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Maximal qubit violation of n-locality inequalities in a star-shaped quantum network

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Abstract
Bell's theorem was a cornerstone for our understanding of quantum theory and the establishment of Bell non-locality played a crucial role in the development of quantum information. Recently, its extension to complex networks has been attracting growing attention, but a deep characterization of quantum behavior is still missing for this novel context. In this work we analyze quantum correlations arising in the bilocality scenario, that is a tripartite quantum network where the correlations between the parties are mediated by two independent sources of states. First, we prove that non-bilocal correlations witnessed through a Bell-state measurement in the central node of the network form a subset of those obtainable by means of a local projective measurement. This leads us to derive the maximal violation of the bilocality inequality that can be achieved by arbitrary two-qubit quantum states and arbitrary local projective measurements. We then analyze in details the relation between the violation of the bilocality inequality and the CHSH inequality. Finally, we show how our method can be extended to the n-locality scenario consisting of n two-qubit quantum states distributed among $n + 1$ nodes of a star-shaped network.

1. Introduction

Since its establishment in the early decades of the last century, quantum theory has been elevated to the status of the 'most precisely tested and most successful theory in the history of science' [1]. And yet, many of its consequences have puzzled—and still do—most of the physicists confronted with it. At the heart of many of the counter-intuitive features of quantum mechanics is quantum entanglement [2], nowadays a crucial resource in quantum information and computation [3] but that also plays a central role in the foundations of the theory. For instance, as shown by the celebrated Bell's theorem [4], quantum correlations between distant parts of an entangled system can violate Bell inequalities, thus precluding its explanation by any local hidden variable (LHV) model, the phenomenon known as quantum non-locality.

Given its fundamental importance and practical applications in the most varied tasks of quantum information [5], not surprisingly many generalizations of Bell's theorem have been pursued over the years. Bell's original scenario involves two distant parties that upon receiving their shares of a joint physical system can measure one out of possible dichotomic observables. Natural generalizations of this simple scenario include more measurements per party [6] and sequential measurements [7], more measurement outcomes [8], more parties [9, 10] and also stronger notions of quantum non-locality [11–14]. All these different generalizations share the common feature that the correlations between the distant parties are assumed to be mediated by a single common source of states (see, for instance, figure 1(a)). However, as it is often in quantum networks [15], the correlations between the distant nodes is not given by a single source but by many independent sources which distribute entanglement in a non-trivial way across the whole network and generate strong correlations among its nodes (figures 1(b)–(d)). Surprisingly, in spite of its clear relevance, such networked scenario is far less explored.

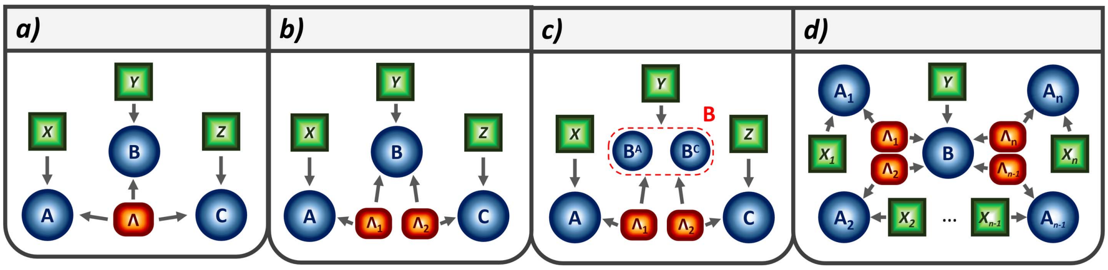

Figure 1. Description of the causal structure of some different networks. (a) LHV model representing a tripartite scenario with a single source of states. (b) BLHV model describing the bilocality counterpart of an entanglement swapping scenario. (c) Causal structure of a bilocality scenario where the local projective measurements performed in B are represented by the presence of the two substations B^A and B^C. (d) Extension of the bilocality scenario to a network consisting of n different stations sharing a quantum state with a central node, i.e. the so-called n-local star network.

The simplest networked scenario is provided by entanglement swapping [16], where two distant parties, Alice and Charlie, share entangled states with a central node Bob (see figure 1(b)). Upon measuring in an entangled basis and conditioning on his outcomes, Bob can generate entanglement and non-local correlations among the two other distant parties even though they had no direct interactions. To contrast classical and quantum correlation in this scenario, it is natural to consider classical models consisting of two independent hidden variables (figures 1(b)), the so-called bilocality assumption [17, 18]. The bilocality scenario and generalizations to networks with an increasing number n of independent sources of states (figures 1(d)), the so called n-locality scenario [19–26], allow for the emergence of a new kind of non-local correlations. For instance, correlations that appear classical according to usual LHV models can display non-classicality if the independence of the sources is taken into account, a result experimentally demonstrated in [27, 28]. However, previous works on the topic have mostly focused on developing new tools for the derivation of inequalities characterizing such scenarios and much less attention has been given to understand what are the quantum correlations that can be achieved in such networks.

That is precisely the aim of the present work. We consider in details the bilocality scenario and the bilocality inequality derived in [17, 18] and characterize the non-bilocal behavior of general qubit quantum states when the parties perform different kinds of projective measurements. First of all we show that the correlations arising in an entanglement swapping scenario, i.e. when Bob performs a Bell-state measurement (BSM), form a strict subclass of those correlations which can be achieved by performing local projective measurements in all stations. Focusing on this wider class of correlations, we derive a theorem characterizing the maximal violation of the bilocality inequality [17, 18] that can be achieved from general two-qubit quantum states shared among the parties. This leads us to obtain a characterization for the violation of the bilocality inequality in relation to the violation of the CHSH inequality [29]. Finally we show how our maximization method can be extended to the star network case [19], a n-partite generalization of the bilocality scenario, deriving thus the maximum violation of the n-locality inequality that can be extracted from this network.

2. Scenario

In the following we will mostly consider the bilocality scenario, which classical description in terms of a directed acyclic graph is shown in figure 1(b). It consists of three spatially separated parties (Alice, Bob and Charlie) whose correlations are mediated by two independent sources of states. In the quantum case, Bob shares two pairs of entangled particles, one with Alice and another with Charlie. Upon receiving their particles Alice, Bob and Charlie perform measurements labeled by the random variables X, Y and Z obtaining, respectively, the measurement outcomes A, B and C. The difference between Bob and the other parties is the fact that the first has in his possession two particles and thus can perform a larger set of measurements including, in particular, measurements in an entangled basis.

Any probability distribution compatible with the bilocality assumption (i.e. independence of the sources) can be decomposed as

$$p(a, b, c|x, y, z) = \int \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 p(\lambda_1)p(\lambda_2)p(a|x, \lambda_1)p(b|y, \lambda_1, \lambda_2)p(c|z, \lambda_2). \tag{1}$$

In particular, if we consider that each party measures two possible dichotomic observables ($x, y, z, a, b, c = 0, 1$), it follows that any bilocal hidden variable (BLHV) model described by equation (1) must fulfill the bilocality inequality

$$\mathcal{B} = \sqrt{|I|} + \sqrt{|J|} \leqslant 1, \tag{2}$$

with

$$I = \frac{1}{4} \sum_{x,z=0,1} \langle A_x B_0 C_z \rangle, \quad J = \frac{1}{4} \sum_{x,z=0,1} (-1)^{x+z} \langle A_x B_1 C_z \rangle, \tag{3}$$

and where

$$\langle A_x B_y C_z \rangle = \sum_{a,b,c=0,1} (-1)^{a+b+c} p(a, b, c|x, y, z). \tag{4}$$

As shown in [17, 18], if we impose the same causal structure to quantum mechanics (e.g. in an entanglement swapping experiment) we can nonetheless violate the bilocality inequality (even though the data might be compatible with LHV models), thus showing the existence of a new form of quantum *non-locality* called quantum *non-bilocality*.

To that aim let us consider the entanglement swapping scenario with an overall quantum state $|\psi^-\rangle_{AB} \otimes |\psi^-\rangle_{BC}$, with $|\psi^-\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$. We notice that equation (2) is also valid if Bob performs a single measurement with four possible outcomes (instead of two dichotomic measurements). Then, we can choose the measurements operators for the different parties in the following way. Stations A and C perform single qubit measurements defined by

$$A_x = \frac{\sigma_z + (-1)^x \sigma_x}{\sqrt{2}}, \qquad C_z = \frac{\sigma_z + (-1)^z \sigma_x}{\sqrt{2}}. \tag{5}$$

Station B, instead, performs a complete BSM, assigning to the two bits $b_0 b_1$ the values

$$00 \text{ for } |\phi^+\rangle, \qquad 01 \text{ for } |\phi^-\rangle, \qquad 10 \text{ for } |\psi^+\rangle, \qquad 11 \text{ for } |\psi^-\rangle. \tag{6}$$

The binary measurement B_y is then defined such that it returns $(-1)^{b_y}$, with respect to the value of $y = 0, 1$. This leads to

$$\begin{aligned} \langle A_x B_y C_z \rangle &= \sum_{a,b_0,b_1,c=0,1} (-1)^{a+b_y+c} p(a, b_0, b_1, c|x, z) \\ &= \sum_{a,b_y,c=0,1} (-1)^{a+b_y+c} p(a, b_y, c|x, z) \equiv \sum_{a,b,c=0,1} (-1)^{a+b+c} p(a, b, c|x, y, z), \end{aligned} \tag{7}$$

where, in the last steps, we made explicit use of the marginalization of probability $p(a, b_0, b_1, c|x, z)$ over $b_{k\neq y}$. With these state and measurements, the quantum mechanical correlations achieve a value $\mathcal{B} = \sqrt{2} > 1$, which violates the bilocality inequality and thus proves quantum non-bilocality.

3. Results

3.1. Non-bilocal correlations with local projective measurements

As reproduced above, in an entanglement swapping scenario Quantum Mechanics (QM) can exhibit correlations which cannot be reproduced by any BLHV model. In turn, it was recently proved [22] that an equivalent form of the bilocality inequality (equation (2)), can be violated by QM when all parties only perform a a particular case of single qubit operations (i.e. $\sigma_x, \sigma_z, \sigma_y$ and linear combinations). Here we will prove that, given the bilocality inequality of equation (2), the non-bilocal correlations arising in an entanglement swapping scenario (exploiting the BSM protocol described in section 2) are a strict subclass of those obtainable by means of local projective measurements. This latter scenario differs from the former since station B performs the two-spin projective measurements described by

$$\begin{aligned} &B_y = (1 - y)\, \vec{\lambda}_1 \cdot \vec{\sigma} \otimes \vec{\lambda}_2 \cdot \vec{\sigma} + y\, \vec{\delta}_1 \cdot \vec{\sigma} \otimes \vec{\delta}_2 \cdot \vec{\sigma}, \qquad y = 0, 1, \\ &\vec{\lambda}_1 \cdot \vec{\lambda}_1 = \vec{\lambda}_2 \cdot \vec{\lambda}_2 = \vec{\delta}_1 \cdot \vec{\delta}_1 = \vec{\delta}_2 \cdot \vec{\delta}_2 = 1, \qquad \vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z), \end{aligned} \tag{8}$$

which do not require any entangled operation (differently from the BSM case) and can be obtained by only performing local projective measurements.

The main motivation for considering local projective measurements is the fact that they correspond to the simplest set of measurements implementable in most physical setups used to study the violation of Bell inequalities. This means that station B can be seen as composed of two independent substations B^A and B^C (figure 1(c)). On the contrary, a general Local Operations and Classical Communication (LOCC) process would require substantially more experimental efforts, for instance the implementation of a feedback system which drives the settings choice in substation B^C, with respect to the results of substation B^A's measurement. As a matter of fact, when considering linear optical setups, even complete BSMs cannot be implemented [30]. This implies that recent experiments violating the bilocality inequality [27, 28] have to rely on assumptions about the

Table 1. Expected values for the operator B_y, as implicitly defined in equation (7).

y	$b_0 b_1$			
	00 (ϕ^+)	01 (ϕ^-)	10 (ψ^+)	11 (ψ^-)
$y = 0$	1	1	−1	−1
$y = 1$	1	−1	1	−1

measurement device and being thus device-dependent in clear opposition to the general aim of Bell's theorem. Finally, as pointed out in [31], when shared reference frames are added in the scenario, some particular situations may require that no internal communication is ensured between B^A and B^C, i.e. the presence of the local measurements scenario. From a more theoretical perspective, we stress that, so far, most of works on the topic has been focused on the case where station B performs a BSM. The possibility of emulating the non-bilocal results of a BSM (or rotated BSM) by means of an experimentally friendly subset of the LOCC operations is, in our view, a notable point.

We can now proceed with the evaluation of the main result of this section. The core of the bilocality parameter $\mathcal{B}$ is the computation of the expected value $\langle A_x B_y C_z \rangle$ (equation (4)), that in the quantum case is given by

$$\langle A_x B_y C_z \rangle = \mathrm{Tr}[(A_x \otimes B_y \otimes C_z)(\varrho_{AB} \otimes \varrho_{BC})]. \tag{9}$$

For the entanglement swapping scenario we can summarize the measurements in stations A and C by

$$\begin{aligned} A_x &= (1 - x)\, A_0 \; + \; x\, A_1 \qquad x = 0,\, 1, \\ C_z &= (1 - z)\, C_0 \; + \; z\, C_1 \qquad z = 0,\, 1, \end{aligned} \tag{10}$$

where A_x and C_z are general single qubit projective measurements with eigenvalues 1 and -1. When dealing with station B, it is suitable to consider its operatorial definition which is implicit in equation (7). Indeed we can consider that $(-1)^{b_y}$ is the outcome of our measurement, leading to values shown in table 1.

The quantum mechanical description of the operator B_y (in an entanglement swapping scenario) is thus given by

$$B_y = |\phi^+\rangle\langle\phi^+| + (1 - 2y)|\phi^-\rangle\langle\phi^-| + (2y - 1)|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-| \tag{11}$$

which relates each value of $y = 0,\;\; 1$ with its correct set of outcomes. This leads to the following theorem.

Theorem 1 (Non-bilocal correlations and local projective measurements). *Given the general set of local projective measurements described in equation (8), QM predictions for the bilocality parameter $\mathcal{B}$ which arise in an entanglement swapping scenario (where Bob performs the measurement described in equation (11)) are completely equivalent to those obtainable by performing a strict subclass of equation (8), i.e.*

$$\{\mathcal{B}\}_{\mathrm{BSM}} \subset \{\mathcal{B}\}_{\mathrm{LPM}}. \tag{12}$$

Proof. Let us write the Bell basis of a two qubit Hilbert space in terms of the computational basis $(|00\rangle,\;\; |01\rangle,\;\; |10\rangle,\;\; |11\rangle)$. From equation (11), we obtain

$$\begin{aligned} B_y &= |\phi^+\rangle\langle\phi^+| + (1 - 2y)|\phi^-\rangle\langle\phi^-| + (2y - 1)|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-| \\ &= (1 - y)\; (|00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|) \\ &\quad + y\; (|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|) \\ &= (1 - y)\; \sigma_z \otimes \sigma_z + y\; \sigma_x \otimes \sigma_x. \end{aligned} \tag{13}$$

This shows that the entanglement swapping scenario is equivalent to the one where station B only performs the two local projective measurements $B_0 = \sigma_z \otimes \sigma_z$ and $B_1 = \sigma_x \otimes \sigma_x$, which form a strict subclass of the general set of local measurements given by equation (8). Moreover, if we consider a rotated Bell basis, then we obtain

$$\begin{aligned} B'_y &= U_{B^A} \otimes U_{B^C} B_y U^\dagger_{B^A} \otimes U^\dagger_{B^C} = (1 - y)\; U_{B^A}\sigma_x U^\dagger_{B^A} \otimes U_{B^C}\sigma_x U^\dagger_{B^C} + y\; U_{B^A}\sigma_z U^\dagger_{B^A} \otimes U_{B^C}\sigma_z U^\dagger_{B^C} \\ &= (1 - y)\; \vec{b}_0^{\,A} \cdot \vec{\sigma} \otimes \vec{b}_0^{\,C} \cdot \vec{\sigma} + y\; \vec{b}_1^{\,A} \cdot \vec{\sigma} \otimes \vec{b}_1^{\,C} \cdot \vec{\sigma}, \end{aligned} \tag{14}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{b}_0^{\,A},\;\; \vec{b}_1^{\,A}\;\; (\vec{b}_0^{\,C},\;\; \vec{b}_1^{\,C})$ are orthonormal vectors. Due to the constraints $\vec{b}_0^{\,A} \perp \vec{b}_1^{\,A}$ and $\vec{b}_0^{\,C} \perp \vec{b}_1^{\,C}$, this case still represents a strict subset of equation (8). □

To further illustrate the relevance of theorem 1 we show next an example of correlations that can be achieved with local projective measurements but cannot with rotated BSMs. Let us consider the overall quantum state

$\varrho_{AB} \otimes \varrho_{BC}$, where

$$\varrho_{AB} = \varrho_{BC} = \frac{\sqrt{11}}{6}|\phi^+\rangle\langle\phi^+| + \left(1 - \frac{\sqrt{11}}{6}\right)|00\rangle\langle 00|. \tag{15}$$

Moreover, let us define the measurements performed in stations A and C as

$$A_0 = C_0 = \frac{\sqrt{99}\,\sigma_x + \sigma_z}{10}, \qquad A_1 = C_1 = \sigma_z. \tag{16}$$

We can analyze the set of non-bilocal correlations in terms of the correlation parameters I and J (defined in section 2). As we show in appendix, if we consider the rotated BSM case (constrained by $\vec{b}_0^A \perp \vec{b}_1^A$ and $\vec{b}_0^C \perp \vec{b}_1^C$), it is possible to show that the value $|I| = 121/320 \sim 0.378$ can only be obtained by means of unitary rotations $U_{B^A} \otimes U_{B^C}$ which, nonetheless, lead to the constraint $|J| \leqslant 1/5 = 0.2$. On the contrary, relaxing these orthogonality assumptions (i.e. exploring the local projective measurements scenario), some measurement settings exist such that the two values $|I| = 121/320 \sim 0.378$, $|J| = 89/320 \sim 0.278 > 0.2$ can be reached at the same time. Since this latter couple of I, J values leads to $\mathcal{B} \sim 1.14 > 1$, then it proves how, in this scenario, some peculiar non-bilocal correlations can only be obtained by means of local projective measurements, rather than performing a rotated BSM.

It is interesting to stress that the proof of theorem 1 also allows for convex combinations of local projective measurements. However, in principle, more general separable measurement are obtained through LOCCs.

As it turns out, this theorem has strong implications in our understanding of the non-bilocal behavior of QM. Indeed, it shows how the usual BSM protocol described in section 2, which is characteristic of the entanglement swapping scenario, is not capable of exploring the whole set of quantum non-bilocal correlations, since its expected value for the parameter $\mathcal{B}$ is equivalent to a subclass of those obtainable by means of local projective measurements. Moreover, it also shows that neither its extension to a rotated Bell basis is able to provide an enhancement, compared to the local projective measurement protocol. As we will show next, a better characterization of quantum correlations within the bilocality context must thus in principle take into account more general forms for Bob's measurements, especially when dealing with different types of quantum states.

3.2. Non-bilocality maximization criterion

We will now explore the maximization of the bilocality inequality considering that Bob performs the projective measurements described by equation (8). It is convenient to consider station B as a unique station composed of the two substations B^A and B^C, which perform single qubit measurements on one of the qubits belonging to the entangled state shared, respectively, with station A or C (see figure 1(c)).

Let A perform a general single qubit measurement and similarly for B^A, B^C and C. We can define these measurements as

$$\text{Station A} \longrightarrow \vec{a}_x \cdot \vec{\sigma}, \qquad \text{Station B} \longrightarrow \vec{b}_y^A \cdot \vec{\sigma} \otimes \vec{b}_y^C \cdot \vec{\sigma}, \qquad \text{Station C} \longrightarrow \vec{c}_z \cdot \vec{\sigma}, \tag{17}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Let us now define a general 2-qubit quantum state density matrix as

$$\varrho = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + \vec{r} \cdot \vec{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \vec{s} \cdot \vec{\sigma} + \sum_{n,\ m=1}^{3} t_{nm}\ \sigma_n \otimes \sigma_m). \tag{18}$$

The coefficients t_{nm} can be used to define a real matrix T_ϱ that lead to the following result:

Lemma 1 (Bilocality Parameter with Local Projective Measurements). *Given the set of general local projective measurements described in equation (17) and defined the general quantum state $\varrho_{AB} \otimes \varrho_{BC}$ accordingly to equation (18), the bilocality parameter $\mathcal{B}$ is given by*

$$\mathcal{B} = \frac{1}{2}\sqrt{|(\vec{a}_0 + \vec{a}_1) \cdot T_{\varrho_{AB}}\vec{b}_0^A|\ \ |\vec{b}_0^C \cdot T_{\varrho_{BC}}(\vec{c}_0 + \vec{c}_1)|} + \frac{1}{2}\sqrt{|(\vec{a}_0 - \vec{a}_1) \cdot T_{\varrho_{AB}}\vec{b}_1^A|\ \ |\vec{b}_1^C \cdot T_{\varrho_{BC}}(\vec{c}_0 - \vec{c}_1)|}. \tag{19}$$

Proof. Let us consider two operators O_i in the form $O_i = \vec{v}_i \cdot \vec{\sigma}$ and a two qubit quantum state ϱ described by equation (18). We can write

$$\langle O_1 \otimes O_2\rangle_\varrho = \mathrm{Tr}[(O_1 \otimes O_2)\varrho] = \mathrm{Tr}[\sum_{j,k=1,2,3}(v_1^j v_2^k \sigma_j \otimes \sigma_k)\varrho] = \sum_{j,k=1,2,3} v_1^j v_2^k t_{jk} = \vec{v}_1 \cdot (T_\varrho \vec{v}_2), \tag{20}$$

where we made use of the properties of the Pauli matrices σ_i. Given the set of local measurements described in equation (17), and the definitions of I and J (showed in equation (3)), the proof comes from a direct application of equation (20) to the quantum mechanical expectation value:

$$\langle A_x \otimes B_y^A \otimes B_y^C \otimes C_z\rangle_{\varrho_{AB}\otimes\varrho_{BC}} = \langle A_x \otimes B_y^A\rangle_{\varrho_{AB}}\langle B_y^C \otimes C_z\rangle_{\varrho_{BC}}. \tag{21}$$

□

Next we proceed with the maximization of the parameter $\mathcal{B}$ over all possible measurement choices, that is, the maximum violation of bilocality we can achieve with a given set of quantum states. To that aim, we introduce the following Lemma.

Lemma 2. *Given a square matrix M and defined the two symmetric matrices $\mathcal{M}_1 = M^{\mathbf{T}}M$ and $\mathcal{M}_2 = MM^{\mathbf{T}}$, each* non-null *eigenvalue of $\mathcal{M}_1$ is also an eigenvalue of $\mathcal{M}_2$, and vice versa.*

Proof. Let λ be an eigenvalue of $\mathcal{M}_1$

$$M^{\mathbf{T}}M\vec{v} = \lambda\vec{v}. \tag{22}$$

If $\lambda \neq 0$ we must have $M\vec{v} \neq \vec{0}$. We can then apply the operator M from the left, obtaining

$$MM^{\mathbf{T}}(M\vec{v}) = \lambda(M\vec{v}), \tag{23}$$

which shows that $M\vec{v}$ is an eigenvector of $\mathcal{M}_2$ with eigenvalue λ.

The *opposite* statement can be analogously proved. □

We can now enunciate the main result of this section.

Theorem 2 (Bilocality Parameter Maximization). *Given the set of measurements described in equation (17), the maximum bilocality parameter that can be extracted from a quantum state $\varrho_{AB} \otimes \varrho_{BC}$ can be written as*

$$\mathcal{B}_{\max} = \sqrt{\sqrt{t_1^A t_1^C} + \sqrt{t_2^A t_2^C}}, \tag{24}$$

where t_1^A and t_2^A (t_1^C and t_2^C) are the two greater (and positive) eigenvalues of the matrix $T_{\varrho_{AB}}^{\mathbf{T}} T_{\varrho_{AB}}$ ($T_{\varrho_{BC}}^{\mathbf{T}} T_{\varrho_{BC}}$), with $t_1^A \geqslant t_2^A$ and $t_1^C \geqslant t_2^C$.

Proof. We will prove theorem 2, following a scheme similar to the one used by Horodecki [32] for the CHSH inequality. Let us introduce the two pairs of mutually orthogonal vectors

$$\begin{aligned} (\vec{a}_0 + \vec{a}_1) &= 2\cos\alpha\ \vec{n}_A \quad \text{and} \quad (\vec{a}_0 - \vec{a}_1) = 2\sin\alpha\ \vec{n}'_A, \\ (\vec{c}_0 + \vec{c}_1) &= 2\cos\gamma\ \vec{n}_C \quad \text{and} \quad (\vec{c}_0 - \vec{c}_1) = 2\sin\gamma\ \vec{n}'_C, \end{aligned} \tag{25}$$

and let us apply equation (25) to (19)

$$\begin{aligned} \mathcal{B}_{\max} &= \max\left(\sqrt{|(\vec{n}_A\cdot T_{\varrho_{AB}}\vec{b}_0^{\,A})\,(\vec{b}_0^{\,C}\cdot T_{\varrho_{BC}}\vec{n}_C)\cos\alpha\ \cos\gamma|} + \sqrt{|(\vec{n}'_A\cdot T_{\varrho_{AB}}\vec{b}_1^{\,A})\,(\vec{b}_1^{\,C}\cdot T_{\varrho_{BC}}\vec{n}'_C)\sin\alpha\ \sin\gamma|}\right) \\ &= \max\left(\sqrt{|(\vec{b}_0^{\,A}\cdot T_{\varrho_{AB}}^{\mathbf{T}}\vec{n}_A)\,(\vec{b}_0^{\,C}\cdot T_{\varrho_{BC}}\vec{n}_C)\cos\alpha\ \cos\gamma|} + \sqrt{|(\vec{b}_1^{\,A}\cdot T_{\varrho_{AB}}^{\mathbf{T}}\vec{n}'_A)\,(\vec{b}_1^{\,C}\cdot T_{\varrho_{BC}}\vec{n}'_C)\sin\alpha\ \sin\gamma|}\right), \end{aligned} \tag{26}$$

where the maximization is done over the variables $\vec{n}_A$, $\vec{n}'_A$, $\vec{b}_0^{\,A}$, $\vec{b}_1^{\,A}$, $\vec{n}_C$, $\vec{n}'_C$, $\vec{b}_0^{\,C}$, $\vec{b}_1^{\,C}$, α and γ. We can choose $\vec{b}_0^{\,A}$, $\vec{b}_1^{\,A}$, $\vec{b}_0^{\,C}$, and $\vec{b}_1^{\,C}$ so that they maximize the scalar product. Defining

$$||M\vec{v}||^2 = M\vec{v}\cdot M\vec{v} = \vec{v}\cdot M^{\mathbf{T}}M\vec{v}, \tag{27}$$

and remembering that $\vec{b}_0^{\,A}$, $\vec{b}_1^{\,A}$, $\vec{b}_0^{\,C}$, and $\vec{b}_1^{\,C}$ are unitary vectors, we obtain

$$\mathcal{B}_{\max} = \max\left(\sqrt{||T_{\varrho_{AB}}^{\mathbf{T}}\vec{n}_A||\ ||T_{\varrho_{BC}}\vec{n}_C||\ |\cos\alpha\ \cos\gamma|} + \sqrt{||T_{\varrho_{AB}}^{\mathbf{T}}\vec{n}'_A||\ ||T_{\varrho_{BC}}\vec{n}'_C||\ |\sin\alpha\ \sin\gamma|}\right). \tag{28}$$

Next we have to choose the optimum variables α and γ. This leads to the set of equations

$$\begin{aligned} \frac{\partial\mathcal{B}(\alpha,\gamma)}{\partial\alpha} &= \frac{1}{2}\sqrt{||T_{\varrho_{AB}}^{\mathbf{T}}\vec{n}'_A||\ ||T_{\varrho_{BC}}\vec{n}'_C||\ |\sin(\alpha)\sin(\gamma)|}\ \cot(\alpha) \\ &\quad - \frac{1}{2}\sqrt{||T_{\varrho_{AB}}^{\mathbf{T}}\vec{n}'_A||\ ||T_{\varrho_{BC}}\vec{n}'_C||\ |\cos(\alpha)\cos(\gamma)|}\ \tan(\alpha) = 0, \\ \frac{\partial\mathcal{B}(\alpha,\gamma)}{\partial\gamma} &= \frac{1}{2}\sqrt{||T_{\varrho_{AB}}^{\mathbf{T}}\vec{n}'_A||\ ||T_{\varrho_{BC}}\vec{n}'_C||\ |\sin(\alpha)\sin(\gamma)|}\ \cot(\gamma) \\ &\quad - \frac{1}{2}\sqrt{||T_{\varrho_{AB}}^{\mathbf{T}}\vec{n}'_A||\ ||T_{\varrho_{BC}}\vec{n}'_C||\ |\cos(\alpha)\cos(\gamma)|}\ \tan(\gamma) = 0. \end{aligned} \tag{29}$$

This system of equations admits only solutions constrained by

$$\tan(\alpha)^2 = \tan(\gamma)^2 \quad \leftrightarrow \quad \gamma = \pm\alpha + n\pi,\ \ n\in\mathbb{Z}, \tag{30}$$

leading to

$$\mathcal{B}_{\max} = \max\,(|\cos\alpha|\sqrt{||T^{\mathrm{T}}_{\varrho_{AB}}\vec{n}_A||\;||T_{\varrho_{BC}}\vec{n}_C||} + |\sin\alpha|\sqrt{||T^{\mathrm{T}}_{\varrho_{AB}}\vec{n}'_A||\;||T_{\varrho_{BC}}\vec{n}'_C||}\,)$$
$$= \max\,(\sqrt{||T^{\mathrm{T}}_{\varrho_{AB}}\vec{n}_A||\;||T_{\varrho_{BC}}\vec{n}_C|| + ||T^{\mathrm{T}}_{\varrho_{AB}}\vec{n}'_A||\;||T_{\varrho_{BC}}\vec{n}'_C||}\,). \quad (31)$$

Next, we must take into account the constraints $\vec{n}_A \perp \vec{n}'_A$ and $\vec{n}_C \perp \vec{n}'_C$. Since these two couples of vectors are, however, independent, we can proceed with a first maximization which deals only with the two set of variables $\vec{n}_A$ and $\vec{n}'_A$. Since $T_{\varrho_{AB}} T^{\mathrm{T}}_{\varrho_{AB}}$ is a symmetric real matrix, it is diagonalizable. Let us call λ_1, λ_2 and λ_3 its eigenvalues and let us write $\vec{n}_A$ and $\vec{n}'_A$ in an eigenvector basis. If we define $k_1 = ||T_{\varrho_{BC}}\vec{n}_C|| > 0$ and $k_2 = ||T_{\varrho_{BC}}\vec{n}'_C|| > 0$, our problem can be written in terms of Lagrange multipliers related to the maximization of a function f, given the constraints g_i

$$f(\vec{n}_A,\ \vec{n}'_A) = k_1\sqrt{\sum_{i=1,2,3}\lambda_i (n^i_A)^2} + k_2\sqrt{\sum_{i=1,2,3}\lambda_i ({n'}^i_A)^2},$$
$$g_1(\vec{n}_A) = \vec{n}_A\cdot\vec{n}_A - 1,\qquad g_2(\vec{n}'_A) = \vec{n}'_A\cdot\vec{n}'_A - 1,\qquad g_3(\vec{n}_A,\ \vec{n}'_A) = \vec{n}_A\cdot\vec{n}'_A, \quad (32)$$

where we considered that finding the values that maximize $\sqrt{|f(x)|}$ is equivalent to find these values for $|f(x)|$. Let us now introduce the scaled vectors $\vec{\eta}_A = k_1\,\vec{n}_A$ and $\vec{\eta}'_A = k_2\,\vec{n}'_A$. We obtain

$$f(\vec{\eta}_A,\ \vec{\eta}'_A) = \sqrt{\sum_{i=1,2,3}\lambda_i(\eta^i_A)^2} + \sqrt{\sum_{i=1,2,3}\lambda_i({\eta'}^i_A)^2},$$
$$g_1(\vec{\eta}_A) = \vec{\eta}_A\cdot\vec{\eta}_A - (k_1)^2,\qquad g_2(\vec{\eta}'_A) = \vec{\eta}'_A\cdot\vec{\eta}'_A - (k_2)^2,\qquad g_3(\vec{\eta}_A,\ \vec{\eta}'_A) = \vec{\eta}_A\cdot\vec{\eta}'_A, \quad (33)$$

whose solution is given by vectors with two null components, out of three. If we define $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ and if $k_1 > k_2$, the solution related to the maximal value is then given by

$$f_{\max} = k_1\sqrt{\lambda_1} + k_2\sqrt{\lambda_2} \quad (34)$$

which leads to

$$\mathcal{B}_{\max} = \max_{\vec{n}_C,\ \vec{n}'_C}(\sqrt{||T_{\varrho_{BC}}\vec{n}_C||\sqrt{t_1^A} + ||T_{\varrho_{BC}}\vec{n}'_C||\sqrt{t_2^A}}\,), \quad (35)$$

where we made use of the lemma 2.

The maximization over the last two variables leads to an analogous Lagrange multipliers problem with similar solutions, thus proving the theorem. □

This theorem generalizes the results of [33] (which dealt with some particular classes of quantum states in the entanglement swapping scenario) to the more generic case of any quantum state in the local measurements scenario (which, in a bilocality context, includes the correlations obtained through entanglement swapping). It represents an extension of the Horodecki criterion [32] to the bilocality scenario, taking into account a general class of local measurements which can be performed in station B. Our result thus shows that, as far as we are concerned with the optimal violations of the bilocality inequality provided by given quantum states, local projective measurements or a BSM (in the right basis) are fully equivalent.

3.3. The relation between the non-bilocality and non-locality of sources

We will now characterize quantum non-bilocal behavior with respect to the usual non-locality of the states shared between A, B and B, C. Let us start from equation (24) and separately consider Bell non-locality of the states ϱ_{AB} and ϱ_{BC}. We can quantify it by evaluating the greatest CHSH inequality violation that can be obtained with these states. Let us define the CHSH inequality as

$$\mathcal{S}^{UV} \equiv \frac{1}{2}|\langle U_0V_0 + U_0V_1 + U_1V_0 - U_1V_1\rangle| \leqslant 1. \quad (36)$$

If we apply the criterion by Horodecki *et al* [32], we obtain

$$\mathcal{S}^{AB}_{\max} = \sqrt{t_1^A + t_2^A},\qquad \mathcal{S}^{BC}_{\max} = \sqrt{t_1^C + t_2^C}, \quad (37)$$

where we defined t_1^A, t_2^A, t_1^C and t_2^C accordingly to equation (24). From a direct comparison of (24) and (37) we can write

Proposition 1.

$$\mathcal{S}^{AB}_{\max} \leqslant 1 \ \text{and}\ \mathcal{S}^{BC}_{\max} \leqslant 1 \ \longrightarrow\ \mathcal{B}_{\max} \leqslant 1. \quad (38)$$

Proof. Applying the Cauchy–Schwarz inequality we obtain

$$\mathcal{B}_{\max}^2 \leqslant \mathcal{S}_{\max}^{AB}\mathcal{S}_{\max}^{BC} \leqslant 1. \tag{39}$$

□

This result shows that if the two sources cannot violate the CHSH inequality then they will also not violate the bilocality inequality. Thus, in this sense, if our interest is to check the non-classical behavior of sources of states, it is just enough to check for CHSH violations (at least if Bob performs a BSM or local projective measurements). Notwithstanding, we highlight that this does not mean that the bilocality inequality is useless, since there are probability distributions that violate the bilocality inequality but nonetheless are local according to a LHV model and thus cannot violate any usual Bell inequality [18, 27]. This fact can be explained considering that the bilocality inequality is derived under the assumption of independence of the two sources of states, which introduces a supplementary constraint. A probability distribution which violates the bilocality inequality, then, could nonetheless admit a decomposition in term of LHV models, since they do not rely on this supplementary assumption and can then include a wider class of probability distributions.

Next we consider the reverse case: is it possible to have quantum states that can violate the CHSH inequality but cannot violate the bilocality inequality? That turns out to be the case. To illustrate this phenomenon, we start considering two Werner states in the form $\varrho = v(|\psi^-\rangle\langle\psi^-|) + (1-v)\mathbb{I}/4$. In this case, indeed, in order to have a non-local behavior between A and B (B and C) we must have $v_{AB} > 1/\sqrt{2}$ ($v_{BC} > 1/\sqrt{2}$) while it is sufficient to have $\sqrt{v_{AB}v_{BC}} > 1/\sqrt{2}$ in order to witness non-bilocality. This example shows that on one hand it might be impossible to violate the bilocality inequality although one of ϱ_{AB} or ϱ_{BC} is Bell non-local (for instance $v_A = 1$ and $v_C = 0$). It also shows that, when one witnesses non-locality for only one of the two states, it can be possible, at the same time, to have non-bilocality by considering the entire network (for instance $v_A = 1$ and $1/2 < v_C < 1/\sqrt{2}$). Another possibility is the one described by the following Proposition

Proposition 2. *Given a tripartite scenario*

$$\exists\ \varrho_{AB} \text{ and } \varrho_{BC} \text{ such that } \mathcal{S}_{\max}^{AB} > 1,\ \ \mathcal{S}_{\max}^{BC} > 1\ \ \&\ \ \mathcal{B}_{\max} \leqslant 1. \tag{40}$$

Proof. We will prove this point with an example. Let us take

$$\varrho_{AB} = \frac{3}{5}|\psi^+\rangle\langle\psi^+| + \frac{2}{5}|\phi^+\rangle\langle\phi^+| = \begin{pmatrix} 0.2 & 0 & 0 & 0.2 \\ 0 & 0.3 & 0.3 & 0 \\ 0 & 0.3 & 0.3 & 0 \\ 0.2 & 0 & 0 & 0.2 \end{pmatrix},$$
$$\varrho_{BC} = \varrho\left(v = \frac{7}{10},\ \lambda = \frac{1}{3}\right) = \begin{pmatrix} 0.05 & 0 & 0 & 0 \\ 0 & 0.45 & -0.35 & 0 \\ 0 & -0.35 & 0.45 & 0 \\ 0 & 0 & 0 & 0.05 \end{pmatrix}, \tag{41}$$

where we defined $\varrho(v, \lambda)$ as

$$\varrho(v, \lambda) = v\ |\psi^-\rangle\langle\psi^-| + (1-v)\left[\lambda\frac{|\psi^-\rangle\langle\psi^-| + |\psi^+\rangle\langle\psi^+|}{2} + (1-\lambda)\frac{\mathbb{I}}{4}\right]. \tag{42}$$

For these two quantum states one can check that

$$t_1^A = 1, \quad t_2^A = 0.04, \quad t_1^C = 0.64, \quad t_2^C = 0.49, \tag{43}$$

which leads to

$$\mathcal{S}_{\max}^{AB} \simeq 1.02, \quad \mathcal{S}_{\max}^{BC} \simeq 1.06, \quad \mathcal{B}_{\max} \simeq 0.97. \tag{44}$$

□

This shows how it is possible to have non-local quantum states which nonetheless cannot violate the bilocality inequality (with local measurements). However, we highlight that, since the bilocal set is a subset of the local one, certainly there are bilocality inequalities other than (2) that are violated by these non-local states.

All these statements provide a well-defined picture of the relation between the CHSH inequality and the bilocality inequality in respect to the quantum states $\varrho_{AB} \otimes \varrho_{BC}$. We indeed derived all the possible cases of quantum non-local correlations which may be seen between couples of nodes, or in the whole network (according to the CHSH and bilocality inequalities). This characterization is shown in figure 2, in terms of a Venn diagram.

We finally notice that if A and B share a maximally entangled state while B and C share a generic quantum state, then it is easier to obtain a bilocality violation in the tripartite network rather than a CHSH violation between the nodes B^C and C. Indeed it is possible to derive

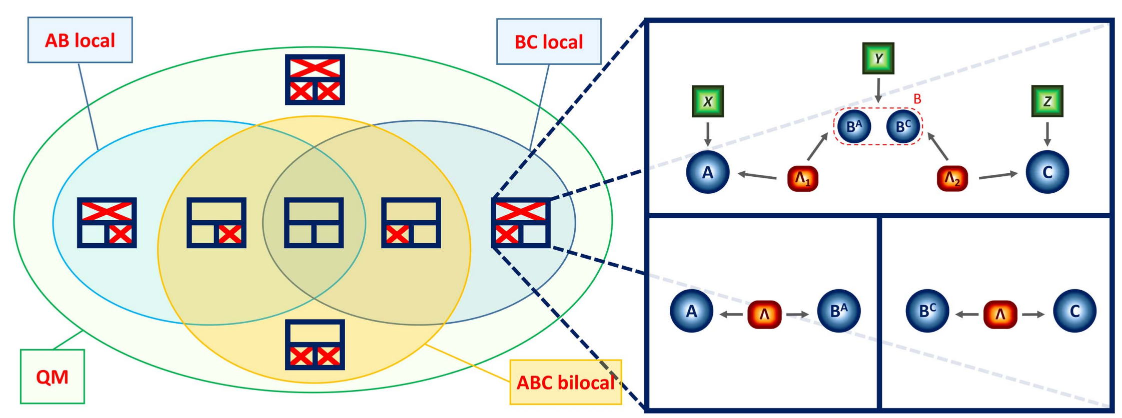

Figure 2. Venns diagram representing quantum correlations in a bilocality scenario. Possible quantum correlations that may be witnessed given a quantum state $\varrho_{AB} \otimes \varrho_{BC}$. The blue sets represent quantum states that do not violation the CHSH inequality for ϱ_{AB} (AB local) or ϱ_{BC} (BC local). The orange set includes, instead, these states whose correlations do not violate the bilocality inequality, while the whole set of quantum correlations is represented in green. For all different regions a blue square shows those decompositions which are not allowed (crossed with red lines), accordingly to the greater square on the right. All violations are related to local projective measurements.

$$\mathcal{B}_{\max}(|\Phi^+\rangle\langle\Phi^+| \otimes \varrho_{BC}) = \sqrt{\sqrt{t_1^C} + \sqrt{t_2^C}} \geqslant \sqrt{t_1^C + t_2^C} = \mathcal{S}_{\max}^{BC}, \tag{45}$$

where we made use of the following Lemma

Lemma 3. *Given the parameters $t_1^A,\ t_2^A,\ t_1^C$ and t_2^C defined in equation (24), it holds*

$$0 \leqslant t_1^A,\ t_2^A,\ t_1^C,\ t_2^C \leqslant 1 \tag{46}$$

Proof. This proof will be divided in two main points.

(1) $\forall\, \varrho, \quad \exists\ \varrho' = U^\dagger \varrho U$ *such that* $T_{\varrho'}$ *is diagonal.*

As discussed in [34], if we apply a local unitary $U = U_1 \otimes U_2$ to the initial quantum state ϱ, the matrix T_ϱ will transform accordingly to

$$T_\varrho \longrightarrow \tilde{U}_1 T_\varrho \tilde{U}_2^{\mathrm{T}}, \tag{47}$$

where $\tilde{U}_1$ and $\tilde{U}_2$ are orthogonal 3×3 real matrices which represent, respectively, the spin rotations properties of U_1 and U_2. According to the Singular Decomposition Theorem, it is always possible to choose $\tilde{U}_1$ and $\tilde{U}_2$ such that $\tilde{U}_1 T_\varrho \tilde{U}_2^{\mathrm{T}}$ is diagonal, thus demonstrating point 1.

It is important to stress that we can always rotate our Hilbert space in a way that $\varrho \to U^\dagger \varrho U$ so we can take ϱ' without loss of generality.

(2) *If T_ϱ is diagonal, then the eigenvalues of $T_\varrho^{\mathrm{T}} T_\varrho$ are less or equal to 1.*

It was shown in [32] that, for every quantum state ϱ, we have $|t_{nm}| \leqslant 1,\ \ t_{nm} \in \mathbb{R}$ regardless to the basis chosen for our Hilbert space. If T_ϱ is diagonal then $T_\varrho^{\mathrm{T}} T_\varrho = T_\varrho^2$ and its eigenvalues t_i can be written as $t_i = t_{ii}^2 \leqslant 1$.

Given the definitions of t_1^A and t_2^A (t_1^C and t_2^C) described in equation (24), the lemma is proved. □

3.4. Extension to the star network scenario

We now generalize the results of theorem 2, to the case of a *n-partite* star network. This network is the natural extension of the bilocality scenario, and it is composed of n sources sharing a quantum state between one of the n stations A_i and a central node B (see figure 1(d)). The bilocality scenario corresponds to the particular case where $n = 2$. The classical description of correlations in this scenario is characterized by the probability decomposition

$$p(\{a_i\}_{i=1,\ldots,n}, b|\{x_i\}_{i=1,n}, y) = \int \left(\prod_{i=1}^{n} \mathrm{d}\lambda_i p(\lambda_i) p(a_i|x_i, \lambda_i)\right) p(b|y, \{\lambda_i\}_{i=1,n}). \tag{48}$$

As shown in [19], assuming binary inputs and outputs in all the stations, the following n-locality inequality holds

$$\mathcal{N}_{\text{star}} = |I|^{1/n} + |J|^{1/n} \leqslant 1, \tag{49}$$

where

$$I = \frac{1}{2^n}\sum_{x_1\ldots x_n}\langle A^1_{x_1}\ldots A^n_{x_n}B_0\rangle, \qquad I = \frac{1}{2^n}\sum_{x_1\ldots x_n}(-1)^{\sum_i x_i}\langle A^1_{x_1}\ldots\ A^n_{x_n}B_1\rangle,$$
$$\langle A^1_{x_1}\ldots A^n_{x_n}B_y\rangle = \sum_{a_1\ldots a_n,b}(-1)^{b+\sum_i a_i}p(\{a_i\}_{i=1,n},\ b|\{x_i\}_{i=1,n},\ y). \tag{50}$$

We will now derive a theorem showing the maximal value of parameter $\mathcal{N}_{\text{star}}$ that can be obtained by local projective measurements on the central node and given arbitrary bipartite states shared between the central node and the n parties.

Theorem 3 (Optimal violation of the n-locality inequality). *Given single qubit projective measurements and defined the generic quantum state $\varrho_{A_1B}\otimes\ldots\otimes\varrho_{A_nB}$ accordingly to equation (18), the maximal value of $\mathcal{N}_{\text{star}}$ is given by*

$$\mathcal{N}^{\max}_{\text{star}} = \sqrt{\left(\prod_{i=1}^n t_1^{A_i}\right)^{1/n} + \left(\prod_{i=1}^n t_2^{A_i}\right)^{1/n}}, \tag{51}$$

where $t_1^{A^i}$ and $t_2^{A^i}$ are the two greater (and positive) eigenvalues of the matrix $T^{\mathbf{T}}_{\varrho_{A_iB}}T_{\varrho_{A_iB}}$ with $t_1^{A_i}\geqslant t_2^{A_i}$.

Proof. In our single qubit measurements scheme the operator B can be written as

$$B_y = \bigotimes_{i=1}^n B_y^i = \bigotimes_{i=1}^n \vec{b}_y^{\,i}\cdot\vec{\sigma}. \tag{52}$$

As pointed out in [19], this allows us to write

$$\mathcal{N}_{\text{star}} = \left|\prod_{i=1}^n \frac{1}{2}(\langle A_0^iB_0^i\rangle + \langle A_1^iB_0^i\rangle)\right|^{1/n} + \left|\prod_{i=1}^n \frac{1}{2}(\langle A_0^iB_1^i\rangle - \langle A_1^iB_1^i\rangle)\right|^{1/n}, \tag{53}$$

which leads to

$$\mathcal{N}_{\text{star}} = \left|\prod_{i=1}^n \frac{1}{2}(\vec{a}_0^{\,i}+\vec{a}_1^{\,i})\cdot T_{\varrho_{A_iB}}\vec{b}_0^{\,i}\right|^{1/n} + \left|\prod_{i=1}^n \frac{1}{2}(\vec{a}_0^{\,i}-\vec{a}_1^{\,i})\cdot T_{\varrho_{A_iB}}\vec{b}_1^{\,i}\right|^{1/n}. \tag{54}$$

Introducing the pairs of mutually orthogonal vectors

$$(\vec{a}_0^{\,i}+\vec{a}_1^{\,i}) = 2\cos\alpha_i\ \vec{n}_i \quad \& \quad (\vec{a}_0^{\,i}-\vec{a}_1^{\,i}) = 2\sin\alpha_i\ \vec{n}_i', \tag{55}$$

allows us to write

$$\mathcal{N}_{\text{star}} = \left|\prod_{i=1}^n \cos\alpha_i\ \vec{n}_i\cdot T_{\varrho_{A_iB}}\vec{b}_0^{\,i}\right|^{1/n} + \left|\prod_{i=1}^n \sin\alpha_i\ \vec{n}_i'\cdot T_{\varrho_{A_iB}}\vec{b}_1^{\,i}\right|^{1/n}. \tag{56}$$

We can choose the parameters $\vec{b}_y^{\,i}$ so that they maximize the scalar products. We obtain

$$\mathcal{N}^{\max}_{\text{star}} = \max\left(\left|\prod_{i=1}^n \cos\alpha_i\ ||T^{\mathbf{T}}_{\varrho_{A_iB}}\vec{n}_i||\right|^{1/n} + \left|\prod_{i=1}^n \sin\alpha_i\ ||T^{\mathbf{T}}_{\varrho_{A_iB}}\vec{n}_i'||\right|^{1/n}\right). \tag{57}$$

We can now proceed to the maximization over the parameters α_i. Let us define the function

$$K(\alpha_1,\ldots\alpha_n) = \left|\lambda_1\prod_{i=1}^n\cos\alpha_i\right|^{1/n} + \left|\lambda_2\prod_{i=1}^n\sin\alpha_i\right|^{1/n}. \tag{58}$$

We can write

$$\frac{\partial K(\alpha_1,\ldots\alpha_n)}{\partial\alpha_j} = \frac{|\lambda_2\prod_{i=1}^n\sin\alpha_i|^{1/n}}{n}cot\alpha_j - \frac{|\lambda_1\prod_{i=1}^n\cos\alpha_i|^{1/n}}{n}\tan\alpha_j = 0, \tag{59}$$

which, similarly to equation (29), admits only solutions constrained by

$$\tan(\alpha_j)^2 = \tan(\alpha_k)^2 \quad\leftrightarrow\quad \alpha_j = \pm\alpha_k + n\pi,\ \ n\in\mathbb{Z}\quad \forall j, k. \tag{60}$$

This leads to

$$K(\alpha_1,\ldots\alpha_n)_{\max} = \max_\alpha(|\lambda_1^{1/n}\cos\alpha| + |\lambda_2^{1/n}\sin\alpha|) = \sqrt{\lambda_1^{2/n}+\lambda_2^{2/n}}, \tag{61}$$

which allows us to write

$$\mathcal{N}_{\text{star}}^{\max} = \max\sqrt{\left|\prod_{i=1}^{n} ||T_{\varrho_{A_iB}}^{\mathbf{T}}\vec{n}_i||\right|^{2/n} + \left|\prod_{i=1}^{n} ||T_{\varrho_{A_iB}}^{\mathbf{T}}\vec{n}_i'||\right|^{2/n}}. \tag{62}$$

Let us now define

$$k_1 = \left|\prod_{i=2}^{n} ||T_{\varrho_{A_iB}}^{\mathbf{T}}\vec{n}_i||\right|, \qquad k_2 = \left|\prod_{i=2}^{n} ||T_{\varrho_{A_iB}}^{\mathbf{T}}\vec{n}_i'||\right|, \tag{63}$$

we have that

$$\mathcal{N}_{\text{star}}^{\max} = \max\sqrt{k_1^{2/n}||T_{\varrho_{A_1B}}^{\mathbf{T}}\vec{n}_1||^{2/n} + k_2^{2/n}||T_{\varrho_{A_1B}}^{\mathbf{T}}\vec{n}_1'||^{2/n}}. \tag{64}$$

Labeling λ_1, λ_2 and λ_3 as the eigenvalues of $T_{\varrho_{A_1B}}T_{\varrho_{A_1B}}^{\mathbf{T}}$ (which is real and symmetric) and writing $\vec{n}_1$ and $\vec{n}_1'$ in an eigenvector basis we obtain the Lagrange multipliers problem related to the maximization of a function f, given the constraints g_i:

$$\begin{gathered} f(\vec{n}_1, \vec{n}_1') = \sqrt{\left(k_1^2 \sum_{i=1,2,3} \lambda_i (n_1^i)^2\right)^{2/n}} + \sqrt{\left(k_2^2 \sum_{i=1,2,3} \lambda_i (n'^i_1)^2\right)^{2/n}}, \\ g_1(\vec{n}_1) = \vec{n}_1\cdot\vec{n}_1 - 1, \qquad g_2(\vec{n}_1') = \vec{n}_1'\cdot\vec{n}_1' - 1, \qquad g_3(\vec{n}_1, \vec{n}_1') = \vec{n}_1\cdot\vec{n}_1', \end{gathered} \tag{65}$$

where we considered that the values which maximize $|f(x)|$ also maximize $\sqrt{|f(x)|}$.

This Lagrangian multipliers problem can be treated similarly to equation (32), giving the same results. If $k_1 > k_2$, we obtain

$$f_{\max} = (k_1\sqrt{\lambda_1})^{2/n} + (k_2\sqrt{\lambda_2})^{2/n} \tag{66}$$

which leads to

$$\mathcal{N}_{\text{star}}^{\max} = \max\left(\sqrt{(t_1^{A_1})^{1/n}\left(\prod_{i=2}^{n} ||T_{\varrho_{A_iB}}^{\mathbf{T}}\vec{n}_i||\right)^{2/n} + (t_2^{A_1})^{1/n}\left(\prod_{i=2}^{n} ||T_{\varrho_{A_iB}}^{\mathbf{T}}\vec{n}_i'||\right)^{2/n}}\right). \tag{67}$$

The proof is concluded by applying iteratively this procedure. □

We notice that the bilocality scenario can be seen as a particular case ($n = 2$) of a star network, where $A_2 \equiv C$ and $x_2 \equiv z$. Moreover we emphasize that equation (51) gives the same results that would be obtained if one performed an optimized CHSH test on a 2-qubit state were t_1 and t_2 are given by the geometric means of the parameters $t_1^{A_i}$ and $t_2^{A_i}$.

4. Conclusions

Generalizations of Bell's theorem to complex networks offer a new theoretical and experimental ground for further understanding quantum correlations and its practical applications in information processing. Similarly to usual Bell scenarios, understanding the set of quantum correlations we can achieve and in particular what are the optimal quantum violation of Bell inequalities is of primal importance.

In this work we have taken a step forward in this direction, deriving the optimal violation of the bilocality inequality proposed in [17, 18] and generalized in [19] for the case of a star-shaped network with n independent sources. Considering that the central node in the network performs arbitrary local projective measurements and that the other parties perform projective measurements we have obtained the optimal value for the violation of the bilocality and n-locality inequalities. Our results can be understood as the generalization for complex networks of the Horodecki's criterion [32] valid for the CHSH inequality [29]. We have analyzed in details the relation between the bilocality inequality and in particular shown that if both the quantum states cannot violate the CHSH inequality then the bilocality inequality also cannot be violated, thus precluding, in this sense, its use as a way to detect quantum correlations beyond the CHSH case. Moreover, we have shown that some quantum states can separately exhibit Bell non-local correlations, but nevertheless cannot violate the bilocality inequality when considered as a whole in the network, thus proving that not all non-local states can be used to witness non-bilocal correlations (at least according to this specific inequality).

However, all these conclusions are based on the assumption that the central node in the network performs local measurements (that in such scenario include as a particular case the results obtained through the usual complete BSM protocol). This immediately opens a series of interesting questions for future research. Can we achieve better violations by employing more general measurements in the central station, for instance, entangled measurements in different basis, non-maximally entangled, non-projective or more general separable measurements? To exemplify, we notice that if Bob applies the protocol described in section 2 to the non-maximally entangled basis $\{|\phi^+\rangle, |\phi^-\rangle, |01\rangle, |10\rangle\}$, then its measurement operator will be given by

$$B_y = (1-y)\sigma_z \otimes \sigma_z + y\frac{\sigma_z \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_z + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y}{2}, \tag{68}$$

which does not admit the decomposition defined in equation (8), and thus cannot be reproduced by means of local projective measurements. Although this fact does not prove that non-maximally entangled basis can provide an enhancement over neither the BSM nor the local projective measurements protocols, it shows that their analysis could lead to potential advantages.

Related to that, it would be highly relevant to derive new classes of network inequalities [21, 22, 35]. One of the goals of generalizing Bell's theorem for complex networks is exactly the idea that since the corresponding classical models are more restrictive, it is reasonable to expect that we can find new Bell inequalities allowing us to probe the non-classical character of correlations that are local according to usual LHV models. Can it be that local projective measurements or measurement in the Bell basis allow us to detect such kind of correlations if new bilocality or n-locality inequalities are considered? And what would happen if we considered general POVM measurements in all our stations? Could we witness a whole new regime of quantum states, which at the moment, instead, admit a n-local classical description? Finally, one can wonder whether quantum states of higher dimensions (qudits) would allow for higher violations of the n-locality inequalities.

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Note added in proof. During the preparation of this manuscript which contains results of a master thesis [36], we became aware of an independent work [37] preprinted in February 2017.

Appendix. Example of local projective measurements correlations

Let us consider the scenario described by equations (15) and (16). Assuming equation (14) for the rotated BSM case, and equation (17) for the local projective measurement case, we can evaluate the correlations parameters $I,\ J$, similarly to lemma 1. The two quantum states are characterized by

$$T_{\varrho_{AB}} = T_{\varrho_{BC}} = \begin{pmatrix} \sqrt{11}/6 & 0 & 0 \\ 0 & -\sqrt{11}/6 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A1}$$

while the A, C measurement vectors are

$$\vec{a}_0 = \vec{c}_0 = \frac{1}{10}\begin{pmatrix} \sqrt{99} \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{a}_1 = \vec{c}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A2}$$

Let us consider the rotated BSM case. The measurement vectors $\vec{b}_0^{\,A},\ \vec{b}_1^{\,A}$ ($\vec{b}_0^{\,C},\ \vec{b}_1^{\,C}$) will be constrained by their mutual orthogonality condition. If we define a general 3-dimensional vector $\vec{v}$ and the generic vector $\vec{v}_\perp$ belonging to its orthogonal plane as

$$\vec{v}(\theta,\ \phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \cos\theta \\ \sin\theta\sin\phi \end{pmatrix}, \qquad \vec{v}_\perp(\theta,\ \phi,\ \psi) = \cos\psi\begin{pmatrix} \cos\theta\cos\phi \\ -\sin\theta \\ \cos\theta\sin\phi \end{pmatrix} + \sin\psi\begin{pmatrix} -\sin\phi \\ 0 \\ \cos\phi \end{pmatrix}, \tag{A3}$$

then we can consider

$$\vec{b}_0^{\,A} = \vec{v}(\theta_A,\phi_A), \qquad \vec{b}_1^{\,A} = \vec{v}_\perp(\theta_A,\phi_A,\psi_A), \qquad \vec{b}_0^{\,C} = \vec{v}(\theta_C,\phi_C), \qquad \vec{b}_1^{\,C} = \vec{v}_\perp(\theta_C,\phi_C,\psi_C). \tag{A4}$$

The evaluation of the correlation parameters $I,\ J$ for the rotated BSM case will then lead to

$$I_{\mathrm{BSM}} = \Gamma(\theta_A,\phi_A)\Gamma(\theta_C,\phi_C)\left(\frac{11}{40}\right)^2, \qquad J_{\mathrm{BSM}} = \Lambda(\theta_A,\phi_A,\psi_A)\Lambda(\theta_C,\phi_C,\psi_C)\left(\frac{1}{40}\right)^2, \tag{A5}$$

where we defined

$$\Gamma(\theta,\,\phi) \equiv (\cos\phi + 2\sin\phi)\sin\theta,$$
$$\Lambda(\theta,\,\phi,\,\psi) \equiv (11\cos\phi - 18\sin\phi)\cos\theta\cos\psi - (18\cos\phi + 11\sin\phi)\sin\psi. \tag{A6}$$

We can now evaluate the maximum value $I_{\mathrm{BSM}}^{\max}$ of $|I_{\mathrm{BSM}}|$. This calculation leads to $I_{\mathrm{BSM}}^{\max} = 121/320 \sim 0.378$, which can be obtained if and only if

$$\theta_A = \frac{\pi}{2} + k_A\pi,\quad \theta_C = \frac{\pi}{2} + k_C\pi,\quad \phi_A = \arctan 2 + n_A\pi,\quad \phi_C = \arctan 2 + n_C\pi, \tag{A7}$$

where $k_A,\ k_C,\ n_A,\ n_C \in \mathbb{Z}$.

Nevertheless, when assuming equation (A7), one obtains $|J_{\mathrm{BSM}}| = (1/5)|\sin\psi_A\,\sin\psi_C| \leqslant 1/5 = 0.2$.

On the contrary, we can consider the local projective measurement case by relaxing the orthogonality assumption of vectors $\vec{b}_0^{\,A},\ \vec{b}_1^{\,A}$ ($\vec{b}_0^{\,C},\ \vec{b}_1^{\,C}$). This means that we can define the four generic independent vectors

$$\vec{b}_0^{\,A} = \vec{v}(\theta_A,\,\phi_A),\quad \vec{b}_1^{\,A} = \vec{v}(\alpha_A,\,\beta_A),\quad \vec{b}_0^{\,C} = \vec{v}(\theta_C,\,\phi_C),\quad \vec{b}_1^{\,C} = \vec{v}(\alpha_C,\,\beta_C), \tag{A8}$$

leading to

$$I_{\mathrm{LPM}} = I_{\mathrm{BSM}} = \Gamma(\theta_A,\,\phi_A)\Gamma(\theta_C,\,\phi_C)\left(\frac{11}{40}\right)^2,\qquad J_{\mathrm{LPM}} = \Phi(\alpha_A,\,\beta_A)\Phi(\alpha_C,\,\beta_C)\left(\frac{1}{40}\right)^2, \tag{A9}$$

where we defined

$$\Phi(\alpha,\,\beta) \equiv (11\cos\beta - 18\sin\beta)\sin\alpha. \tag{A10}$$

In this scenario, then, I_{LPM} and J_{LPM} are not correlated by some measurements' orthogonality assumption, and can thus be independently maximized with a proper measurements choice. Since $J_{\mathrm{LPM}}^{\max} = \max|J_{\mathrm{LPM}}| = 89/320 \sim 0.278 > 0.2$ (which can be obtained by setting $\alpha_A = \alpha_C = \pi/2$ and $\beta_A = \beta_C = \arctan[-11/18]$), if we choose, at the same time, the values of $\theta_A,\ \theta_C,\ \phi_A,\ \phi_C$ described in equation (A7), then we are able to obtain non-bilocal correlations which could not be reached by means of a rotated BSM.

We stress that this fact does not prove that some scenarios exist where the maximal amount of non-bilocality can be obtained only with local projective measurements rather than by performing a rotated BSM. It proves, instead, that some peculiar set of non-bilocal correlations (defined by the corresponding values of the correlation parameters I and J), can only be addressed by means of local projective measurements.

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