N(\omega)d\omega$ corresponds to the number of modes $\psi_\Omega(\underline{x})$ are in the range ω and $\omega + d\omega$ for $\Omega = \omega, \mathfrak{m}$ and S is associated with the simplified density matrix. Considering (60).

$$\ln|Z| = \sum_{\eta=\pm}\sum_{\mathfrak{m}}\mathfrak{m}\ln\left|\frac{1}{1-e^{-\beta\omega}}\right|. \tag{64}$$

The wavenumber is required to be real (F23)

$$\mathbf{T}^{*2}(r,\omega,\mathfrak{m}) \geq 0. \tag{65}$$

This is under the WKB (F25) approach

$$\psi_\Omega(r) = \frac{1}{\sqrt[4]{4\omega^2\mathbf{T}^{*2}}}\sin\left[\int_{r_++\epsilon}^{r_{\mathfrak{m},\omega}}\mathbf{T}^*(r,\omega,\mathfrak{m})dr\right]. \tag{66}$$

Which implies that the field modes cancel each other out near the gravitational radius r_+ when Neumann-Dirichlet boundary conditions are considered

$$\int_{r_++\epsilon}^{r_{\mathfrak{m},\omega}}\mathbf{T}^*(r,\omega,\mathfrak{m})dr = n\pi, \tag{67}$$

such that

$$\mathbf{T}^{*\mathbf{2}}(r,\omega,\mathfrak{m}) = 0,\ \ \mathbf{T}^{*\mathbf{2}}(r,\omega,\mathfrak{m}) \geq 0 \longrightarrow r' < r, \tag{68}$$

thus $r = r_+ + \epsilon$, $r = R \longrightarrow \omega \geq 0$ and $m \geq 0$. As a result, $T^{*2}(r, \omega, \mathfrak{m})$ is defined by (F23). Taking (67)

$$n(\omega, \mathfrak{m}) = \frac{1}{\pi} \int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} \mathbf{T}^*(r, \omega, \mathfrak{m}) dr. \tag{69}$$

Considering the variation $\frac{\partial n(\omega, \mathfrak{m})}{\partial \omega}$

$$\begin{aligned} \frac{\partial n(\omega, \mathfrak{m})}{\partial \omega} &= \frac{1}{\pi} \int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} \frac{\partial \mathbf{T}^*(r(\omega, \mathfrak{m}); \omega, \mathfrak{m})}{\partial \omega} dr(\omega, \mathfrak{m}) \\ &+ \frac{1}{\pi} \int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} \mathbf{T}^*(r(\omega, \mathfrak{m}); \omega, \mathfrak{m}) \frac{\partial}{\partial \omega} \left[dr(\omega, \mathfrak{m}) \right], \end{aligned} \tag{70}$$

where $dr(\omega, \mathfrak{m}) \gg \frac{\partial}{\partial \omega} \left[dr(\omega, \mathfrak{m}) \right]$. Consequently

$$\frac{\partial n(\omega, \mathfrak{m})}{\partial \omega} \approx \frac{1}{\pi} \int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} \frac{\partial \mathbf{T}^*(r(\omega, \mathfrak{m}); \omega, \mathfrak{m})}{\partial \omega} dr(\omega, \mathfrak{m}). \tag{71}$$

On the other hand, in the continuum limit for (64)

$$\ln |Z| = \frac{1}{\pi} \int_{\omega} \int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} \int_{\mathfrak{m}} d\omega d\mathfrak{m} dr(\omega, \mathfrak{m}) \mathfrak{m} \; \ln \left| \frac{1}{1 - e^{-\beta \omega}} \right| \frac{\partial \mathbf{T}^*(r(\omega, \mathfrak{m}); \omega, \mathfrak{m})}{\partial \omega}. \tag{72}$$

Integrating by parts (72)

$$\begin{aligned} \ln |Z| &= \frac{1}{\pi} \int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} dr \int_{\mathfrak{m}} d\mathfrak{m} \; \mathfrak{m} \, \mathbf{T}^* \; \ln \left| \frac{1}{1 - e^{-\beta \omega}} \right| \\ &+ \frac{1}{\pi} \int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} dr \int_{\mathfrak{m}} d\mathfrak{m} \; \mathfrak{m} \int_{\omega} \frac{\beta \mathbf{T}^*}{e^{\beta \omega} - 1} d\omega. \end{aligned} \tag{73}$$

Regarding (73), the following observations are made

1. The partition function contains two contributions: one proportional to the perimeter

$$\int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} dr \int_{\mathfrak{m}} d\mathfrak{m} \; \mathfrak{m} \, \mathbf{T}^* \; \ln \left| \frac{1}{1 - e^{-\beta \omega}} \right|. \tag{74}$$

2. And another to the area over the phase space

$$\int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} dr \int_{\mathfrak{m}} d\mathfrak{m} \; \mathfrak{m} \int_{\omega} \frac{\beta \mathbf{T}^*}{e^{\beta \omega} - 1} d\omega. \tag{75}$$

With the condition that the area contribution is much larger than the perimeter contribution in the partition function (73), which enables us to approximate

$$\ln |Z| \approx \frac{1}{\pi} \int_{r_+ + \epsilon}^{r_{\mathfrak{m},\omega}} dr \int_{\mathfrak{m}} d\mathfrak{m} \; \mathfrak{m} \int_{\omega} \frac{\beta \mathbf{T}^*}{e^{\beta \omega} - 1} d\omega. \tag{76}$$

According to (F23)

$$\frac{\mathfrak{m}^2}{r} = \frac{L}{r} = \frac{\omega^2}{f(r)} - m^2 + B, \tag{77}$$

thus, for (77), it is possible to define $L = \mathfrak{m}$ and $L_{max} = \mathfrak{m}_{max}$. In other words,

$$\frac{L_{max}}{r} = \frac{\omega^2}{f(r)} - m^2 + B. \tag{78}$$

Therefore,

$$\mathfrak{m} = \pm \sqrt{L}, \;\; d\mathfrak{m} = \frac{dL}{2\sqrt{L}}. \tag{79}$$

then, the integral with respect to $\mathfrak{m}$in (76)

$$\begin{aligned} \int_{\mathfrak{m}} d\mathfrak{m} \; \mathfrak{m} \mathbf{T}^* &= \frac{1}{2} \int_L dL \; \mathbf{T}^* \\ &= \frac{1}{2\sqrt{r f(r)}} \int_0^{L_{max}} dL \sqrt{L_{max} - L} \\ &= \frac{1}{3\sqrt{r f(r)}} L_{max}^{3/2}. \end{aligned} \tag{80}$$

Inserting (80) in (76)

$$\ln |Z| = \int_0^{\infty} \frac{\beta N(\omega)}{e^{\beta \omega} - 1} d\omega. \tag{81}$$

Taking (81) and considering (78)

$$\begin{aligned} N(\omega) &= \frac{1}{3\pi} \int_{r_+ + \epsilon}^{R} dr \frac{1}{\sqrt{r f(r)}} L_{max}^{3/2} \\ &= \frac{1}{3\pi} \int_{r_+ + \epsilon}^{R} dr \frac{1}{\sqrt{r f(r)}} r^{3/2} p^3, \end{aligned} \tag{82}$$

where it possible to define

$$N(\omega) = \int_{r_+ + \epsilon}^{R} dr \frac{1}{3\pi} \frac{1}{\sqrt{r f(r)}} r^{3/2} \left[\frac{\omega^2}{f(r)} - m^2 + B \right]^{3/2}$$
$$= \int_{r_+ + \epsilon}^{R} dr N^*(\omega). \quad (83)$$

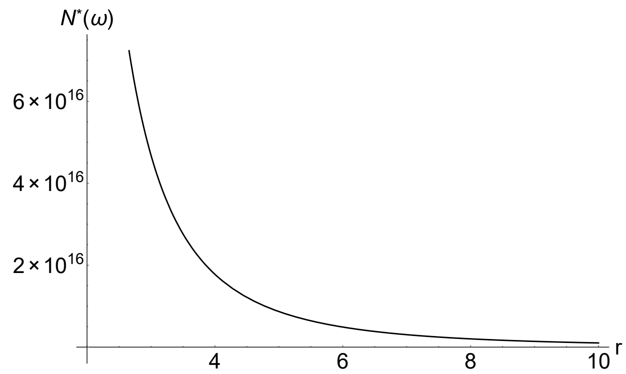

FIG. 4. Occupation number $N^*(\omega)$ for a BTZ spacetime.

On the other hand, the Helmholtz free energy F, is related to the partition function Z as

$$F = -\frac{1}{\beta} \ln |Z|$$
$$= -\int_0^\infty \frac{N(\omega)}{e^{\beta\omega} - 1} d\omega. \quad (84)$$

In that same direction, the internal energy of the scalar field is $U = -\frac{\partial}{\partial \beta} \ln |Z|$, as a result

$$U = \int_0^\infty d\omega \frac{\omega}{e^{\beta\omega} - 1} N'(\omega) \quad (85)$$

where $N'(\omega) = \frac{\partial N(\omega)}{\partial \omega}$ has been defined. According to (82)

$$\frac{\partial N(\omega)}{\partial \omega} = \frac{1}{\pi} \int_{r_+ + \epsilon}^{R} dr \frac{1}{\sqrt{r f(r)}} \frac{r\omega}{f(r)} \sqrt{r \left(\frac{\omega^2}{f(r)} - m^2 + B \right)}. \quad (86)$$

Inserting (86) in (85)

$$U = \int d\mathcal{A} \; \sigma(r) \quad (87)$$

where $\sigma(r)$ is defined by (F51). The entropy S of the field is obtained as

$$S = \beta[U - F]$$
$$= \int_{r_+ + \epsilon}^{R} d\mathcal{A} \; s(r) \; \mathbb{H}, \quad (88)$$

where is the area integral in BTZ spacetime. In addition,

$$s(r) = \frac{3\beta}{2} \int_0^\infty \frac{E}{e^{E/T(r)} - 1} \frac{2\pi p dp}{h^2} \quad (89)$$

is the field entropy density

$$\mathbb{H} = \frac{2p^2}{9\pi\omega(e^{\beta\omega} - 1)} \quad (90)$$

and is the coupling factor. From the foregoing, it is possible to obtain

$$s(r) = \beta[\sigma(r) + P]. \quad (91)$$

Let $x = \frac{E}{T}$, then (89)

$$s(r) = \frac{3}{2} T^3(r) \frac{2\pi}{h^2} \int_0^\infty \frac{x^2}{e^x - 1} dx.$$
$$= \frac{6\pi\zeta[3]}{h^2} T^2(r) \quad (92)$$

where $\zeta[3]$ corresponds to the Riemann Zeta function. Taking (88)

$$S = \int \frac{2\pi r dr}{\sqrt{f(r)}} \frac{6\pi\zeta[3]}{h^2} T^2(r)$$
$$= \frac{6\pi\zeta[3]}{h^2} T_\infty^2 \mathcal{P} \int_{r_+}^{r_+ + \epsilon} \frac{dr}{f(r)^{3/2}}, \quad (93)$$

where $r_+ = \sqrt{M l^2}$ and $\mathcal{P} = 2\pi\sqrt{M l^2}$. Furthermore, the metric factor $f(r)$ can be expressed in terms of the surface gravity

$$f(r) \approx f'(r_+)(r - r_+), \;\; \epsilon = r - r_+. \quad (94)$$

On the other hand, let the proper distance above the horizon α be

$$\alpha = \int_{r_+}^{r_1} \frac{dr}{\sqrt{f(r)}}$$
$$= \frac{2}{\sqrt{f'(r_+)}} \sqrt{\epsilon}, \quad (95)$$

which makes it possible to establish [16, 17]

$$\epsilon = \frac{1}{2} \kappa_0 \alpha^2. \quad (96)$$

This makes it possible to evaluate (93)

$$S = \frac{6\pi\zeta[3]}{h^2} T_\infty^2 \mathcal{P} \int_{r_+}^{r_+ + \epsilon} \frac{dr}{f(r)^{3/2}}$$
$$= \frac{3\zeta[3]}{2h^2\pi} \gamma \left[\frac{T_\infty}{\kappa_0 / 2\pi} \right]^2 \frac{\mathcal{P}}{\alpha}, \quad (97)$$

where $\gamma = \sqrt{2}(\sqrt{2} - 1)$. In the case where $S_{\text{Ent}} = S_{BH}$, implies that $T_\infty = T_H$, makes it possible to obtain

$$\alpha = \frac{3\zeta[3] \, \gamma \, \mathcal{P}}{4\pi h^2} \sqrt{\frac{2G_3}{l^2 M}}$$
$$= \frac{3\zeta[3] \, \gamma}{2\pi h^2} \sqrt{2G_3}. \quad (98)$$

Then,

$$S_{BH} = 2\pi\sqrt{\frac{Ml^2}{2G_3}}. \tag{99}$$

It is possible, from (97) to calculate the specific heat of BTZ as [34]

$$\begin{aligned} C_V &= C_P \\ &= T_\infty \left(\frac{\partial S}{\partial T_\infty}\right)_P \\ &= \frac{3\zeta[3]}{h^2\pi}\gamma\left[\frac{T_\infty}{\kappa_0/2\pi}\right]^2 \frac{\mathcal{P}}{\alpha}. \end{aligned} \tag{100}$$

VIII. DISCUSSIONS AND CONCLUSIONS

A thermal source of the Bekenstein- Hawking entropy S_{BH} for a BTZ black hole was modeled as an entropy of entanglement of a real massive scalar field. Entropy associated with a thin spherical dust shell, collapsing gravitationally from a specific distance to a radius slightly larger than the gravitational radius, in an AdS_3 space-time, and described by a FIDO observer.

Section III describes the equation of motion of the contracting shell in the BTZ space-time, measured by an observer comoving to the shell in terms of the proper time τ. Then, we obtained the equation of motion of the shell in the same spacetime, expressed by equation (B11), in terms of the coordinate time t, according to the measurement of a FIDO observer.

Section IV presents the developments of the quantization of a real massive scalar field over the BTZ spacetime. Specifically, in a Thermo Field Dynamics context, we calculated the expected value for the component $T_{00}(x, x')$ of the momentum-energy tensor with respect to the Boulware and Hartle-Hawking vacuum states, for the mentioned scalar field. Thus, we show in expression (F52), under the WKB approximation, that the thermal entropy is strongly located near the outer surface of the shell, at the collapse limit, according to the FIDO observer. In these terms, we obtained a well-defined energy density, of a hot scalar field.

Additionally in this section performs a thermodynamic analysis of the thermal environment found in the previous sections. In greater detail, we determined from the partition function corresponding to the hot scalar field, the occupation number $N(\omega)$ in proximity to the outer surface of the shell, which, according to the FIDO observer, is very close to the corresponding event horizon. Then we calculated the internal energy U and the entropy density $s(r)$, expressed by equations (87) and (92) .

At the end of IV , the entropy of entanglement was calculated, resorting to the cutoff introduced by 't Hooft in his wall model, and corrected by Mukohyama and Israel. This modified model involved a reinterpretation of the vacuum state and of the described object. To an outside observer, the object described is star-like and not a black hole. However, the external observer cannot distinguish between one and the other.

With the previously described collapsing shell model, in agreement with the FIDO observer, a gap is established between the inner surface of the shell and the event horizon, which is identified with ϵ, related to the proper distance above the horizon α, in the modified wall model. Thus, the 't Hooft wall coincides with the shell in proximity to the event horizon. The possibility of calculating a microscopic parameter with macroscopic criteria is noteworthy. Thus, we calculated the finite entropy of entanglement given by equation (97).

What we show with the gravitational collapse shell model is that the thermal source of the Bekenstein-Hawking entropy, in the Mukohyama-Israel two-source model, based on a complementarity principle [9], is external and fully corresponds to the thermal entropy of entanglement. Using the euclidean Gibbons-Hawking technique, Bekenstein-Hawking entropy can be obtained for the euclidean BTZ spacetime and an euclidean BTZ black shell, where these results coincide with those obtained using the entanglement technique and Thermo Field Dynamics. This result could be the key to relate the black hole entanglement formalism in the context of Mukohyama-Israel complementary principle and the black hole entanglement thermal entropy formalism for the conformal fields in the context of the AdS/CFT duality, resorting to the common topological property S^1 that their respective spacetimes have.

The results obtained in this research complement the progress of the study of the Bekenstein-Hawking entropy in the context of asymptotic symmetries, reinforcing the considerations made by Fursaev [14]. An interesting common geometric background between the model developed in this article and the modeling of asymptotic symmetries is the eternal black hole in AdS_3 [35].

In this background, we present two types of properties related to the thermal character of the Bekenstein-Hawking entropy, intimately associated with two natural boundaries, near the horizon and the asymptotic region when $r \longrightarrow \infty$. The near-horizon boundary has the same properties as its corresponding horizon in an asymptotically flat static spacetime, so in effective terms, it is possible to consider the Boulware and Hartle-Hawking vacuum states and obtain an entropy related to the Bekenstein-Hawking entropy for the thermal source. The other boundary substantiates the calculations with the AdS/CFT duality, resorting to the metric's Euclidean properties. The calculation of the Bekenstein-Hawking entropy with this formalism is related to the one presented in this article, resorting to the complementary source of Mukohyama-Israel, based on the Gibbons-Hawking formalism [9].

Appendix A: BTZ SPACETIME STRUCTURE

Consider a BTZ black hole, proposed by Banados *et. al.* [36] spacetime in $(2+1)$ dimensions. The outer metric is of the form

$$ds^2 = -\left(N^{\perp}\right)^2 dt^2 + \frac{1}{f(r)^2}dr^2 + r^2\left(d\phi + N^{\phi}dt\right)^2 \quad \text{(A1)}$$

where

$$N^{\perp} = f(r) = \sqrt{-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}}, \quad \text{(A2)}$$

$$N^{\phi} = -\frac{J}{2r^2}, \ \ |J| \leq Ml. \quad \text{(A3)}$$

Identifying M the mass of the black hole, J the angular momentum, l associated with the cosmological constant of the form

$$l^2 = \frac{1}{-\Lambda}. \quad \text{(A4)}$$

Then, it is possible to consider the $g_{\mu\nu}$ component and find the ergosphere region

$$r_{erg} = \sqrt{r_+ - r_-} = l\sqrt{M}, \quad \text{(A5)}$$

and

$$r_{\pm}^2 = \frac{Ml^2}{2}\left[1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right], \quad \text{(A6)}$$

Where $r_- \leq r_+ \leq r_{erg}$. It is possible to obtain the mass of the black hole M from (A6), as

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad \text{(A7)}$$

which is equivalent to

$$J = \frac{2r_+r_-}{l}. \quad \text{(A8)}$$

A coordinate transformation for the metric (A1) allows obtaining the Carter-Penrose diagram for the BTZ black hole

$$\frac{T^2}{1/2} - \frac{R^2}{1/2} = 1. \quad \text{(A9)}$$

This is essential given that a FIDO moves on a hyperbolic trajectory in BTZ spacetime [36–38].

Appendix B: SOLUTION TO THE SHELL MOTION EQUATION

This section discusses a possible solution to equation (33).

Let

$$a = \frac{M}{\mu} = \frac{2\pi\lambda R}{2\pi\lambda R_0} = 1, \quad \text{(B1)}$$

where μ is the mass of the shell in R_0 and M is the mass of the shell in R, with the condition that $R_0 \gg R$.

$$\frac{\mu}{R_0} = 2\pi\lambda. \quad \text{(B2)}$$

Then (33) is simplified to

$$\frac{dF}{d\tau} = \frac{2}{l}\sqrt{(F-M)F\left[\frac{4M^2l^2F}{a^2R_0^2(F-M)} - 1\right]}. \quad \text{(B3)}$$

It is useful to express the proper time τ in terms of the coordinate time t, as

$$d\tau = \sqrt{F}dt. \quad \text{(B4)}$$

Therefore,

$$\frac{dF}{dt} = \alpha\sqrt{F^2\left[F - \vartheta(F-M)\right]}, \quad \text{(B5)}$$

where

$$\alpha = \frac{4M}{aR_0}, \ \ \vartheta = \left[\frac{aR_0}{2Ml}\right]^2. \quad \text{(B6)}$$

Integrating (B5) yields

$$\int_{F_0}^{F} \frac{1}{\sqrt{F^2\left[F - \vartheta(F-M)\right]}} = \alpha\int_{t_0}^{t} dt \quad \text{(B7)}$$

$$\operatorname{arctanh}\left[\sqrt{\frac{F + \vartheta M - FM}{\vartheta M}}\right]_{F_0}^{F} = -\frac{\alpha\sqrt{\vartheta M}}{2}(t - t_0), \quad \text{(B8)}$$

where $\operatorname{arctanh}[x] = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = x + \frac{x^3}{3} + \frac{x^5}{7} + \ldots$

$$F = M + \left(\frac{R}{l}\right)^2, \ \ F_0 = M + \left(\frac{R_0}{l}\right)^2 \quad \text{(B9)}$$

and

$$\mathbf{A} = \frac{\sqrt{\vartheta M} + \sqrt{F_0 + \vartheta M - F_0\vartheta}}{\sqrt{\vartheta M} - \sqrt{F_0 + \vartheta M - F_0\vartheta}}, \quad \text{(B10)}$$

allows to obtain

$$R(t) = \sqrt{\frac{l^2\vartheta M}{1-\vartheta}\left\{\left[\frac{\mathbf{A}e^{-\alpha\sqrt{\vartheta M}(t-t_0)}-1}{\mathbf{A}e^{-\alpha\sqrt{\vartheta M}(t-t_0)}+1}\right]^2+1\right\}}. \tag{B11}$$

Let (B11) correspond to the shell motion equation in BTZ spacetime, as seen by a FIDO observer measuring a coordinate time t. Let r_+ correspond to the position of the horizon, which allows establishing that the horizon not only depends on the mass of the BTZ black hole, but also on the value of the cosmological constant (A4) [16, 17, 22–25, 39–44].

Appendix C: ENTANGLEMENT MODES FOR BTZ

Considering BTZ spacetime, it can be rewritten as

$$ds^2 = \frac{l^4}{r_+^2}\frac{(r+r_+)^2}{l^2}dUdV + r^2 d\phi \tag{C1}$$

where the coordinates (U, V) are null coordinates. Then, the geometry of the BTZ black hole is described by Figure 1. Under such conditions, let a scalar field Φ [45]

$$\mathcal{L}_M = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m^2\Phi^2\right]. \tag{C2}$$

The action that includes the space-time component and the matter fields is [36]

$$S = \int\left[\frac{1}{2\pi}\sqrt{-g}\left(R+\frac{2}{l^2}\right)+\mathcal{L}_M\right]d^3x. \tag{C3}$$

The variation δS leads to the KGE. Consider a possible solution for KGE

$$\Phi(t,r,\phi) = \frac{\varphi_\Omega(r)}{\sqrt{2\omega}}e^{-i\omega}e^{i\mathfrak{m}\phi}, \tag{C4}$$

where $\mathfrak{m}$ corresponds to the magnetic quantum number, associated to the angular part of the scalar field Φ. These orthogonal modes under Klein-Gordon inner product. Expanding (C4) into KGE leads to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rf(r)\frac{\partial\varphi_\Omega(r)}{\partial r}\right] + \varphi_\Omega(r)\left[\frac{\omega^2}{f(r)} - \frac{\mathfrak{m}^2}{r} - m^2\right] = 0, \tag{C5}$$

with $dr* = \frac{1}{f(r)}dr$. Allows simplifying

$$\frac{d}{dr*}\left[\frac{d\varphi_\Omega(r)}{dr*}\right] + \mathbf{T}(\omega,\mathfrak{m},m,r)\varphi_\Omega(r) = 0, \tag{C6}$$

where

$$\mathbf{T}(\omega,\mathfrak{m},m,r) = \omega^2 - \left(m^2+\frac{\mathfrak{m}}{r}\right)f(r). \tag{C7}$$

Under the WKB approximation [46], consider a harmonic solution of the form

$$\varphi_\Omega(r) = e^{-i\phi(r)}, \tag{C8}$$

with the condition that $\mathbf{T}(\omega,\mathfrak{m},m,r)$ varies very slightly, which is why $\phi''(r)$is very small

$$\phi(r) = \int\sqrt{\mathbf{T}(\omega,\mathfrak{m},m,r)}dr. \tag{C9}$$

(C6) is satisfied when

$$\phi''(r) \cong \frac{1}{2}\left|\frac{\mathbf{T}'}{\sqrt{\mathbf{T}}}\right| \ll \mathbf{T}, \quad \mathbf{T} = \mathbf{T}(\omega,\mathfrak{m},m,r). \tag{C10}$$

Considering one of the solutions of (C8), it is possible to write

$$\Phi(t,r,\phi) = \frac{e^{-i\left(\int\sqrt{\mathbf{T}}dr - \mathfrak{m}\phi\right)}e^{-i\omega t}}{\sqrt[4]{4\omega^2\mathbf{T}}}. \tag{C11}$$

If, in addition, $\underline{x} = r, \phi$ and $\Omega = \omega, \mathfrak{m}$

$$\Phi_\Omega(\underline{x}) = \frac{e^{-i\left(\int\sqrt{\mathbf{T}}dr - \mathfrak{m}\phi\right)}}{\sqrt[4]{4\omega^2\mathbf{T}}}, \tag{C12}$$

$$\Phi_\Omega(t,\underline{x}) = \Phi_\Omega(\underline{x})e^{-i\omega t}. \tag{C13}$$

Using tortoise Coordinates

$$\Phi(t,\underline{x}) = \Phi_\Omega(r*,\underline{x}) = \Phi_\Omega(r*,\underline{x})e^{-i\omega r*}. \tag{C14}$$

For (C14), it is possible to define the incoming and outgoing modes of the scalar field over BTZ space-time as

$$\Phi_\Omega^{(+)}(r*,\underline{x}) = \Phi_\Omega(\underline{x})e^{i\omega r*}, \text{ incoming modes.} \tag{C15}$$

$$\Phi_\Omega^{(-)}(r*,\underline{x}) = \Phi_\Omega(\underline{x})e^{-i\omega r*}, \text{ outgoing modes.} \tag{C16}$$

Thus, the modes (C14)

$$\Phi(u,v,\underline{x}) = \Phi_\Omega(\underline{x})e^{-\frac{i\omega v}{2}}e^{\frac{i\omega u}{2}}. \tag{C17}$$

Figure 3 shows a Carter-Penrose diagram for a BTZ black hole. From (C17), the outgoing modes with $u = 0$ and the incoming modes $v = 0$ are defined

$$\Phi^\epsilon(t,\underline{x}) = \Phi_\Omega^{in}(\underline{x})e^{-\frac{i\omega v}{2}} = \Phi_\Omega^{in}(v,\underline{x}). \tag{C18}$$

$$\Phi^\epsilon(t,\underline{x}) = \Phi_\Omega^{out}(\underline{x})e^{\frac{i\omega u}{2}} = \Phi_\Omega^{out}(u,\underline{x}). \tag{C19}$$

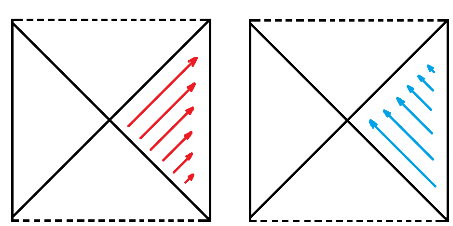

FIG. 5. Left: Outgoing modes of the scalar field in the Carter-Penrose diagram for a BTZ black hole. Right: Incoming modes of the scalar field in the Carter-Penrose diagram for a BTZ black hole.

The modes $\Phi_{\Omega}^{in}(v,\underline{x})$ and $\Phi_{\Omega}^{out}(v,\underline{x})$ are written in terms of the Kruskal null coordinates U, V as

$$\Phi_{\Omega}^{\epsilon}\left(U,\underline{x}\right) = \Theta(\epsilon,V)\Phi_{\Omega}^{in}(v,\underline{x}) = \Theta(\epsilon,V)\Phi_{\Omega}^{in}(\underline{x})e^{-\frac{i\omega v}{2}}, \tag{C20}$$

$$\Phi_{\Omega}^{\epsilon}\left(V,\underline{x}\right) = \Theta(-\epsilon,V)\Phi_{\Omega}^{out}(u,\underline{x}) = \Theta(-\epsilon,U)\Phi_{\Omega}^{out}(\underline{x})e^{-\frac{i\omega u}{2}}, \tag{C21}$$

where the function $\Theta_{\epsilon}(x)$ is

$$\Theta_{\epsilon}(x) = \frac{1}{2}\left[\Theta(-\epsilon U) + \Theta(\epsilon V)\right]. \tag{C22}$$

This allows defining $(+) \in \mathbb{R}$ and $(-) \in \mathbb{L}$ of the Carter-Penrose diagram in Figure (3). Consequently, the complete modes for the region R are

$$\Phi_{\Omega}^{(+)}\left(u,v,\underline{x}\right) = \Phi_{\Omega}^{(+)}\left(v,\underline{x}\right) + \Phi_{\Omega}^{(+)}\left(u,\underline{x}\right). \tag{C23}$$

And the complete modes for the region L are

$$\Phi_{\Omega}^{(-)}\left(u,v,\underline{x}\right) = \Phi_{\Omega}^{(-)}\left(v,\underline{x}\right) + \Phi_{\Omega}^{(-)}\left(u,\underline{x}\right). \tag{C24}$$

Consequently, it is possible to obtain the modes of the scalar field in BTZ, which are contained in the R and L regions as

$$\Phi_{\Omega}\left(u,v,\underline{x}\right) = \Phi_{\Omega}^{(-)}\left(u,v,\underline{x}\right) + \Phi_{\Omega}^{(+)}\left(u,v,\underline{x}\right). \tag{C25}$$

Therefore, there are two representations in terms of the Killing-Boulware modes (KB^{*})

$$\Phi_{\Omega}^{(\epsilon)}(U,\underline{x}), \;\; \Phi_{\Omega}^{(\epsilon)}(V,\underline{x}), \tag{C26}$$

that are seen by a FIDO observer. And another representation of Hartle-Hawking modes

$$\Psi_{\Omega}^{(\epsilon)}(U,\underline{x}), \;\; \Psi_{\Omega}^{(\epsilon)}(V,\underline{x}), \tag{C27}$$

where such modes are seen by a FFO observer[1] [9, 10, 14, 25, 37] . The modes are orthogonal for a FIDO observer when

$$\left(\Phi_{\Omega}^{(\epsilon)+}(\underline{x}), \Phi_{\Omega'}^{(\epsilon')+}(\underline{x}')\right) = \left(\Phi_{\Omega}^{(\epsilon)-}(\underline{x}), \Phi_{\Omega'}^{(\epsilon')-}(\underline{x}')\right) = \epsilon(\omega)\delta_{\Omega\Omega'}\delta_{\epsilon\epsilon'}. \tag{C28}$$

[1] The KB^{*} modes have been defined as those modes of the scalar field that are measured by a FIDO observer. This type of observer is accelerated to a fixed distance above the BTZ spacetime horizon. And which are not the KB modes of the quantum field for Schwarzschild space-time that were originally defined. The HH^{*} modes are the modes of the quantum field, which are seen by a FFO observer, falling radially in the direction of the black hole in BTZ spacetime. And not to be confused with the HH modes defined for Schwarzschild spacetime .

and they are null when [31]

$$\left(\Phi_{\Omega}^{(\epsilon)\pm}(\underline{x}), \Phi_{\Omega'}^{(\epsilon')\mp}(\underline{x}')\right) = 0. \tag{C29}$$

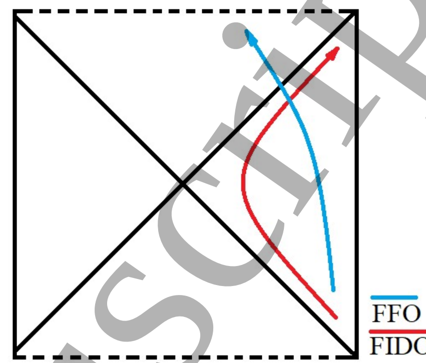

FIG. 6. Types of FIDO and FFO observers in the Carter-Penrose diagram for a BTZ black hole..

And, for a FFO it is

$$\left(\Psi_{\Omega}^{(\epsilon)+}(\underline{x}), \Psi_{\Omega'}^{(\epsilon')+}(\underline{x}')\right) = \left(\Psi_{\Omega}^{(\epsilon)-}(\underline{x}), \Psi_{\Omega'}^{(\epsilon')-}(\underline{x}')\right) = \epsilon(\omega)\delta_{\Omega\Omega'}\delta_{\epsilon\epsilon'} \tag{C30}$$

And they are null when [31]

$$\left(\Psi_{\Omega}^{(\epsilon)\pm}(\underline{x}), \Psi_{\Omega'}^{(\epsilon')\mp}(\underline{x}')\right) = 0. \tag{C31}$$

The relationship between the KB^{*} and los modos HH^{*} modes is mediated by a Bogoliubov transformation of the form [45]

$$\Psi_{\Omega}^{(\epsilon)}(\underline{x}) = \Phi_{\Omega}^{(\epsilon)}(\underline{x})\cosh(x) + \Phi_{\Omega}^{(-\epsilon)}(\underline{x})\sinh(x). \tag{C32}$$

To this end, consider the maximally extended BTZ spacetime as

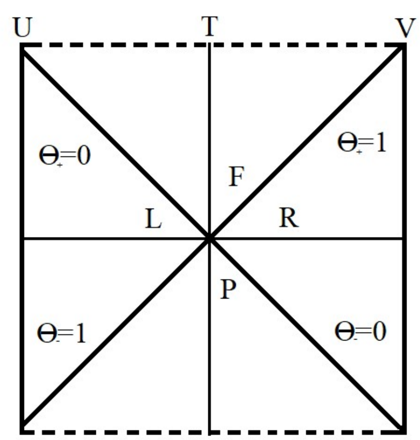

FIG. 7. The Heaviside step function $\Theta_{\epsilon(x)}$ for the Carter-Penrose diagram of a BTZ black hole.

Figure 7 defines the Heaviside step function $\Theta_{\epsilon(x)}$ for the Carter-Penrose diagram of a BTZ black hole. Taking the null coordinates U, V which are rewritten in terms of the surface gravity

$$U = -e^{-\kappa_0 u}, \;\; V = e^{\kappa_0 v}. \tag{C33}$$

This allows rewriting

$$e^{-i\omega t_{\epsilon\epsilon'}} = e^{-i\omega t}\left[e^{\frac{\pi\omega\epsilon'}{2\kappa_0}}\Theta_\epsilon + e^{\frac{-\pi\omega\epsilon'}{2\kappa_0}}\Theta_{-\epsilon}\right]. \tag{C34}$$

Let

$$e^{-\frac{\pi|\omega|}{\kappa_0}} = \tanh\chi, \tag{C35}$$

then

$$e^{-\frac{\pi|\omega|}{2\kappa_0}} = \sqrt{\frac{\sinh\chi}{\cosh\chi}}. \tag{C36}$$

So (C34) is simplified to

$$e^{-i\omega t_{\epsilon\epsilon'}} = \frac{e^{-i\omega t}}{\sqrt{\sinh\chi\cosh\chi}}\left[\cosh\chi\;\Theta_\epsilon + \sinh\chi\;\Theta_{-\epsilon}\right], \tag{C37}$$

$$e^{-i\omega t_{\epsilon\epsilon(\omega)}}\sqrt{\sinh\chi\cosh\chi} = e^{-i\omega t}\left[\cosh\chi\;\Theta_\epsilon + \sinh\chi\;\Theta_{-\epsilon}\right]. \tag{C38}$$

When considering the modes of the field by regions R and L, according to (C14), it follows that

$$\Phi_\Omega^{(\epsilon)} = \Phi_\Omega(\underline{x})e^{-i\omega t}\Theta_\epsilon(x) \;\in R, \tag{C39}$$

$$\Phi_\Omega^{(-\epsilon)} = \Phi_\Omega(\underline{x})e^{-i\omega t}\Theta_{-\epsilon}(x) \;\in L. \tag{C40}$$

From the foregoing, it follows that (C38) is simplified to

$$e^{-i\omega t_{\epsilon\epsilon(\omega)}}\sqrt{\sinh\chi\cosh\chi}\;\Phi_\Omega = \Phi_\Omega^{(\epsilon)}\cosh\chi + \Phi_\Omega^{(-\epsilon)}\sinh\chi. \tag{C41}$$

For (C41), the HH^* modes are recognized for the scalar field in BTZ as

$$\Psi_\Omega^\epsilon(\underline{x}) = e^{-i\omega t_{\epsilon\epsilon(\omega)}}\sqrt{\sinh\chi\cosh\chi}\;\Phi_\Omega(\underline{x}). \tag{C42}$$

From the foregoing, it is possible to obtain the Bogoliubov transformation between HH^* modes and KB^* modes defined as [10, 25, 28, 30, 45]

$$\Psi_\Omega^\epsilon(\underline{x}) = \Phi_\Omega^\epsilon(\underline{x})\cosh\chi + \Phi_\Omega^{-\epsilon}(\underline{x})\sinh\chi. \tag{C43}$$

Appendix D: QUANTUM FORMULATION

In this section, a quantum approximation to the scalar field in BTZ spacetime is considered. Let the scalar field modes be of the form [47]

$$\Phi_\Omega(t,\underline{x}) = \frac{e^{-i\left[\omega t - \int\sqrt{\mathbf{T}}dr + \mathfrak{m}\phi\right]}}{\sqrt[4]{4\omega^2\mathbf{T}V^2}}, \tag{D1}$$

where V is the 2-Volume for BTZ spacetime. From (D1), it is possible to define

$$F_\Omega = \frac{e^{-i\left[\omega t - \int\sqrt{\mathbf{T}}dr + \mathfrak{m}\phi\right]}}{\sqrt[4]{4\omega^2\mathbf{T}V^2}}, \tag{D2}$$

and its conjugate hermitian. Therefore, the field operator $\Phi_\Omega(t,\underline{x})$, is rewritten as

$$\Phi_\Omega(t,\underline{x}) = \sum_\Omega\left[a_\Omega F_\Omega + b_\Omega^\dagger F_\Omega^*\right] \tag{D3}$$

and its conjugate hermitian. Therefore, this allows obtaining the Hamiltonian of the scalar field in the BTZ spacetime

$$H = \sum_\Omega\frac{1}{\sqrt{\mathbf{T}}}\omega\left[a_\Omega^\dagger a_\Omega + b_\Omega^\dagger b_\Omega\right] + Z.P.E., \tag{D4}$$

where $Z.P.E$ is recognized as the zero-point energy, $N = a_\Omega^\dagger a_\Omega$ and $\bar{N} = b_\Omega^\dagger b_\Omega$. Thus, the Hamiltonian operator has been estimated for the scalar field in the proximity of the BTZ hole [10, 25, 32, 45, 48, 49].

Appendix E: THERMO FIELD DYNAMICS ON BTZ BLACK HOLE

Considering that the BTZ spacetime was written as (C1), where r_+ and the null coordinates U, V. Which allows to build the Carter-Penrose diagram, see Figure 3 [32, 36, 38, 50]. Where, $(+) \in R$ and $(-) \in L$. In this context, the scalar field $\Phi(t,\underline{x})$ has a Hamiltonian of eigenvalues constituting the eigenvalues as

$$H^{(+)}, |n\rangle^{(+)} \in R, \;\; H^{(-)}, |n\rangle^{(-)} \in L, \tag{E1}$$

and the eigenvalue equation

$$H^{(+)}\,|n\rangle^{(+)} = E^{(+)}\,|n\rangle^{(+)} \;\;\in R, \tag{E2}$$

$$H^{(-)}\,|n\rangle^{(-)} = E^{(-)}\,|n\rangle^{(-)} \;\;\in L. \tag{E3}$$

The TFD technique establishes the field Hamiltonian as a state of entanglement between the field

$$\Phi^{(+)}(t,\underline{x}) \in R, \;\; \Phi^{(-)}(t,\underline{x}) \in L. \tag{E4}$$

Consequently, the Hamiltonian H of the complete field is determined as

$$H = H^{(+)} - H^{(-)}, \tag{E5}$$

where the modes of the field $\Phi_\Omega(t,\underline{x})$ are expressed by regions R, L in terms of the creation and annihilation operators of the particles and corresponding antiparticles (D3)

$$\Phi_\Omega^{(\pm)}(t,\underline{x}) = \sum_\Omega\left[a_\Omega^{(\pm)}F_\Omega + b_\Omega^{\dagger(\pm)}F_\Omega^*\right]. \tag{E6}$$

From the foregoing, eight modes are necessary to describe the scalar field $\Phi_\Omega(t,\underline{x})$ in BTZ. Consequently, having the commutation relations for the operators of creation and annihilation per region R, L defined as $\left[a_\Omega^{(\pm)}, a_\Omega^{\dagger(\pm)}\right] = \left[b_\Omega^{(\pm)}, b_\Omega^{\dagger(\pm)}\right] = 1$ and the other possible combinations are null.

On the other hand, the normalization condition on the quantum states of the field is

$$\left|m^{(+)}, n^{(+)}, m^{(-)}, n^{(-)}\right\rangle = \left|m^{(+)}\right\rangle \left|n^{(+)}\right\rangle \left|m^{(-)}\right\rangle \left|n^{(-)}\right\rangle, \tag{E7}$$

for modes $\omega < 0$ and $\omega > 0$.

The completeness relation

$$I = \sum_{m^{(\pm)}, n^{(\pm)}} \left|m^{(+)}, n^{(+)}, m^{(-)}, n^{(-)}\right\rangle \left\langle m^{(+)}, n^{(+)}, m^{(-)}, n^{(-)}\right|. \tag{E8}$$

In addition, the empty states per region are defined as

$$a_\Omega^{(\pm)} \left|0\right\rangle_B^{(\pm)\pm} = 0, \tag{E9}$$

$$b_\Omega^{(\pm)} \left|0\right\rangle_B^{(\pm)\pm} = 0, \tag{E10}$$

Vacuum state for particles with $\omega > 0$ and $\omega < 0$ in the regions R and L. This makes it possible to write the temperature-dependent vacuum state as

$$\left|0(\beta)\right\rangle_B^{\pm} = \left|0^{(+)}(\beta), 0^{(-)}(\beta)\right\rangle_B^{\pm}. \tag{E11}$$

Consequently, the thermal vacuum state is

$$\left|0(\beta)\right\rangle_B^{+} = \sum_n \frac{e^{\frac{-\beta E_n}{2}}}{n!\sqrt{Z(\beta)}} \left[a_\Omega^{\dagger(+)}\right]^n \left[a_\Omega^{\dagger(-)}\right]^n \left|0^{(+)}, 0^{(-)}\right\rangle_B^{+}, \tag{E12}$$

$$\left|0(\beta)\right\rangle_B^{-} = \sum_m \frac{e^{\frac{-\beta E_m}{2}}}{m!\sqrt{Z(\beta)}} \left[b_\Omega^{\dagger(+)}\right]^m \left[b_\Omega^{\dagger(-)}\right]^m \left|0^{(+)}, 0^{(-)}\right\rangle_B^{-}. \tag{E13}$$

and the thermal vacuum state complete for R and L regions on BTZ spacetime, $\left|0(\beta)\right\rangle_B = \left|0(\beta)\right\rangle_B^{+} \left|0(\beta)\right\rangle_B^{-}$

$$\left|0(\beta)\right\rangle_B = \sum_{m,n} \frac{e^{\frac{\beta}{2}(E_n+E_m)}}{m!n!Z(\beta)} \left[a_\Omega^{\dagger(+)}\right]^n \left[a_\Omega^{\dagger(-)}\right]^n \left[b_\Omega^{\dagger(+)}\right]^m \left[b_\Omega^{\dagger(-)}\right]^m \left|0^{(+)}, 0^{(-)}\right\rangle_B^{+} \left|0^{(+)}, 0^{(-)}\right\rangle_B^{-}. \tag{E14}$$

If the vacuum states are subject to the normalization condition, such that it is possible to establish that the partition function is

$$Z^{\pm}(\beta) = \frac{1}{1 - e^{-\beta|\omega|}}, \tag{E15}$$

which makes it possible to obtain the vacuum state as

$$\left|0(\beta)\right\rangle_B^{+} = \sqrt{1 - e^{-\beta|\omega|}} \exp\left[e^{-\frac{\beta|\omega|}{2}} a^{\dagger(+)} a^{\dagger(-)}\right] \left|0^{(+)}, 0^{(-)}\right\rangle, \tag{E16}$$

$$\left|0(\beta)\right\rangle_B^{-} = \sqrt{1 - e^{-\beta|\omega|}} \exp\left[e^{-\frac{\beta|\omega|}{2}} b^{\dagger(+)} b^{\dagger(-)}\right] \left|0^{(+)}, 0^{(-)}\right\rangle. \tag{E17}$$

And the expected value of the occupation number as

$$\left\langle N\right\rangle^{+} = {}_B^{+}\left\langle 0(\beta)\right| a^{\dagger(+)} a^{(+)} \left|0(\beta)\right\rangle_B^{+}, \tag{E18}$$

$$\left\langle \bar{N}\right\rangle^{-} = {}_B^{-}\left\langle 0(\beta)\right| b^{\dagger(+)} b^{(-)} \left|0(\beta)\right\rangle_B^{+}. \tag{E19}$$

To this end, it is necessary to express the temperature-dependent creation and annihilation operators β as a Bogoliubov transformation [29]. This enables us to compute the expected value of the occupation number [7, 10, 25, 27, 29, 30, 51, 52].

$$\left\langle N\right\rangle^{+} = \left\langle \bar{N}\right\rangle^{-} = \frac{1}{e^{\beta|\omega|} - 1}. \tag{E20}$$

Appendix F: WIGHTMAN FUNCTION FOR SCALAR FIELD IN BTZ SPACETIME

The Wightman function corresponds to a two-point Green's function of the form [10, 25, 53, 54]

$$\left\langle T_{\mu\nu}(x, x')\right\rangle = \mathcal{D}_{\mu\nu'} W(x, x'), \tag{F1}$$

where

$$\mathcal{D}_{\mu\nu'} = \partial_{(\mu}\partial_{\nu')} - \frac{g^{\mu\nu}}{2}\left(\partial^{\beta}\partial_{\beta'} - m^2\right) \tag{F2}$$

and

$$\Phi^2(x, x') = W(x, x'). \tag{F3}$$

Such that $W(x, x')^{+}$, is defined as the Wightman function for positive frequency modes, in other

$$W(x, x') = \left\langle 0\right| \Phi(\underline{x}) \Phi^{*}(\underline{x}') \left|0\right\rangle. \tag{F4}$$

Under the Killing-Boulware (KB^*) vacuum state scheme

$$W(x, x')_{KB^*} = {}_{KB^*}\left\langle 0\right| \Phi_\Omega^{(\epsilon)}(\underline{x}) \Phi_{\Omega'}^{*(\epsilon)}(\underline{x}') \left|0\right\rangle_{KB^*}. \tag{F5}$$

And also, under the Hartle-Hawking (HH^*) vacuum state scheme

$$W(x,x')_{HH^*} = {}_{HH^*}\langle 0|\, \Psi_\Omega^{(\epsilon)}(\underline{x})\Psi_{\Omega'}^{*(\epsilon)}(\underline{x}')\,|0\rangle_{HH^*}. \quad \text{(F6)}$$

Moreover, considering that the modes of the field have been written as (D3) for each of the regions R, L, in other words

$$\Phi_\Omega^{(\epsilon)}(t,\underline{x}) = \sum_{\epsilon,\Omega}\left[a_\Omega^{(\epsilon)}F_\Omega^{(\epsilon)} + b_\Omega^{\dagger(\epsilon)}F_\Omega^{*(\epsilon)}\right] \quad \text{(F7)}$$

and its conjugate hermitian, where $\epsilon = \pm$.

Considering the KB^* vacuum state, it is expressed as

$$\begin{aligned}|0\rangle_{KB^*} &= \left|0^{(+)},0^{(-)}\right\rangle^+_{KB^*}\otimes\left|0^{(+)},0^{(-)}\right\rangle^-_{KB^*}\\ &= \left|0^{(\epsilon)}\right\rangle_{KB^*}\otimes\left|0^{(-\epsilon)}\right\rangle_{KB^*}.\end{aligned} \quad \text{(F8)}$$

And the commutation rules

$$\left[a_\Omega^{(\epsilon)}, a_{\Omega'}^{\dagger(\epsilon')}\right] = \left[b_\Omega^{(\epsilon)}, b_{\Omega'}^{\dagger(\epsilon')}\right] = \epsilon\epsilon(\omega)\delta_{\epsilon\epsilon'}\delta_{\Omega\Omega'}. \quad \text{(F9)}$$

Therefore

$$W(x,x')_{KB^*} = \sum_{\epsilon,\Omega,\epsilon',\Omega'} F_\Omega^{*(\epsilon)}F_{\Omega'}^{(\epsilon)}\Theta(\epsilon\omega), \quad \text{(F10)}$$

where $\epsilon(\omega) = \operatorname{sgn}\omega = \Theta\epsilon\omega$, $\delta_{\Omega\Omega'} = \delta_{\omega\omega'}\delta_{\mathfrak{m}\mathfrak{m}'}$, $\underline{x} = r,\phi$ and $\Omega = \omega,\mathfrak{m}$. The Wightman function under the HH^* scheme is

$$W(x,x')_{HH^*} = {}_{HH^*}\langle 0|\, \Psi_\Omega^{*(\epsilon)}(\underline{x})\Psi_{\Omega'}^{(\epsilon)}(\underline{x}')\,|0\rangle_{HH^*}, \quad \text{(F11)}$$

where the scalar field has been written as

$$\Psi_\Omega^{(\epsilon)}(U,V,x) = \sum_{\epsilon,\Omega}\left[d_\Omega^{(\epsilon)}G_\Omega + f_\Omega^{\dagger(\epsilon)}G_\Omega^*\right] \quad \text{(F12)}$$

and its conjugate hermitian. At this point it should be clarified that there is an equivalence between the KB^* modes and the HH^* modes. Such equivalence is mediated by a Bogoliubov transformation (C38). Considering the KB^* vacuum state, it is expressed as

$$\begin{aligned}|0\rangle_{HH^*} &= \left|0^{(+)},0^{(-)}\right\rangle^+_{HH^*}\otimes\left|0^{(+)},0^{(-)}\right\rangle^-_{HH^*}\\ &= \left|0^{(\epsilon)}\right\rangle_{HH^*}\otimes\left|0^{(-\epsilon)}\right\rangle_{HH^*}\end{aligned} \quad \text{(F13)}$$

and the commutation rules $\left[d_\Omega^{(\epsilon)}, d_{\Omega'}^{\dagger(\epsilon')}\right] = \left[f_\Omega^{(\epsilon)}, f_{\Omega'}^{\dagger(\epsilon')}\right] = \epsilon\epsilon(\omega)\delta_{\epsilon\epsilon'}\delta_{\Omega\Omega'}$. This makes it possible to estimate

$$\begin{aligned}W(x,x')_{HH^*} &= {}_{HH^*}\langle 0|\, \Psi_\Omega^{(\epsilon)}(\underline{x})\Psi_{\Omega'}^{*(\epsilon)}(\underline{x}')\,|0\rangle_{HH^*}\\ &= \sum_{\epsilon,\Omega,\epsilon',\Omega'} G_\Omega^{*(\epsilon)}G_{\Omega'}^{(\epsilon)}\Theta(\epsilon\omega).\end{aligned} \quad \text{(F14)}$$

Therefore, it is possible to obtain

$$\begin{aligned}W(x,x')_{HH^*} = \sum_{\epsilon,\Omega}&\Theta(\epsilon\omega)\times\\ &\left[F_\Omega^{(\epsilon)}F_\Omega^{*(\epsilon)}\cosh^2\chi + F_\Omega^{(-\epsilon)}F_\Omega^{*(-\epsilon)}\sinh^2\chi\right].\end{aligned} \quad \text{(F15)}$$

The difference between (F15) and (F10) is

$$(W_{HH^*} - W_{KB^*})(x,x') = \sum_{\pm,\Omega}\Theta(+\omega)\sinh^2\chi\left[F_\Omega^{(+)}F_\Omega^{*(+)} + F_\Omega^{(-)}F_\Omega^{*(-)}\right] + \Theta(-\omega)\sinh^2\chi\left[F_\Omega^{(+)}F_\Omega^{*(+)} + F_\Omega^{(-)}F_\Omega^{*(-)}\right] \quad \text{(F16)}$$

It is possible to constrain (F16) to one of the regions

$$\begin{aligned}(W_{HH^*} - W_{KB^*})(x,x') = \sum_\Omega &\sinh^2\chi\\ &\left[F_\Omega^{*(\epsilon)}F_\Omega^{(\epsilon)} + F_\Omega^{*(-\epsilon)}F_\Omega^{(-\epsilon)}\right].\end{aligned} \quad \text{(F17)}$$

It follows that (F17) is simplified to

$$(W_{HH^*} - W_{KB^*})(x,x') = \sum_\Omega \frac{1}{e^{\beta\omega}-1}F_\Omega^{*(+)}F_\Omega^{(+)}. \quad \text{(F18)}$$

From the foregoing, it is feasible to obtain the component $\langle T_{00}(x,x')\rangle$ as

$$\partial_0\partial_{0'}\left(W_{HH^*}-W_{KB^*}\right)(x,x') = \frac{1}{2}\sum_{\omega}\frac{\omega}{e^{\beta\omega}-1}\sum_{\mathfrak{m}}\varphi^*_{\Omega}(r)\varphi(r)_{\Omega} \tag{F19}$$

the sum over ω, in the limit to the continuum can be expressed as an integral

$$\sum_{\omega}\frac{\omega}{e^{\beta\omega}-1}\longrightarrow\int_{-\infty}^{\infty}\frac{1}{e^{\beta\omega}-1}d\omega = 2\int_0^{\infty}\frac{1}{e^{\beta\omega}-1}d\omega. \tag{F20}$$

From the foregoing, it follows that

$$\partial_0\partial_{0'}\left(W_{HH^*}-W_{KB^*}\right)(x,x') = \int_0^{\infty}\frac{\omega}{e^{\beta\omega}-1}d\omega\sum_{\mathfrak{m}}\varphi^*_{\Omega}(r)\varphi(r)_{\Omega}. \tag{F21}$$

Considering the BTZ metric (C1), where it follows that for the scalar field it is possible to obtain the KGE and whose modes (C4) make it possible to obtain the radial equation determined. It is possible to consider a new transformation of the radial component as $\varphi(r)_{\Omega} = \frac{1}{\sqrt{rf(r)}}\psi_{\Omega}(r)$. Which makes it possible to obtain

$$\frac{d^2\psi(r)_{\Omega}}{dr^2} + \mathbf{T}^{*2}\psi(r)_{\Omega} = 0 \tag{F22}$$

where

$$\mathbf{T}^{*2} = \frac{1}{f(r)}\left\{\frac{\omega^2}{f(r)} - m^2 - \frac{\mathfrak{m}^2}{r} + B\right\} \tag{F23}$$

and

$$B = \frac{1}{2}\left[-\frac{d}{dr}\left(\frac{df(r)}{dr} - \frac{1}{2}f(r)\right) + \frac{d}{dr}\left(\ln|f(r)^{1/2}|\right) - \frac{f(r)}{2r^2}\right]. \tag{F24}$$

Under the WKB approximation, the component $\varphi(r)_{\Omega}$is

$$\psi_{\Omega}(r) = \frac{1}{\sqrt[4]{4\omega^2\mathbf{T}^{*2}}}e^{-i\int\mathbf{T}^*dr}. \tag{F25}$$

This allows for $\varphi^*_{\Omega}(r)\varphi(r)_{\Omega} = \frac{1}{2\omega rf(r)\mathbf{T}^*}$. Therefore, it follows that

$$\partial_0\partial_{0'}\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'} = \frac{1}{2r}\int_0^{\infty}\frac{\omega}{e^{\beta\omega}-1}d\omega\times\frac{1}{\omega}\sum_{\mathfrak{m}}\frac{1}{f(r)\mathbf{T}^*}. \tag{F26}$$

The sum over $\mathfrak{m}$ in the limit to the continuum is rewritten as

$$\sum_{\mathfrak{m}}\frac{1}{f(r)\mathbf{T}^*}\longrightarrow\int_0^{\mathfrak{m}_{max}}\frac{1}{f(r)\mathbf{T}^*}d\mathfrak{m}, \tag{F27}$$

which makes it possible to rewrite

$$\partial_0\partial_{0'}\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'} = \frac{1}{2r}\int_0^{\infty}\frac{\omega}{e^{\beta\omega}-1}d\omega\frac{1}{\omega}\times\int_0^{\mathfrak{m}_{max}}\frac{1}{f(r)\mathbf{T}^*}d\mathfrak{m}. \tag{F28}$$

Choosing that

$$\mathbf{T}^{*2} = \mathbf{T}^{*2}(r,\omega,\mathfrak{m})|_{\mathfrak{m}=\mathfrak{m}_{max}} = 0. \tag{F29}$$

$$\mathbf{T}^{*2}_{max} = \frac{1}{f(r)}\left\{\frac{\omega^2}{f(r)} - m^2 - \frac{\mathfrak{m}^2_{max}}{r} + B\right\} = 0 \tag{F30}$$

in other words, $\frac{\mathfrak{m}^2_{max}}{r} = \frac{\omega^2}{f(r)} - m^2 + B = p^2$. Therefore, it follows that (F29) is simplified to

$$\mathbf{T}^*(r,\omega,\mathfrak{m})|_{\mathfrak{m}=\mathfrak{m}_{max}} = \sqrt{\frac{1}{f(r)}\left[p^2 - \frac{\mathfrak{m}^2}{r}\right]}. \tag{F31}$$

Inserting (F31) in

$$\int_0^{\mathfrak{m}_{max}}\frac{1}{f(r)\mathbf{T}^*}d\mathfrak{m} = \int_0^{\mathfrak{m}_{max}}\frac{1}{f(r)\sqrt{\frac{1}{f(r)}\left[p^2-\frac{\mathfrak{m}^2}{r}\right]}}d\mathfrak{m}, \tag{F32}$$

And then (F28)

$$\partial_0\partial_{0'}\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'}=\frac{1}{2r}\int_0^\infty\frac{\omega}{e^{\beta\omega}-1}d\omega\frac{1}{\omega}\frac{\sqrt{rf(r)}}{f(r)}\int_0^{\mathfrak{m}_{max}}\frac{1}{\sqrt{[p^2r-\mathfrak{m}^2]}}d\mathfrak{m}. \tag{F33}$$

Integrating by $\mathfrak{m}$ is

$$\int_0^{\mathfrak{m}_{max}}\frac{1}{\sqrt{[p^2r-\mathfrak{m}^2]}}d\mathfrak{m}=\arctan\left[\frac{\mathfrak{m}_{max}}{\sqrt{p^2r-\mathfrak{m}^2_{max}}}\right], \tag{F34}$$

in a Taylor series

$$\arctan\left[\frac{\mathfrak{m}_{max}}{\sqrt{p^2r-\mathfrak{m}^2_{max}}}\right]\approx 1. \tag{F35}$$

Consequently, (F33) is simplified to

$$\partial_0\partial_{0'}\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'}=\frac{1}{2r}\int_0^\infty\frac{\omega}{e^{\beta\omega}-1}d\omega \times\frac{1}{\omega}\frac{\sqrt{rf(r)}}{f(r)}. \tag{F36}$$

With local energy per mode

$$E=\frac{\omega}{\sqrt{f(r)}} \tag{F37}$$

and the local temperature

$$T(r)=\frac{T_H}{\sqrt{f(r)}},\ \ T_H=\frac{1}{\beta}. \tag{F38}$$

Then

$$\beta\omega=\frac{E}{T(r)}. \tag{F39}$$

Considering the relativistic energy $E^2=m^2+p^2$, with the condition that $E\gg m$, makes it possible to obtain

$$\frac{\omega d\omega}{f(r)}=pdp. \tag{F40}$$

Then

$$\partial_0\partial_{0'}\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'}=\int_0^\infty\frac{E}{e^{E/T(r)}-1}pdpf(r)\Delta \tag{F41}$$

where $\Delta=\frac{2\pi}{\omega^2\sqrt{r}}$. It is possible to assert that $\Delta|_{r=r_+}=\frac{2\pi}{\omega^2\sqrt{r_+}}\approx cte$. This is possible, since the shell is in a meta-stable phase according to (B11) [16, 17]. Moreover, it follows that

$$g^{00}g_{00}+g^{0\beta}g_{0\beta}=1. \tag{F42}$$

Then $g^{00}g_{00}=1$, also $g^{00}\partial_0=\partial^0$, then(F41)

$$-\partial^0\partial_{0'}\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'}=\int_0^\infty\frac{E}{e^{E/T(r)}-1}pdp\,\Delta. \tag{F43}$$

Considering (F2) with $\mu=\nu=0$

$$\mathcal{D}_{00'}\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'}=\left[\partial({}_0\partial_{0'})-\frac{g_{00}}{2}\left(\partial^0\partial_{0'}-m^2\right)\right]\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'} \tag{F44}$$

and $\left(\partial^\beta\partial_{\beta'}-m^2\right)\left(W_{HH^*}-W_{KB^*}\right)(x,x')|_{x=x'}$, where $|\partial_\mu\Phi|^2=\partial_\mu\Phi^*\partial_\mu\Phi$. Therefore

$$g^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi=\left[-\frac{\omega^2}{f(r)}\varphi^2_\Omega(r)+f(r)\left|\frac{\partial\varphi_\Omega(r)}{\partial r}\right|^2\right]\frac{1}{2\omega}+\frac{1}{r^2}\frac{\varphi^2_\Omega(r)}{2\omega}\mathfrak{m}^2, \tag{F45}$$

where $\Omega=\omega,\mathfrak{m}$ for BTZ. In addition, considering that $\varphi_\Omega(r)$ is broken down into the incoming (C17) and outgoing (C18) modes. Consequently,

$$g^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi+\Phi^2_\Omega(r)m^2=\Phi^2_\Omega(r)\frac{\alpha_{\omega,\mathfrak{m}}(r)}{2\omega}. \tag{F46}$$

Which makes it possible to write

$$g^{\mu\nu}\partial_\mu\partial_\nu+m^2=\frac{\alpha_{\omega,\mathfrak{m}}(r)}{2\omega}. \tag{F47}$$

In the limit to the continuum over ω

$$\left[\partial^{\mu'}\partial_\mu + m^2\right]\left(W_{HH^*} - W_{KB^*}\right)(x,x') = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\omega^{-1}}{e^{\beta\omega}-1}\sum_{\mathfrak{m}} F_\Omega^{*(+)} F_\Omega^{(+)}\alpha_{\omega,\mathfrak{m}}(r). \tag{F48}$$

Consequently

$$\langle T_{\mu\nu}(x,x')\rangle = -\frac{g_{00}}{2}\left(\partial^{\beta'}\partial_\beta - m^2\right)\left(W_{HH^*} - W_{KB^*}\right)(x,x'). \tag{F49}$$

For the time components of the tensor $T_{\mu\nu}$

$$\begin{aligned} g^{00}\langle T_{00}(x,x')\rangle = &-\partial^0\partial_{0'}\left(W_{HH^*} - W_{KB^*}\right)(x,x') \\ &-\left(\partial^\beta\partial_{\beta'} - m^2\right)\left(W_{HH^*} - W_{KB^*}\right)(x,x') \end{aligned} \tag{F50}$$

where, the first term on the right is determined by (F41) and is proportional to the frequency ω, while the second term is proportional to $1/\omega$. Consequently, in limit of high frequencies, the second term can be disregarded, in other words

$$\left\langle T_0^0(x,x')\right\rangle = \sigma(r) = -\int_0^\infty \frac{E}{e^{E/T(r)}-1} p dp. \tag{F51}$$

With the condition that $\Delta|_{r_+} \approx cte$, allows simplifying (F51) even more as

$$\sigma(r) = -\int_0^\infty \frac{E}{e^{E/T(r)}-1}\frac{2\pi p\, dp}{h^2}. \tag{F52}$$

Supposing an ideal gas model and $E = pv$

$$\begin{aligned} P &= \frac{1}{2}\sigma(r) \\ &= \frac{1}{2}\int_0^\infty \frac{pv}{e^{E/T(r)}-1}\frac{2\pi p\, dp}{h^2}. \end{aligned} \tag{F53}$$

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