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The free-electron laser harmonic cascade

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Abstract. Free-electron laser (FEL) devices based on a sequence of amplifiers with a harmonic relation between the resonant frequencies of each section have been proposed to extend to shorter wavelengths the FEL operating range. Because of the practical limit on the tunability of undulator magnets, the e-beam energy still represents the main constraint on the shortest reachable wavelength of the cascade. In this paper, we propose a scheme where the undulators of the cascade are tuned at different, not-harmonic, fundamental frequencies having instead one of the higher order harmonics at a common frequency. A short and intense seed pulse in such a system creates a superradiant pulse which harmonically seeds the following undulator at the common multiple frequency. The microbunching at the higher harmonic in the second undulator is enhanced by the modulation of the previous undulator so that lasing at shorter wavelengths may be obtained with a relatively low-energy electron beam.

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## 1. Introduction

A cascaded free-electron laser (FEL) is a sequence of FEL amplifiers where the beam and/or radiation from an undulator are injected into a following one tuned at a frequency which is a higher harmonic of the fundamental resonance of the first. Such schemes have been proposed in order to extend the spectral range of an FEL and are promising in the VUV/soft x-ray region of the electromagnetic spectrum, where the FEL oscillator configuration is ruled out by the lack of suitable optics and where the conditions to enable the development of the self-amplified spontaneous emission (SASE) FEL instability impose stringent requirements on the electron beam quality. Moreover, the amplified FEL signal in a cascade preserves the coherence/spectral properties of the input seed, thus offering a solution to the problem of reduced longitudinal coherence intrinsic to the SASE amplification of beam shot-noise.

The main elements of a FEL cascade, whose idea can be traced back to [1], can be summarized by considering an example composed by two undulators, where the first is resonant at $\omega_1$ and the second is resonant at $\omega_2 = n\omega_1$ with $n$ integer. In this paper, the first section serves to modulate the current density, while in the second one the emission process at the selected higher order harmonic takes place. The working principle lies in the fact that the current density modulation (bunching), which occurs during the FEL optical amplification in the first stage, has higher order harmonic components at frequency $n\omega_1$. These are responsible for the enhancement of radiation emission when the beam is injected in a second undulator with frequency $\omega_2$ matching the beam density modulation periodicity. Many variations of this scheme have been proposed and studied over the years [2]–[5]. At the same time many experimental tests have been successfully carried out (with $n = 2$ [6], $n = 3$ [7] and recently $n = 4$ [8]). The idea of a cascade initiated at visible/UV wavelength from solid state laser sources has stimulated several proposals based on multiple stage FEL cascades [9]–[12].

One of the aspects limiting the possibility of extension to higher frequency multiplication factors is related to the energy modulation that is introduced in the electron beam every time the beam is modulated at a wavelength $\lambda_n = \lambda_1 / n$. There is a minimum induced energy spread which is required to reach coherent emission at the $n$th harmonic of the fundamental wavelength $\lambda_1$. This limit can be calculated by imposing that the beam phase space area before and after the bunching process is preserved (Liouville theorem). The minimum energy spread necessary to obtain a modulation at $\lambda_n$, is $\sigma'_e = 2n\sigma_e$, where $\sigma_e$ is the initial uncorrelated energy spread over a slice of the beam of length $\lambda_1$.

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Such energy modulation is converted by the dispersion section into the desired density modulation. Afterwards, during the emission process in the radiator, the correlated energy spread continues to affect the density distribution because of the radiator dispersion, and the result, in the simplified picture of pure linear dispersion, is a progressive de-bunching and a reduction of the coherent emission. The de-bunching $R_{56} \sigma'_\epsilon$ depends on the $R_{56}$ coefficient of the radiator (with resonance at $\lambda_n$) which is given by twice the slippage distance, i.e. $R_{56} = 2N\lambda_n$. The emission is inhibited when the de-bunching is comparable to half of the emitted wavelength $\lambda_n/2$. For this reason the number of periods of the modulator where efficient emission can be obtained scales inversely with the initial beam energy spread $N \approx 1/8n\sigma'_\epsilon$. As an example, we may consider an initial ‘local’ energy spread $\sigma'_\epsilon \approx 5 \times 10^{-4}$, a harmonic multiplication factor of 6 and we get coherent emission on a length of about 40 periods. This limitation may be more stringent in a multiple stage cascade where the initial energy spread is increased by the interaction in one of the previous stages. In order to overcome the problem it was proposed to use, in different positions along the cascade, different parts of the unperturbed high quality electron beam [13]. In this proposed configuration, a seed pulse shorter than the electron beam may be up-converted in frequency in a single stage cascade, then, with a dispersive section, it may be shifted over a fresh portion of the beam which has not yet been heated by the FEL interaction, and re-injected in a new cascade stage.

Another limitation in preventing high multiplication factors is constituted by the propagation of the signal-to-noise ratio through each stage of the cascade [14]. In fact, any phase fluctuation of the field at $\omega$ is magnified by $n$ at the harmonic frequency $\omega_n = n\omega_1$. In terms of intensities, the effect scales unfavourably as $n^2$ thus limiting to few stages the possibility of preserving the signal-to-noise ratio of the seed through the cascade. The situation appears less critical when the limited bandwidth associated to the FEL process is taken into account [12], but the problems related to the stability and preservation of longitudinal coherence in a multiple stage cascade are still an issue.

A possibility to reach the soft x-ray region of the spectrum with a cascade with a relatively low multiplication factor is provided by the recent progress in laser technology. The high-order harmonics generated by an intense laser when focused on to a gas jet (HHG) are promising sources to seed an FEL amplifier cascade scheme with a final operating wavelength down to the water window range. This technique is proposed in the framework of the Arc En Ciel [15] and 4GLS [16] future user facilities and experiments aiming at verifying its feasibility are foreseen in the near future at SCSS (Japan) [17] and SPARC (Italy) [18].

An alternative to these schemes has been proposed in [19] where the cascade of undulators is designed to induce the transition of the propagating pulse in the superradiant regime. This idea, which will be briefly reviewed in the next section, is based on the concept that, once saturation is reached, the superradiant regime is the natural evolution of any short seed pulse injected in a FEL amplifier. In superradiance a solitary wave-like pulse shorter than the electron beam is amplified in the undulator. Due to the different propagation velocities of the electron beam and of the electromagnetic field, the radiation pulse slips over the high quality electron beam and is fed by the fresh electrons it finds on its course. For this reason the system dynamics includes a self-induced ‘fresh bunch injection’ without the need of dispersive elements alternated to the cascade sections to overcome the energy spread budget limitation. At the same time the superradiant pulse is up-converted in frequency at the transitions between the stages of the cascade, because of the strong modulation components at all the higher order harmonics of the fundamental, located at
its leading edge. A cascade based on this process is less sensitive to the characteristics of the seed signal and very short, self-similar pulses may be produced with high harmonic multiplication factors in a multiple stage device.

In all the above examples, since the undulator magnet has period length in the cm range, and the wavelength tuning is obtained by varying only the undulator period $\lambda_w$ and strength $K$, a very high energy electron beam is still needed to access the far UV and x-ray region of the electromagnetic spectrum. One way to reduce the necessary beam energy consists of exploiting the FEL harmonic interaction. The FEL resonance condition, i.e. the condition for efficient energy exchange between the transverse EM wave and the electrons, takes place at electron energies such that the wiggling induced by the laser field has the same frequency as the wiggling induced by the undulator in the beam rest frame. However, a planar undulator resonance can also occur when the laser frequency is a multiple of the undulator fundamental frequency has been experimentally demonstrated in the oscillator configuration where the wavelength is selected by the reflectivity of the resonator optics [23]. In a single pass FEL, the selection of one of the higher order harmonics may be obtained by introducing suitable phase jumps along the undulator in order to suppress the gain of the fundamental frequency which otherwise would grow and saturate before the higher harmonic field. This principle has been studied in simulations both in steady state [24] and in time-dependent mode, assuming startup from e-beam shot noise (SASE) and from a coherent external seed [25].

In the harmonic cascade scheme, a similar suppression of the fundamental harmonic field is obtained by tuning the two stages of a cascade with their fundamental resonant wavelength in a ratio corresponding to a rational number $n/m$. The pulse generated in the first stage of the cascade, is off resonance in the second stage, but its $n$th order harmonic is still resonant with the $m$th frequency of the second stage and may be amplified. The microbunching at the higher harmonic in the second undulator is enhanced by the modulation in the previous undulator so that lasing at shorter wavelengths may be obtained before the growth of the SASE signal at the fundamental resonance of the second undulator. In principle the harmonic cascade is a concept which can be useful in all the conventional schemes of FEL frequency multiplication devices because of the possibility of generating short wavelength radiation pulses with low-energy electron beams. It turns out that a particular advantage comes in the superradiant scheme since in this case, the efficiency is proportional to the slippage length $\Delta = N\lambda_1$. In a superradiant harmonic cascade, despite the fact that the amplified pulse has wavelength $\lambda_m$ the slippage distance and the energy extracted from the electron beam are proportional to the fundamental wavelength which is $m$ times larger.

In the next section, we will review the superradiant cascade scheme and section 3 is devoted to an introduction of the harmonic cascade concept. Simulations of this FEL scheme have been performed in one-dimension (1D) and in 3D using the existing well-known codes PERSEO and GENESIS 1.3, the latter suitably modified to follow the dynamics of the harmonic field. Finally a discussion of some specific configurations is given in the fifth section of the paper, where it is shown how a relatively low-energy electron beam (200 MeVs) may be used to achieve coherent radiation pulses at very short wavelengths (sub 50 nm).
2. Superradiant FEL cascade

In the superradiant regime a solitary wave-like, short and intense pulse of radiation propagates through the undulator. The pulse leading edge travels at the velocity of light and slips over the beam current by $N$ resonant optical periods every $N$ undulator periods. Because of nonlinear effects the pulse length is related to its intensity and it is approximately given by the expression

$$\tau \approx \frac{\tau_s \lambda}{2 \lambda_w},$$

where $\tau$ is the pulse length, $\lambda$ is the optical wavelength, $\lambda_w$ is the undulator period and $\tau_s$ is the synchrotron oscillation period. The latter can be expressed as a function of the dimensionless normalized vector potential associated with the radiation wave $K_r = eE_0\lambda_0/2\pi m_0 c^2$ as

$$\tau_s = \frac{2\pi}{\omega_s} = \frac{\lambda_w}{c} \sqrt{1 + K^2/2} f_B(n, \xi K),$$

where $K$ is the normalized undulator strength and $f_B(n, \xi) = J(n-1/2)(n\xi) - J(n+1/2)(n\xi)$ is the planar undulator Bessel function factor with $\xi = K^2/(4 + 2K^2)$ and $n = 1$.

Superradiance in FELs has been observed in experiments both in oscillators [26, 27], and in single pass devices. At the SDL (Brookhaven) FEL test facility a direct measurement of the second order FROG trace of the superradiant pulse has been recorded and the transition from the exponential regime to the superradiant regime has been documented [28].

In this regime, the pulse dynamics obeys a number of scaling laws derived in the limit of a 1D model. The pulse energy grows as $z^{3/2}$, the peak power grows as $z^2$ and the pulse length decreases as $z^{-1/2}$, where $z$ is the longitudinal position of the pulse along the undulator. The pulse leading edge modulates the e-beam energy, emission takes place in the front part of the pulse where the strong optical field and the undulator dispersion has converted the energy modulation into a density modulation. In the rear side the particles following synchrotron orbits in the longitudinal phase space, shift out of phase with the radiation and the radiation energy is re-absorbed by the electrons. This mechanism explains why the pulse length is strictly related to the synchrotron period and it is determined by the laser intensity (equation (1)). For a more comprehensive view of particles and field dynamics in FEL in superradiant regime the reader is addressed to other papers on the subject [19, 29, 30].

In a cascaded FEL operating in the superradiant regime the distinction between the modulator and the radiator typical of common HGHG-FEL schemes is no longer appropriate: the steps of energy modulation, density modulation and emission take place simultaneously at different locations along the optical pulse. As in steady state FEL dynamics, the density modulation induces intense emission of radiation at higher order harmonics, but this occurs during the pulse propagation along all the undulator. When the pulse passes from one stage of a cascade to another, the modulation at the higher order harmonic emits coherently a short pulse of radiation at the new resonant wavelength. This pulse slips on a fresh portion of the electron beam and may be amplified in the exponential gain regime or, as happens in most cases, may be intense enough to saturate and enter the superradiant regime.

The pulse intensity and phase become quickly independent from those of the seed pulse that initiated the cascade and are determined only by the dynamical properties of the system. The relation between the pulse length and the peak power makes this approach very well suited for...
the generation of short pulses. The pulse length may become shorter than the FEL cooperation length, which is the length of a single radiation spike occurring before saturation in the SASE amplification. The pulse evolution in superradiance is not governed by the FEL exponential gain process, and the resulting cascade has a lower sensitivity to external parameter variations with respect to a conventional cascade operating below saturation. As an example, a fluctuation in intensity of the seed which initiated the cascade may induce a change of the length over which the same output power is achieved, which scales as $z^2$ and the pulse length at the exit may change by a factor proportional to $z^{-1/2}$. The amplitudes of the bunching coefficients are completely independent from the seed intensity, once the peak intensity has reached saturation. This concept was the basis of a multiple stage cascade design with a frequency multiplication factor exceeding 50, with final wavelength 5 nm illustrated in [19].

3. The harmonic cascade

One of the factors limiting the frequency multiplication of all cascade schemes is related to the necessity of operating all the stages of the cascade with the same electron beam energy. The change in the resonant wavelength for the different sections must be compensated by variations of the undulator period and $K$ parameter. Using the conventional techniques of undulator magnet construction, the span of these variations is limited. Henceforth to access shorter wavelength, one needs a high energy electron beam.

An alternative is the ‘harmonic cascade’ concept. In order to introduce this concept, let us first analyse the case of a conventional cascade composed by two undulators with resonances $\omega_1$ and $\omega_2 = n\omega_1$. In the first undulator, where the modulation at $\omega_1$ occurs, a coupling at the higher frequency $n\omega_1$ is already present for $n$ odd. The FEL parameter $\rho_n$ corresponding to the $n$th harmonic coupling is given by the expression [31]

$$\rho_n = \left( \frac{f_B(n, \xi)}{f_B(1, \xi)} \right)^{2/3} \rho_0,$$

where $\rho_0$ is the FEL parameter for the fundamental harmonic [32]. The parameter $\rho_n$, which is inversely proportional to the saturation length, scales with the harmonic order as the function $f_B(n, \xi)^{2/3}$ plotted in figure 1 for the first seven harmonics, at $K = 2, 3$ and 5.

One of the reasons why the cascade design based on two undulators with the second resonant at $\omega_2 = n\omega_1$ provides higher power radiation at the $n$th harmonic wavelength is that in the second undulator, the field at $\omega_1$ is out of resonance and the modulated e-beam interacts only with the field at the frequency $\omega_2$. The synchrotron oscillations induced by saturation of the field at $\omega_1$, which would inhibit the amplification of all the higher order harmonics, are suppressed and the field at frequency $\omega_2$ may grow up to full saturation. A similar effect can be obtained with two undulators which have their main resonances ($n = 1$) differing by a relatively small amount, by tuning the second undulator in such a way that one of its higher order harmonics is equal to $n\omega_1$, i.e. $m\omega_2 = n\omega_1$, with $m \neq n$. A schematic diagram of the frequency spectrum of the two undulators is sketched in figure 2

The two undulators main resonant frequencies differ by the ratio $n/m$ and in fact can have the same period and slightly different undulator parameter $K$. The output frequency is a higher harmonic in both the undulator sections. As previously pointed out, this scheme is particularly interesting when operated in superradiant regime because the energy exchange between the
current and the laser field is proportional to the slippage length which scales with the fundamental resonant wavelength even if the radiation is amplified at the harmonic wavelength which is $m$ times smaller.

4. Simulation of the harmonic cascade

While in the simulation of FELs operating on the fundamental undulator resonant frequency the self-consistent coupling between particles and fields on the higher order harmonics may be neglected, this is not true in the case of the harmonic cascade. One limitation in current state-of-the-art simulations of the FEL interaction has been the difficulty to model at the same time, time-dependent effects, the evolution of the harmonic electromagnetic field components and 3D effects (see table 1 and [33]). Many codes can solve the 1D FEL equations [34, 35] in time-dependent mode, or the 3D time-dependent [36]–[38], or the 3D harmonic [21]. Only

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**Figure 1.** Higher order harmonic FEL coefficients $f_B(n, \xi)^{2/3}$ for $n = 1, 3, 5$ versus $K$.

**Figure 2.** Scheme of harmonic cascade spectrum.
recently the possibility of time-dependent simulation of FEL oscillator and amplifiers has been introduced in Medusa [39].

The harmonic cascade scheme has been recently simulated in 1D [40] in PERSEO, a library of functions that can be simply programmed to set up FEL simulations in a wide variety of practical configurations. In order to take into account 3D effects, we included the evolution of the harmonic electromagnetic field components in GENESIS 1.3. This code has been widely used in the simulations of many FEL projects around the world and it has been validated against the major experimental results for high gain FELs. GENESIS 1.3 solves the FEL differential equation in a fully 3D Cartesian coordinate system and it computes the transverse particle dynamics due to the natural undulator focusing and/or to an external quadrupole lattice.

In order to implement the higher harmonics in a FEL dynamics simulation, the slowly varying envelope approximation has to be applied to the field components at the different harmonics. This is possible when the field frequency components are different from zero only in narrow spectral windows centred at the undulator resonant frequencies. The field may be then written as

\[ E(z, t) = \sum_n \tilde{E}_n(z, t) e^{i(k_n z - \omega_n t)}. \]  (4)

The \( \tilde{E}_n(z, t) \) terms represent the ‘slow’ components of the field at the different harmonics. After a change of variables from the laboratory frame to the beam frame moving with velocity \( \beta_z \), i.e. \((z, t) \rightarrow (\xi = z - \beta_z c t, \tilde{z} = \beta_z c t)\) the equation for the evolution of the field slowly varying component at the harmonic \( n \) reads

\[ \left[ \frac{\partial}{\partial \tilde{z}} + (1 - \beta_z) \frac{\partial}{\partial \xi} \right] \tilde{E}_n(\tilde{z}, \xi) = -\frac{e_0}{2\varepsilon_0} \frac{K}{\gamma} \rho(\xi) f_\theta(n, \xi) (e^{in\theta_i})_i, \]  (5)

where the variable \( \gamma = \langle \gamma_i \rangle \) is the average beam energy in \( m_0 c^2 \) units, \( e_0 \rho(\xi) \) is the charge density and where \((\gamma_i, \theta_i = (k + k_w) z_i - \omega t_i)\) individuate the \( i \)th particle coordinates in the longitudinal phase space. The bunching coefficients average \( \langle e^{in\theta_i} \rangle \) is intended over a region of size \( \lambda \) at the position with coordinate \( \xi \).
The multi-frequency pendulum-like longitudinal dynamics of the \(i\)th electron is governed by the equations

\[
\frac{d\gamma_i}{d\tilde{z}} = \sum_n K e_0 \tilde{E}_n(f_B(n, \xi) \cos(n\theta + \phi), \quad \frac{d\theta_i}{d\tilde{z}} = k_w - k - \frac{1 + K^2/2}{2\gamma^2}.
\]

(6)

where \(k_w = 2\pi/\lambda_w\) is the wave-vector corresponding to the undulator period.

In the case \(n = 1\), the solution of the coupled equations (5) and (6) is treated extensively in the literature [32, 41, 42]. Since the \(n = 1\) term is usually the dominant term of the sum on the right-hand side of the first of (6), retrieving the evolution of the particle distribution from a single-harmonic FEL code, in principle one could \(a\ posteriori\) calculate the bunching coefficients and the radiation at the harmonics. On the other hand, this post-processing solution is not self-consistent as it is valid in the approximation that the harmonic field does not have an important contribution to the particle dynamics. When the seed (either as electromagnetic field or bunched beam) is at an harmonic of the fundamental undulator frequency, as it is in the harmonic cascade case, this simplified calculation cannot give the correct solution.

Based on these considerations, the extension to the multiple harmonic case is reduced to the inclusion of the terms with \(n \neq 1\) and to the solution of the remaining \(n - 1\) partial differential equations (5). Moreover the equations of motions for the particles in 3D have to take into account the transverse motion in the magnetic lattice consisting of the undulator and eventually focusing quadrupoles. The equation for the evolution of the field components \(\tilde{E}_n(\tilde{z}, \zeta, x, y)\) now functions of the transverse coordinates including the diffraction effects in the paraxial approximation reads

\[
\left(\frac{\partial}{\partial \tilde{z}} + (1 - \beta_z) \frac{\partial}{\partial \zeta} - \frac{i}{2nk} \nabla^2_\perp\right) \tilde{E}_n(\tilde{z}, \zeta) = -\frac{e_0}{2\epsilon_0} \frac{K}{\gamma} \rho(\zeta) f_B(n, \xi) \langle e^{i\theta_i}\rangle_i.
\]

(7)

To extend the computation to the harmonic field case, two major modifications have been applied to Genesis 1.3. From the 3D particle distribution, we computed the \(n\) different source terms on a transverse space grid and solved for the harmonic field the wave equation in the paraxial approximation which takes into account properly the diffraction effects. As a check of the validity of the such implementation, we have calculated the radiation rms size for a free propagation of the first three harmonics. The simulation results match perfectly the dependence of the Rayleigh ranges on the radiation wavelengths (see figure 3).

The complex field components at the harmonic frequencies defined on the spatial grid are then used in the equation for the particle dynamics. The modified version of the code has been checked in a steady state simulation against the 1D code PERSEO. The peak power at first, third and fifth harmonic along the undulator are reported in figure 4. The simulation is done in steady state mode, starting from a seed of 10 kW on the fundamental undulator resonant frequency. The 3D simulation shows a longer lethargy at the beginning of the FEL amplifier justified by the fact that the transverse mode of the seed is not perfectly matched to the electron beam and it takes a longer distance in the undulator to enter the exponential gain regime. The gain lengths of the different harmonics are in very good agreement showing that the coupling coefficients between fields and particles dynamics are calculated consistently in the two codes. Some differences appear at saturation, where the optical mode is no longer guided by the gain process. In these conditions, we expect that the 1D code, where diffraction effects are included only by correcting \(a\ priori\) the coupling coefficients with a filling factor derived from the Xie analytical fit [43],
Figure 3. RMS size of the field at the first three harmonics propagating without gain. The steady-state simulation of the FEL amplifier is performed with GENESIS 1.3 with harmonic calculation implemented. The analytic solution for free space propagation at the three harmonic wavelengths is shown for comparison.

Figure 4. First, third and fifth harmonic in a steady state simulation of an FEL amplifier from a 1D simulation with PERSEO (red) and a 3D simulation with modified GENESIS 1.3 (black).
In the 3D simulation, the off-axis particles that experience a lower field reach saturation later in the undulator. The saturation condition and their contribution to the global irradiated power adds up in a different way with respect to the 1D case.

The simulation of the harmonic cascade requires specific attention to the set-up of the initial particles’ phase space. In the harmonic cascaded FEL, the beam propagates in (at least) two undulators, where the ratio between the resonant wavelengths is a non-integer rational number. The initial electron beam distribution must satisfy a quiet start condition in both the undulator sections that have a different fundamental resonance [33]. To be consistent, the length of the elementary beamlet representing the particles longitudinal phase space on the wavelength scale, must be equivalent to an integer number of optical cycles in both the undulators. In order to satisfy this condition the simulated particles have been distributed in the interval \((0, 2 \pi m)\) in the first undulator and \((0, 2 \pi n)\) in the second undulator. In this situation it is not straightforward to set the coefficients introducing the shot noise consistently in both undulators, whose resonant wavelengths are in the ratio \(\lambda_1/\lambda_2 = m/n\) [44]. On the other hand, the shot noise is not expected to give relevant effects in a simulation starting from a high intensity seed, when the undulator length is shorter than the SASE saturation length. For this reason the beam shot noise has been suppressed.

5. Examples of ‘harmonic cascade’ FELs

For a practical implementation of the harmonic cascade scheme, we have chosen the parameters of the SPARC-FEL project [45]. The seed wavelength is the third harmonic of the Ti:Sa solid state laser which also drives the photocathode. This source is synchronized with the electron bunch and can easily generate the intense and short UV pulses required to reach saturation in a short undulator. The SPARC undulator is composed of six, variable gap, independent sections. In order to implement the harmonic cascade, the first section is set with the resonance at 266 nm. This is the section of the undulator where the superradiant pulse is formed. We give three different examples of harmonic cascade. The input parameters are reported in table 2 and are labelled A, B and C. In case A, the second part of the undulator, composed by the other five sections, is tuned to 200 nm whose third harmonic (66 nm) is resonant with the fourth harmonic of the first undulator. With the notation used in the previous sections, we have \(n = 4\) and \(m = 3\). In case B, the final radiation wavelength is still at 66 nm, but we obtain it by tuning the second undulator of the cascade to a resonant wavelength five times longer, that is 332 nm. In other words, we have \(n = 4, m = 5\). Finally in case C, we fully exploit the advantage of the harmonic cascade scheme generating coherent radiation at 44 nm (which is the sixth harmonic of the first undulator and the fifth harmonic of the second one). Despite the high order of harmonic multiplication, the two undulators always differ by relatively small \(K\) parameter variations which shift the resonance only of about 30%. A summary of the results of the simulations is listed in table 3. For the cases A and B, the radiation parameters are calculated over a spectral region centred at 66 nm, in case C the spectral region of interest is centred at 44 nm. The gain length of the SASE instability at the fundamental resonance in the second undulator is listed in the last row of table 2. At the end of the second undulator, the bunching coefficient at the fundamental resonance induced by SASE is of the order of 0.1 and is not yet substantially perturbing the evolution of the superradiant pulse. This is however a limiting factor to the possibility of further extending the length of the second undulator section.

Table 2. Parameters used for the simulation of the harmonic cascade in three different cases: A with $n = 4$, $m = 3$, B with $n = 4$, $m = 5$ and C with $n = 6$, $m = 5$.

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<td>200 MeV</td>
</tr>
<tr>
<td>Beam energy spread</td>
<td>20 KeV</td>
<td>20 KeV</td>
<td>20 KeV</td>
</tr>
<tr>
<td>Beam current</td>
<td>110 A</td>
<td>110 A</td>
<td>110 A</td>
</tr>
<tr>
<td>Average beta</td>
<td>2.55 m</td>
<td>1.70 m</td>
<td>2.25 m</td>
</tr>
<tr>
<td>Emittance</td>
<td>1 mm mrad</td>
<td>1 mm mrad</td>
<td>1 mm mrad</td>
</tr>
<tr>
<td>Gain length @ 2nd und. $\lambda$</td>
<td>0.57 m</td>
<td>0.53 m</td>
<td>0.55 m</td>
</tr>
</tbody>
</table>

Table 3. Results of the harmonic cascade simulation in the three different reference cases: A with $n = 4$, $m = 3$, B with $n = 4$, $m = 5$ and C with $n = 6$, $m = 5$.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radiation wavelength</td>
<td>66 nm</td>
<td>66 nm</td>
<td>44 nm</td>
</tr>
<tr>
<td>Peak power</td>
<td>140 MW</td>
<td>155 MW</td>
<td>10 MW</td>
</tr>
<tr>
<td>Energy</td>
<td>4 $\mu$J</td>
<td>8 $\mu$J</td>
<td>1 $\mu$J</td>
</tr>
<tr>
<td>FWHM pulse length</td>
<td>40 fs</td>
<td>48 fs</td>
<td>70 fs</td>
</tr>
</tbody>
</table>

In figure 5, the pulse shape after the modulation section for an input seed power of 10 MW is shown. At the end of the first undulator, the seed pulse is just entering the superradiant regime and starts to assume the characteristic self-similar shape. The first bunching coefficient exceeding 0.75 is an indication of saturation at the fundamental wavelength. As expected a strong bunching at the fourth harmonic in the front side of the pulse is present. This fourth harmonic modulation will be responsible for the generation of radiation in the second part of the cascade (case A and B).

In figure 6(a), we show the pulse energy versus $\tilde{z}$ in the second undulator for the cases A and B. In the first part of the undulator, the existing modulation with an harmonic component at 66 nm induces the growth of an intense radiation pulse by coherent harmonic generation. After about 6 m of interaction, the pulse energy scaling versus $\tilde{z}$ becomes that typical of the superradiant regime. The final pulse energy exceeds 4 $\mu$J at 66 nm in case A and 8 $\mu$J at the same wavelength in case B. In case A, the undulator resonant wavelength is shorter and the slippage is smaller. Since in the superradiant regime the total energy of the electromagnetic field is proportional to
how many electrons have been slipped over by the wave, it is understandable why in case B the pulse has a higher energy.

The pulse shape at the end of the second undulator is shown in figure 7(a). The characteristic solitary wave self-similar shape is now evident. The (red) dashed line represents the local energy spread. The pulse length at the end of the harmonic cascade is 40 fs (fwhm) and the peak power exceeds the 140 MW level. The spectrum at the end of the undulator is shown in figure 7(b). The spectral width is 2 nm (fwhm) corresponding to 3% of the central wavelength.
Figure 7. Power profile (a) and spectrum (b) of the pulse at the end of the second undulator for case A. The dashed line (red) in (a) is the local energy spread.

Figure 8. Pulse energy and duration (FWHM) versus $\tilde{z}$ for in the configurations listed in table 2, case A (---- red) and B (—— black). The seed power has been reduced to 2 MW.

An analysis of the system response to a change in the input field intensity can be done by observing figure 8(a) where the pulse energy is shown with an input seed power reduced to 2 MW. Despite the reduction of a factor five in the seed, the output power changes only slightly in both cases A and B. In figure 8(b), the behaviour of the FWHM pulse length versus $\tilde{z}$ is shown. The slow increase in pulse duration at $\tilde{z} \sim 4$ m is an indication of exponential gain growth that is anticipating saturation. A change in the input seed power is compensated by a longer distance of propagation in the exponential gain regime leading to saturation. Superradiance is established only at $\tilde{z} \sim 7$ m where the pulse length starts to decrease according to the scaling law $\tau \propto \tilde{z}^{-1/2}$. The pulse length behaviour at the beginning of the undulator has a different interpretation. The radiation is emitted in the coherent harmonic generation regime and the lengthening is associated to slippage. The different slope of elongation between the traces representing the case A and B respectively, is due to the different fundamental undulator wavelength. Only the region located at the leading edge of the pulse is able to propagate over fresh electrons which had no FEL
interaction (except for the negligible SASE contribution induced in the first segment) and is amplified. This explains the pulse length reduction observed for $\bar{\tau} \sim 2–4\,\text{m}$.

Finally, we studied how the initial beam quality affects the output of such a FEL device. We varied the input energy spread and the input emittance in the simulations for case A and recorded the final peak power. The results of these scans are reported in figure 9.

Despite the stability with respect to a variation of the input seed power, the proposed device is still very sensitive to the beam quality. A ‘slice’ emittance of $1.5\,\text{mm mrad}$ or a ‘slice’ energy spread of $2 \times 10^{-4}$ are sufficient to reduce the output power by a factor $\sim 2–3$. On the other hand, it must be considered that the parameters in table 2 for the SPARC layout are relevant to an uncompressed beam with a peak current of $110\,\text{A}$ and the SPARC-FEL is not designed with the aim of optimizing the superradiant cascade. The power levels listed in table 3 are however competitive with other sources available in the same spectral range [46] and a longer undulator would substantially increase the emitted power.

6. Conclusions

While the design and construction of many FELs is based on the use of the fundamental lasing wavelength, advanced FEL schemes do foresee the use of the harmonics in the quest for shorter and shorter wavelengths and the harmonic cascade FEL is a promising configuration to reduce the operation wavelength of a single pass device. The examples shown have pointed out how lasing at a wavelength of $44\,\text{nm}$ can be obtained with a beam energy of $200\,\text{MeV}$, which is compatible with the SPARC linac energy. A similar configuration could be suitable for extending the spectral range of x-ray FELs as the LCLS [47], or the European X-FEL [48]. In this framework, several ideas for the generation of attosecond intense radiation pulses have been proposed [49]–[52]. For instance, the energy chirped beam operation for the selection of a single SASE temporal spike [51], could be combined in an undulator composed by multiple segments, with the harmonic cascade
concept proposed here. Once such a short pulse has reached saturation in the first segment, it could eventually constitute a suitable initial condition for starting a superradiant cascade aimed at extending the spectral range of coherent FEL sources to the sub-Angstrom regime. One of the aspects that we have pointed out about superradiance is a reduced sensitivity to the seed properties due to the nonlinear evolution of the short pulse in the FEL amplifier. The pulse energy growth is indeed proportional to the slippage length and to the beam current, i.e. to the number of electrons that are trapped by the moving bucket of the saturated FEL and the pulse shape is entirely determined by the nonlinear properties of superradiance. Despite the differences in the dynamics with respect to the exponential gain regime, the quality of the electron beam still plays an important role, as the deteriorating effects of the emittances and the energy spread on the final output power have shown. The role of the emittances and more generally of the inhomogeneous spectral line broadening on the superradiant pulse propagation must be further studied and could constitute the real limit to the application of such a scheme to a hard x-ray FEL.

For these reasons and for the increasing interest in harmonic generation in short wavelength FELs design, there is a strong demand for codes that simulate the 3D time-dependent harmonic generation process. The present study has the additional byproduct of making available a useful tool in the design of future light sources foreseeing the use of higher order harmonics. The GENESIS 1.3 code capabilities can be further extended to take into account the coupling to the even harmonics associated to the particles off-axis betatron motion. Even harmonics in the SPARC simulations have been neglected since the betatron angles of the electron trajectories are very small, but their inclusion in the physical model may be simply obtained by expanding in the small betatron angle limit the double sum of Bessel functions [53].

Acknowledgment

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References


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