Deformation quantization of superintegrable systems and Nambu mechanics

To cite this article: Thomas L Curtright and Cosmas K Zachos 2002 *New J. Phys.* **4** 83

View the article online for updates and enhancements.

Deformation quantization of superintegrable systems and Nambu mechanics

Thomas L Curtright[1] and Cosmas K Zachos[2]

[1] Department of Physics, University of Miami, Box 248046, Coral Gables, FL 33124, USA

[2] High Energy Physics Division, Argonne National Laboratory, Argonne, IL 60439-4815, USA

E-mail: curtright@physics.miami.edu and zachos@hep.anl.gov

Abstract. Phase space is the framework best suited for quantizing superintegrable systems, naturally preserving the symmetry algebras of the respective Hamiltonian invariants. The power and simplicity of the method is fully illustrated through new applications to nonlinear σ-models, specifically for de Sitter N-spheres and chiral models, where the symmetric quantum Hamiltonians amount to compact and elegant expressions. Additional power and elegance is provided by the use of Nambu brackets to incorporate the extra invariants of superintegrable models. Some new classical results are given for these brackets, and their quantization is successfully compared to that of Moyal, validating Nambu's original proposal.

1. Introduction

Highly symmetric quantum systems are often integrable and, in special cases, superintegrable and exactly solvable [1]. A superintegrable system of N degrees of freedom has more than N independent invariants, and a maximally superintegrable one has $2N - 1$ invariants. In the case of velocity-dependent potentials, when quantization of a classical system presents operator ordering ambiguities involving x and p, the general consensus has long been [2]–[5] to select those orderings in the quantum Hamiltonian which maximally preserve the symmetries present in the corresponding classical Hamiltonian. Often, even for simple systems, such as σ-models considered here, such constructions may become involved and needlessly technical.

It is pointed out here that, in contrast to conventional operator quantization, this problem of selecting the quantum Hamiltonian which maximally preserves integrability is addressed most suitably and cogently in Moyal's phase-space quantization formulation [6]–[8]. The reason

is that the variables involved in it ('classical kernels' or 'Weyl transforms of operators') are c-number functions, like those of the classical phase-space theory, and have the same interpretation, although they involve $\hbar$-corrections ('deformations'), in general—so $\hbar \to 0$ reduces them to the classical expressions. It is only the detailed algebraic structure of their respective brackets and composition rules which contrast with the variables of the classical theory. This complete formulation is based on the Wigner function (WF), which is a quasi-probability distribution function in phase space, and comprises the kernel function of the density matrix. Observables and transition amplitudes are phase-space integrals of kernel functions weighted by the WF, in analogy to statistical mechanics. Kernel functions, however, unlike classical functions, compose through the $\star$-product, a noncommutative, associative, pseudodifferential operation, which encodes the entire quantum mechanical action and whose antisymmetrization (commutator) is the Moyal bracket (MB) [6]–[8].

Any arbitrary operator ordering could be brought to Weyl-ordering format by the use of Heisenberg commutations, and through Weyl's transform corresponds invertibly to a specific $\hbar$-deformation in the classical kernel [9, 10]. Thus, two operators of different orderings correspond to kernel functions differing in their deformation terms of $O(\hbar)$. The problem thus reduces to a purely $\star$-product algebraic one, as the resulting preferred orderings are specified and encoded most simply by far through the particular deformation of the resulting c-number kernel expressions.

Hietarinta [11] has investigated in this phase-space quantization language the simplest integrable systems of velocity-dependent potentials. In each system, he has promoted the vanishing of the Poisson bracket (PB) of the (one) classical invariant I (conserved integral) with the Hamiltonian, $\{H, I\} = 0$, to the vanishing of its (quantum) MB with the Hamiltonian, $\{\{H_{qm}, I_{qm}\}\} = 0$. This dictates quantum corrections, addressed perturbatively in $\hbar$: he has found $O(\hbar^2)$ corrections to the Is and H (V), needed for quantum symmetry. The expressions found are quite simple, as the systems chosen are such that the polynomial character of the ps, or suitable balanced combinations of ps and qs, ensure collapse or subleading termination of the MBs. The specification of the symmetric Hamiltonian is then complete, since *the quantum Hamiltonian in terms of classical phase-space variables corresponds uniquely to the Weyl-ordered expression for these variables in operator language*. Berry [12] has also studied the WFs of integrable systems in great depth.

In this paper, nonlinear σ-models (with explicit illustrations on N-spheres and chiral models) are utilized to argue for the general principles of power and convenience in isometry-preserving quantization in phase space, for large numbers of invariants, in principle (as many as the isometries of the relevant manifold). In the cases illustrated, the number of algebraically independent invariants matches or exceeds the dimension of the manifold, leading to superintegrability [1], whose impact is best surveyed through Nambu brackets (NB) [13]–[16]. The procedure of determining the proper symmetric quantum Hamiltonian then yields remarkably compact and elegant expressions.

Briefly, we find that the symmetry generator invariants are undeformed by quantization, but the Casimir invariants of their MB algebras are deformed. Hence, the Hamiltonians are also deformed by terms $O(\hbar^2)$, as they consist of quadratic Casimir invariants. Their spectra are then read off through group theory, properly adapted to phase space. The basic principles are illustrated for the simplest curved manifold, the 2-sphere, in section 2, while generalization to larger classes of symmetric manifolds such as chiral models and N-spheres is provided in sections 3 and 4, which also investigate the underlying distinctive geometry of such models.

Moreover, in section 5, the classical evolution of all functions in phase space for such systems is specified through NBs, whose quantization is briefly outlined and compared to the standard Moyal deformation quantization utilized in this work. This comparison validates Nambu's original quantization proposal. Conclusions are summarized in section 6, while a few geometrical derivations on the classical structure of chiral models are provided in the appendix.

2. Principles and S^2

Consider a particle on a curved manifold, in integrable one-dimensional σ-models considered by Sasaki (unpublished):

$$L(q, \dot{q}) = \tfrac{1}{2} g_{ab}(q)\, \dot{q}^a \dot{q}^b, \tag{1}$$

so that

$$p_a = \frac{\partial L}{\partial \dot{q}^a} = g_{ab}\dot{q}^b, \qquad \dot{q}^a = g^{ab}p_b. \tag{2}$$

Thus

$$H(p, q) = \tfrac{1}{2} g^{ab} p_a p_b \qquad (=L). \tag{3}$$

The isometries of the manifold generate the conserved integrals of the motion [22]. The classical equations of motion are

$$\dot{p}_a = -\frac{g^{bc}_{,a}}{2} p_b p_c = \frac{g_{bc,a}}{2}\, \dot{q}^b \dot{q}^c. \tag{4}$$

As the simplest possible nontrivial illustration, consider a particle on a 2-sphere of unit radius, S^2. In Cartesian coordinates (after the elimination of z, so $q^1 = x, q^2 = y$), one has, for $a, b = 1, 2$:

$$g_{ab} = \delta_{ab} + \frac{q^a q^b}{u}, \qquad g^{ab} = \delta_{ab} - q^a q^b, \qquad \det g_{ab} = \frac{1}{u}, \qquad u \equiv 1 - x^2 - y^2. \tag{5}$$

(u is the sine-squared of the latitude, since this represents the orthogonal projection of the globe on its equatorial plane.) The momenta are then

$$p_a = \dot{q}^a + q^a \frac{h}{u} = \dot{q}^a + q^a (q \cdot p), \qquad h \equiv -\dot{u}/2 = x\dot{x} + y\dot{y}. \tag{6}$$

The classical equations of motion here amount to

$$\dot{p}_a = p_a q \cdot p, \qquad \text{i.e.,} \quad \ddot{q}^a + q^a \left(\frac{\dot{h}}{u} + \frac{h^2}{u^2} \right) = 0. \tag{7}$$

It is then easy to find the three classical invariants, the components of the conserved angular momentum in this nonlinear realization,

$$\begin{aligned} L_z &= x p_y - y p_x, \\ L_y &= \sqrt{u}\, p_x, \\ L_x &= -\sqrt{u}\, p_y. \end{aligned} \tag{8}$$

(The last two are the de Sitter 'momenta', or nonlinearly realized 'axial charges' corresponding to the 'pions' x, y of the σ-model: linear momenta are, of course, not conserved.) Their PBs close into $SO(3)$:

$$\{L_x, L_y\} = L_z, \qquad \{L_y, L_z\} = L_x, \qquad \{L_z, L_x\} = L_y. \tag{9}$$

Thus, it follows algebraically that their PBs with the Casimir invariant $\boldsymbol{L}\cdot\boldsymbol{L}$ vanish. Naturally, since $H = \boldsymbol{L}\cdot\boldsymbol{L}/2$, they are manifested to be time-invariant:

$$\dot{\boldsymbol{L}} = \{\boldsymbol{L}, H\} = 0. \tag{10}$$

In quantizing this system, operator ordering issues arise, given the effective velocity (momentum)-dependent potential. In phase-space quantization, one may insert Groenewold's [10] associative (and noncommutative) $\star$-products:

$$\star \equiv \exp\Big(\frac{\mathrm{i}\hbar}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_{p_x} - \overleftarrow{\partial}_{p_x}\overrightarrow{\partial}_x + \overleftarrow{\partial}_y\overrightarrow{\partial}_{p_y} - \overleftarrow{\partial}_{p_y}\overrightarrow{\partial}_y)\Big), \tag{11}$$

in strategic points and orderings of the variables of (3), to maintain integrability. That is, the classical invariance expressions (PB commutativity):

$$\{I, H\} = 0 \tag{12}$$

are to be promoted to quantum invariances (MB commutativity):

$$\{\{I_{qm}, H_{qm}\}\} \equiv \frac{I_{qm} \star H_{qm} - H_{qm} \star I_{qm}}{\mathrm{i}\hbar} = 0. \tag{13}$$

(As $\hbar \to 0$ the MB reduces to the PB.) Here, this argues for a (*c*-number kernel function) Hamiltonian of the form

$$H_{qm} = \tfrac{1}{2}(L_x \star L_x + L_y \star L_y + L_z \star L_z). \tag{14}$$

The reason is that, in this realization, the algebra (9) is promoted to the corresponding MB expression *without any modification*, since all of its MBs collapse to PBs by the linearity in momenta of the arguments: all corrections $\mathrm{O}(\hbar)$ vanish. Consequently, these particular invariants are undeformed by quantization, $\boldsymbol{L} = \boldsymbol{L}_{qm}$. As a result, given associativity for $\star$, the corresponding quantum quadratic Casimir invariant $\boldsymbol{L}\cdot \star \boldsymbol{L}$ has vanishing MBs with $\boldsymbol{L}$ (but not vanishing PBs), and automatically serves as a symmetry-preserving Hamiltonian. The specification of the maximally symmetric quantum Hamiltonian is thus complete.

The $\star$-product in this Hamiltonian trivially evaluates to yield the quantum correction to (3):

$$H_{qm} = H + \frac{\hbar^2}{8}(\det g - 3). \tag{15}$$

In phase-space quantization [6]–[8], the WF (the kernel function of the density matrix) evolves according to Moyal's equation [6]:

$$\frac{\partial f}{\partial t} = \{\{H_{qm}, f\}\}; \tag{16}$$

in addition to it, the WFs for pure stationary states also satisfy [17, 7] $\star$-genvalue equations specifying the spectrum:

$$H_{qm}(x, p) \star f(x, p) = f(x, p) \star H_{qm}(x, p)$$
$$= H_{qm}\Big(x + \frac{\mathrm{i}\hbar}{2}\overrightarrow{\partial}_p,\ p - \frac{\mathrm{i}\hbar}{2}\overrightarrow{\partial}_x\Big)f(x, p) = Ef(x, p). \tag{17}$$

The spectrum of this Hamiltonian, then, is proportional to the $\hbar^2 l(l+1)$ spectrum of the $SO(3)$ Casimir-invariant $\boldsymbol{L}\cdot \star \boldsymbol{L} = L_+ \star L_- + L_z \star L_z - \hbar L_z$ for integer l [18]. It can be produced algebraically by the identical standard recursive ladder operations in $\star$-space which obtain in the operator formalism Fock space

$$L_z \star L_+ - L_+ \star L_z = \hbar L_+, \tag{18}$$

where $L_\pm \equiv L_x \pm \mathrm{i}L_y$.

To bound the $\star$-spectrum of L_z, an adaptation of the standard argument is needed to expectation values which are WF-weighted phase-space integrals in this formulation. Indeed, from the real $\star$-square theorem [19], it follows that

$$\langle \boldsymbol{L}\cdot \star \boldsymbol{L} - L_z \star L_z \rangle = \langle L_x \star L_x + L_y \star L_y \rangle \geq 0. \tag{19}$$

The $\star$-genvalues of L_z, m, are thus bounded, $|m| \leq l < \sqrt{\langle \boldsymbol{L}\cdot \star \boldsymbol{L}\rangle}/\hbar$, necessitating $L_- \star f_{m=-l} = 0$. Hence

$$L_+ \star L_- \star f_{-l} = 0 = (\boldsymbol{L}\cdot \star \boldsymbol{L} - L_z \star L_z + \hbar L_z) \star f_{-l}, \tag{20}$$

and consequently $\langle \boldsymbol{L}\cdot \star \boldsymbol{L}\rangle = \hbar^2 l(l+1)$. Similar $\star$-ladder arguments and inequalities apply directly in phase space to all Lie algebras.

Classical Hamiltonians are scalar under canonical transformations, but it should not be assumed that the quantum mechanical expression (15) is a canonical scalar. If, instead of the orthogonal projection employed above, the gnomonic projection [4] from the centre of the sphere were used, i.e. PR_2 projective coordinates,

$$Q^a = \frac{q^a}{\sqrt{1-q^2}}, \qquad Q^2 + 1 = \frac{1}{1-q^2}, \qquad P^a = (p^a - q^a q \cdot p)\sqrt{1-q^2}, \tag{21}$$

it would yield

$$G^{ab} = (1+Q^2)(\delta_{ab} + Q^a Q^b), \qquad G_{ab} = \frac{1}{1+Q^2}\left(\delta_{ab} - \frac{Q^a Q^b}{1+Q^2}\right),$$
$$\det G_{ab} = \frac{1}{(1+Q^2)^3}. \tag{22}$$

The Hamiltonian would now be polynomial:

$$H = \tfrac{1}{2}(1+Q^2)(P^2 + (Q\cdot P)^2). \tag{23}$$

Rewritten in terms of its invariants

$$L_Z = XP_Y - YP_X, \qquad L_Y = P_X + XP\cdot Q, \qquad L_X = -(P_Y + YP\cdot Q), \tag{24}$$

which would obey the same MB $SO(3)$ algebra as before, it would specify a quantum Hamiltonian:

$$H_{QM} \equiv \tfrac{1}{2}(L_X \bigstar L_X + L_Y \bigstar L_Y + L_Z \bigstar L_Z), \tag{25}$$

where $\bigstar$ involves Q, P instead of q, p. This then would lead to the polynomial quantum correction

$$\frac{\hbar^2}{8}(5Q^2 - 2). \tag{26}$$

But this would be different from the above correction:

$$\frac{\hbar^2}{8}\left(\frac{1}{1-q^2} - 3\right). \tag{27}$$

For canonical transformations in phase-space quantization see [7]. The $\star$-product and WFs would not be invariant but would transform in a suitable quantum covariant way [7], so as to yield an identical MB algebra and $\star$-genvalue equations, and thus a spectrum, following from the identical group theoretical construction.

3. Chiral models

The treatment of the 3-sphere S^3 is very similar, with some significant differences, since it also accords to the standard chiral model technology. The metric and equations of motion, etc, are identical in form to those above, except now $u \equiv 1 - x^2 - y^2 - z^2 = 1/\det g$, $h \equiv -\dot{u}/2 = x\dot{x} + y\dot{y} + z\dot{z}$, and $a, b = 1, 2, 3$. However, the description simplifies upon utilization of Vielbeine, $g_{ab} = \delta_{ij} V_a^i V_b^j$ and $g^{ab} V_a^i V_b^j = \delta^{ij}$.

Specifically, the Dreibeine are either left-invariant or right invariant [20]:

$$^{(\pm)}V_a^i = \epsilon^{iab} q^b \pm \sqrt{u} g_{ai}, \qquad ^{(\pm)}V^{ai} = \epsilon^{iab} q^b \pm \sqrt{u} \delta^{ai}. \tag{28}$$

The corresponding right- and left-conserved charges (left- and right-invariant, respectively) are then

$$R^i = {}^{(+)}V_a^i \dot{q}^a = {}^{(+)}V^{ai} p_a, \qquad L^i = {}^{(-)}V_a^i \dot{q}^a = {}^{(-)}V^{ai} p_a. \tag{29}$$

More intuitive than those for S^2 are the linear combinations into axial and isospin charges (again linear in the momenta)

$$\frac{\boldsymbol{R} - \boldsymbol{L}}{2} = \sqrt{u}\boldsymbol{p} \equiv \boldsymbol{A}, \qquad \frac{\boldsymbol{R} + \boldsymbol{L}}{2} = \boldsymbol{q} \times \boldsymbol{p} \equiv \boldsymbol{I}. \tag{30}$$

It can be seen that the $\boldsymbol{L}$s and $\boldsymbol{R}$s have PBs closing into standard $SU(2) \otimes SU(2)$, i.e. $SU(2)$ relations within each set, and vanishing between the two sets. Thus they are seen to be constant, since the Hamiltonian (and the Lagrangian) can, in fact, be written in terms of either quadratic Casimir invariant:

$$H = \tfrac{1}{2}\boldsymbol{L}\cdot\boldsymbol{L} = \tfrac{1}{2}\boldsymbol{R}\cdot\boldsymbol{R} = L. \tag{31}$$

Quantization consistent with integrability thus proceeds as above for the 2-sphere, since the MB algebra collapses to PBs again, and so the quantum invariants $\boldsymbol{L}$ and $\boldsymbol{R}$ again coincide with the classical ones, without deformation (quantum corrections). The $\star$-product is now the obvious generalization to six-dimensional phase space. The eigenvalues of the relevant Casimir invariant are now $j(j+1)$, for half-integer j [21]. However, this being a chiral model ($G \otimes G$), the symmetric quantum Hamiltonian is simpler than the previous one, since it can now also be written geometrically as

$$H_{qm} = \frac{1}{2}(p_a V^{ai}) \star (V^{bi} p_b) = \frac{1}{2}\left(g^{ab} p_a p_b + \frac{\hbar^2}{4} \partial_a V^{bi} \partial_b V^{ai}\right). \tag{32}$$

The Dreibeine throughout this formula can be either $^+V_a^i$ or $^-V_a^i$, corresponding to either the right- or the left-acting quadratic Casimir invariant. The quantum correction then amounts to

$$H_{qm} - H = \frac{\hbar^2}{8}(\det g - 7). \tag{33}$$

This expression, $\frac{\hbar^2}{8}(1/(1-q^2) - 7)$, again is not canonically invariant. For example, in gnomonic PR_3 coordinates†, it is $\frac{3}{4}\hbar^2(Q^2 - 1)$, i.e. it has not transformed as a canonical scalar [7].

If one wished to interpret this simple result (32) in operator language (for operators $\mathfrak{x}$ and $\mathfrak{p}$), it would appear somewhat more complex: the first term, $g^{ab}(x)p_a p_b/2$, would correspond to the Weyl-ordered expression

$$\frac{1}{8}(\mathfrak{p}_a \mathfrak{p}_b g^{ab}(\mathfrak{x}) + 2\mathfrak{p}_a g^{ab}(\mathfrak{x})\mathfrak{p}_b + g^{ab}(\mathfrak{x})\mathfrak{p}_a \mathfrak{p}_b) = \frac{1}{2}\mathfrak{p}_a g^{ab}(\mathfrak{x})\mathfrak{p}_b + \frac{3\hbar^2}{4}, \tag{34}$$

† The inverse gnomonic Vielbein is also polynomial, $\mathsf{V}^{aj} = \delta^{aj} + Q^j Q^a + \epsilon^{jab} Q^b$.

in agreement with [3]. The second term, of course, is unambiguous, since it does not contain momenta.

In general, the above discussion also applies to all chiral models, with $G \times G$ replacing $SU(2) \times SU(2)$ above. That is, the Vielbein-momenta combinations $V^{aj}p_a$ represent algebra generator invariants, whose quadratic Casimir group invariants yield the respective Hamiltonians, and hence the properly $\star$-ordered quantum Hamiltonians as above. (We follow the conventions of [22], taking the generators of G in the defining representation to be T_j.)

That is to say, for

$$\mathrm{i}U^{-1}\frac{\mathrm{d}}{\mathrm{d}t}U = {}^{(+)}V_a^j T_j \dot{q}^a = {}^{(+)}V^{aj}p_a T_j, \qquad \mathrm{i}U\frac{\mathrm{d}}{\mathrm{d}t}U^{-1} = {}^{(-)}V^{aj}p_a T_j, \qquad (35)$$

it follows that the PBs of the left- and right-invariant charges ${}^{(\pm)}V^{aj}p_a = \frac{\mathrm{i}}{2}TrT_j U^{\mp 1}\frac{\mathrm{d}}{\mathrm{d}t}U^{\pm 1}$ close to the identical Lie algebras:

$$\{{}^{(\pm)}V^{aj}p_a, {}^{(\pm)}V^{bk}p_b\} = -2f^{jkn}\,{}^{(\pm)}V^{an}p_a, \qquad (36)$$

and PB commute with each other:

$$\{{}^{(+)}V^{aj}p_a, {}^{(-)}V^{bk}p_b\} = 0. \qquad (37)$$

These two statements are proved in the appendix.

MBs collapse to PBs by linearity in momenta as before, and the Hamiltonian is identical in form to (32). From (83) of the appendix, the quantum correction in (32) is seen to amount to

$$H_{qm} - H = \frac{\hbar^2}{8}(\Gamma_{ac}^b g^{cd}\Gamma_{bd}^a - f_{ijk}f_{ijk}), \qquad (38)$$

(reducing to (33) for S^3).

In operator language, this Hamiltonian H_{qm} amounts to Weyl-ordering of all products on the rhs, but, for generic groups, the first term in (32) does not reduce as simply as in (34) above. The spectra are given by the Casimir eigenvalues for the relevant algebras and representations.

4. S^N

For the generic sphere models, S^N, the maximally symmetric Hamiltonians are the quadratic Casimir invariants of $SO(N + 1)$:

$$H = \tfrac{1}{2}P_a P_a + \tfrac{1}{4}L_{ab}L_{ab}, \qquad (39)$$

where

$$P_a = \sqrt{u}p_a, \qquad L_{ab} = q^a p_b - q^b p_a, \qquad (40)$$

for $a = 1, \ldots, N$, the de Sitter momenta and angular momenta of $SO(N + 1)/SO(N)$. All of these $N(N+1)/2$ sphere-translations and rotations are symmetries of the classical Hamiltonian.

Quantization proceeds as in S^2, maintaining conservation of all P_a and L_{ab}:

$$H_{qm} = \tfrac{1}{2}P_a \star P_a + \tfrac{1}{4}L_{ab} \star L_{ab}, \qquad (41)$$

and hence the quantum correction is

$$H_{qm} - H = \frac{\hbar^2}{8}\Big(\frac{1}{u} - 1 - N(N - 1)\Big). \qquad (42)$$

The spectra are proportional to the Casimir eigenvalues $l(l + N - 1)$ for integer l [18]. For $N = 3$ of the previous section, this form is reconciled with the Casimir expression for (31) as $l = 2j$ and agrees with [3]–[5].

A plausible question might arise at this point: whether the above quantum Hamiltonian (41) could be expressed geometrically, in tangent space, as was detailed for the chiral models in the previous section. For the generic sphere models, S^N, the Vielbeine are

$$V_a^i = \delta_{ai} - \frac{q^a q^i}{q^2}\Big(1 \pm \frac{1}{\sqrt{u}}\Big), \qquad V^{ai} = \delta_{ai} - \frac{q^a q^i}{q^2}(1 \pm \sqrt{u}). \tag{43}$$

The classical Hamiltonian also equals

$$H = \tfrac{1}{2}(p_a V^{ai})(V^{bi}p_b), \tag{44}$$

but the quantum Hamiltonian (41) is not equal to the chiral model form:

$$H_{qm} \neq H'_{qm} \equiv \tfrac{1}{2}(p_a V^{ai}) \star (V^{bi}p_b). \tag{45}$$

Equality even fails for the S^3 case of the previous section, as (32) only holds for Dreibeine defined differently, as in that section.

The cotangent bundle currents, for a general manifold, do not have their MBs reduce down to the Vielbein currents as in (36), but, instead,

$$\{\{V^{aj}p_a, V^{bk}p_b\}\} = \omega^{a[jk]}p_a = (V^{bk}\partial_b V^{aj} - V^{bj}\partial_b V^{ak})p_a, \tag{46}$$

where, for the N-sphere, choosing the $-$ sign in (43), so $V^{aj} = \delta_{aj} - q^a q^j w$ for $w \equiv (1 - \sqrt{1 - q^2})/q^2$,

$$\omega^{a[ij]} = w(\delta^{ai}q^j - \delta^{aj}q^i). \tag{47}$$

It follows that

$$H_{qm} - H'_{qm} = \frac{\hbar^2}{8}(N - 1)(1 - 2w - N) = \frac{\hbar^2}{8}(N - 1)\Big(1 + \frac{2}{q^2}(\sqrt{1 - q^2} - 1) - N\Big). \tag{48}$$

H'_{qm} corresponds to a different operator ordering in the conventional Hilbert space formulation and has less symmetry than H_{qm}. H'_{qm} conserves the rotation generators L_{ab} (i.e. it is symmetric under the $SO(N)$ stability subgroup for S^N); however, it does not conserve the Vielbein currents on the N-sphere, nor does it conserve the de Sitter momenta. This last statement follows from the difference $H_{qm} - H'_{qm}$ in (48) dictating

$$\{\{H_{qm} - H'_{qm}, P_c\}\} = \hbar^2 q^c (N - 1)\Big(\frac{1 - 2w}{4q^2}\Big), \tag{49}$$

i.e. $\{\{H'_{qm}, P_c\}\} \neq 0$. Still, even with reduced symmetry, H'_{qm} is maximally superintegrable for $N \geq 5$.

Nevertheless, despite these differences, it can be shown that a $\star$-similarity transformation bridges these Hamiltonians. Consider

$$w^{-\frac{(N-1)}{2}} \star p_a V^{aj} \star w^{\frac{(N-1)}{2}} = w^{-(N-1)} \star (p_a V^{aj} w^{N-1}) = V^{aj}p_a - i\hbar\frac{N-1}{2}V^{aj}\partial_a \ln w, \tag{50}$$

and the complex conjugate transformation

$$w^{\frac{(N-1)}{2}} \star p_a V^{aj} \star w^{-\frac{(N-1)}{2}} = (w^{N-1}V^{aj}p_a) \star w^{-(N-1)} = V^{aj}p_a + i\hbar\frac{N-1}{2}V^{aj}\partial_a \ln w. \tag{51}$$

Associativity of the $\star$-product then allows the maximally symmetric real Hamiltonian to be written as one half of (51) $\star$ (50):

$$H_{qm} = \tfrac{1}{2}(w^{N-1}V^{aj}p_a) \star w^{-2(N-1)} \star (w^{N-1}V^{bj}p_b). \tag{52}$$

This form was discovered by using homogeneous coordinates on the sphere, with $1/w = 1 + \cos(\theta)$, where θ is the polar angle.

5. Maximal superintegrability and the Nambu bracket

All the models considered above have extra invariants beyond the number of conserved quantities in involution (mutually commuting) required for integrability in the Liouville sense. The most systematic way of accounting for such additional invariants, and placing them all on a more equal footing, even when they do not all simultaneously commute, is the NB formalism.

For example, the classical mechanics of a particle on an N-sphere, as discussed above, may be summarized elegantly through Nambu mechanics in phase space [13, 15]. Specifically, [14, 16], in an N-dimensional space, and thus $2N$-dimensional phase space, motion is confined on the constant surfaces specified by the algebraically independent integrals of the motion (e.g. L_x, L_y, L_z for S^2 above.) Consequently, the phase-space velocity $\boldsymbol{v} = (\dot{\boldsymbol{q}}, \dot{\boldsymbol{p}})$ is always perpendicular to the $2N$-dimensional phase-space gradients $\nabla = (\partial_{\boldsymbol{q}}, \partial_{\boldsymbol{p}})$ of all these integrals of motion.

As a consequence, if there are $2N - 1$ algebraically independent such integrals, possibly including the Hamiltonian (i.e. the system is maximally superintegrable [1]), the phase-space velocity must be proportional [14] to the cross-product of all those gradients, and hence the motion is fully specified for any phase-space function $k(\boldsymbol{q}, \boldsymbol{p})$ by a phase-space Jacobian which amounts to the NB:

$$\begin{aligned}
\frac{\mathrm{d}k}{\mathrm{d}t} &= \nabla k \cdot \boldsymbol{v} \\
&\propto \partial_{i_1}k\epsilon^{i_1 i_2 \ldots i_{2N}}\partial_{i_2}L_{i_1} \ldots \partial_{i_{2N}}L_{2N-1} \\
&= \frac{\partial(k, L_1, \ldots, \ldots, L_{2N-1})}{\partial(q_1, p_1, q_2, p_2, \ldots, q_N, p_N)} \\
&\equiv \{k, L_1, \ldots, L_{2N-1}\}.
\end{aligned} \tag{53}$$

For instance, for the above S^2,

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{\partial(k, L_x, L_y, L_z)}{\partial(x, p_x, y, p_y)}. \tag{54}$$

For the more general S^N, one now has a choice of $2N - 1$ of the $N(N+1)/2$ invariants of $SO(N+1)$; one of several possible expressions is

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{(-1)^{(N^2-1)}}{P_2 P_3 \cdots P_{N-1}} \frac{\partial(k, P_1, L_{12}, P_2, L_{23}, P_3, \ldots, P_{N-1}, L_{N-1\,N}, P_N)}{\partial(x_1, p_1, x_2, p_2, \ldots, x_N, p_N)}, \tag{55}$$

where $P_a = \sqrt{u}p_a$, for $a = 1, \ldots, N$, and $L_{a,a+1} = q^a p_{a+1} - q^{a+1}p_a$, for $a = 1, \cdots, N-1$.

In general [15], NBs, being Jacobian determinants, possess all antisymmetries of such; being linear in all derivatives, they also obey the Leibniz rule of partial differentiation:

$$\{k(L, M), f_1, f_2, \ldots\} = \frac{\partial k}{\partial L}\{L, f_1, f_2, \ldots\} + \frac{\partial k}{\partial M}\{M, f_1, f_2, \ldots\}. \tag{56}$$

Thus, an entry in the NB algebraically dependent on the remaining entries leads to a vanishing bracket. For example, it is seen directly from above that the Hamiltonian is constant:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \left\{ \frac{\boldsymbol{L} \cdot \boldsymbol{L}}{2}, \ldots \right\} = 0, \tag{57}$$

since each term of this NB vanishes. Naturally, this also applies to all explicit examples discussed here, as they are all maximally superintegrable.

Finally, the impossibility to antisymmetrize more than $2N$ indices in $2N$-dimensional phase space:

$$\epsilon^{ab\ldots c[i} \epsilon^{j_1 j_2 \ldots j_{2N}]} = 0, \tag{58}$$

leads to the fundamental identity, [15], slightly generalized here:

$$\{f_0\{f_1, \ldots, f_{m-1}, f_m\}, f_{m+1}, \ldots, f_{2m-1}\}$$
$$+ \{f_m, f_0\{f_1, \ldots, f_{m-1}, f_{m+1}\}, f_{m+2}, \ldots, f_{2m-1}\}$$
$$+ \cdots + \{f_m, \ldots, f_{2m-2}, f_0\{f_1, \ldots, f_{m-1}, f_{2m-1}\}\}$$
$$= \{f_1, \ldots, f_{m-1}, f_0\{f_m, f_{m+1}, \ldots, f_{2m-1}\}\}. \tag{59}$$

This $m+1$-term identity works for any m, and not just $m = 2N$ here.

The proportionality constant V in (53):

$$\frac{\mathrm{d}k}{\mathrm{d}t} = V\{k, L_1, \ldots, L_{2N-1}\}, \tag{60}$$

has to be a time-invariant [16] if it has no explicit time dependence. This is seen from the consistency of (60), application of which to

$$\frac{\mathrm{d}}{\mathrm{d}t}(V\{f_1, \ldots, f_{2N}\}) = \dot{V}\{f_1, \ldots, f_{2N}\} + V\{\dot{f_1}, \ldots, f_{2N}\} + \cdots + V\{f_1, \ldots, \dot{f}_{2N}\}, \tag{61}$$

yields

$$V\{V\{f_1, \ldots, f_{2N}\}, L_1, \ldots, L_{2N-1}\} = \dot{V}\{f_1, \ldots, f_{2N}\} + V\{V\{f_1, L_1, \ldots, L_{2N-1}\}, \ldots, f_{2N}\}$$
$$+ \cdots + V\{f_1, \ldots, V\{f_{2N}, L_1, \ldots, L_{2N-1}\}\}, \tag{62}$$

and, by virtue of (59), $\dot{V} = 0$ follows.

Closure under PBs of quantities serving as arguments in the NB does *not* suffice for a NB to vanish, as illustrated in (54) where $\{L_x, L_y\} = L_z$. On the other hand, it is always true that PBs of conserved integrals are themselves conserved integrals, i.e.

$$\frac{\mathrm{d}\{L_a, L_b\}}{\mathrm{d}t} \propto \{\{L_a, L_b\}, L_1, \ldots, L_{2N-1}\} \tag{63}$$

must vanish.

Actually, PBs result from a maximal reduction of NBs, by inserting $2N - 2$ phase-space coordinates and summing over them, thereby taking *symplectic traces*

$$\{L, M\} = \frac{1}{(N-1)!}\{L, M, x_{i_1}, p_{i_1}, \ldots, x_{i_{N-1}}, p_{i_{N-1}}\}, \tag{64}$$

where summation over all $N - 1$ pairs of repeated indices is understood†. Fewer traces lead to relations between NBs of maximal rank, $2N$, and those of lesser rank, $2k$:

---

† Thus, from the identity (59), it follows that

$$\{\{L_a, L_b\}, L_1, \ldots, L_{2N-1}\} + \{L_b, \{L_a, L_1\}, L_2, \ldots, L_{2N-1}\} + \cdots + \{L_b, L_1, \ldots, \{L_a, L_{2N-1}\}\}$$
$$= \{L_a, \{L_b, L_1, \ldots, L_{2N-1}\}\}.$$

Each NB in this consistency relation vanishes separately.

$$\{L_1, \ldots, L_{2k}\} = \frac{1}{(N-k)!}\{L_1, \ldots, L_{2k}, x_{i_1}, p_{i_1}, \ldots, x_{i_{N-k}}, p_{i_{N-k}}\} \tag{65}$$

(which is one way to define the lower rank NBs for $k \neq 1$), or between two lesser rank NBs. A complete theory of these relations has not been developed but, essentially, $\{L_1, \ldots, L_{2k}\}$ acts like a Dirac bracket (DB) up to a normalization, $\{L_1, L_2\}_{DB}$, where the fixed additional entries $L_3, \ldots, L_{2k}$ in the NB play the role of the constraints in the DB. (In effect, this has been previously observed, e.g. [15, 16], for the extreme case $N = k$, without symplectic traces.)

As a simple illustration, consider $N = k = 2$ for the system (54), but now taking L_x, L_y as second-class constraints:

$$\{f, g, L_x, L_y\} = \{f, g\}\{L_x, L_y\} + \{f, L_x\}\{L_y, g\} - \{f, L_y\}\{L_x, g\} \equiv \{L_x, L_y\}\{f, g\}_{DB}. \tag{66}$$

That is,

$$\{f, g\}_{DB} = (\{L_x, L_y\})^{-1}\{f, g, L_x, L_y\}, \tag{67}$$

so that, from (59) with $f_0 = (\{L_x, L_y\})^{-1} = 1/L_z$ (also see [16]), it follows that the DBs satisfy the Jacobi identity

$$\{\{f, g\}_{DB}, h\}_{DB} + \{\{g, h\}_{DB}, f\}_{DB} + \{\{h, f\}_{DB}, g\}_{DB} = 0, \tag{68}$$

a property usually established by explicit calculation [23], in contrast to this derivation. Naturally, $\{f, L_x, L_y, L_z\} = \{f, H\}_{DB}$.

By virtue of this symplectic trace, for a general system—not only a superintegrable one—Hamilton's equations admit an NB expression different than (60):

$$\frac{dk}{dt} = \{k, H\} = \frac{1}{(N-1)!}\{k, H, x_{i_1}, p_{i_1}, \ldots, x_{i_{N-1}}, p_{i_{N-1}}\}. \tag{69}$$

Despite considerable progress in the last six years [24], the deformation quantization of the Nambu formalism is not completely settled: a transparent, user-friendly technique is not at hand. One desirable feature would be a quantized NB which reduces to MBs through symplectic traces. One might call such a deformation *autologous* to the $\star$-product method applied throughout this paper. This is not the case for the abelian deformation [24]. But it is the case for another, older approach to quantization, namely the method considered by Nambu in an operator context [13], when applied to the phase-space formalism.

Define quantum Nambu brackets (QNB):

$$[A, B]_\star \equiv i\hbar\{\{A, B\}\}$$
$$[A, B, C]_\star \equiv A \star B \star C - A \star C \star B + B \star C \star A - B \star A \star C + C \star A \star B - C \star B \star A$$
$$\begin{aligned}[A, B, C, D]_\star &\equiv A \star [B, C, D]_\star - B \star [C, D, A]_\star + C \star [D, A, B]_\star - D \star [A, B, C]_\star \tag{70} \\ &= [A, B]_\star \star [C, D]_\star + [A, C]_\star \star [D, B]_\star + [A, D]_\star \star [B, C]_\star \\ &\quad + [C, D]_\star \star [A, B]_\star + [D, B]_\star \star [A, C]_\star + [B, C]_\star \star [A, D]_\star,\end{aligned}$$

etc, and use these symmetrized $\star$-products in the quantum theory instead of the previous Jacobians.

This approach grants only one of the three mathematical desiderata: full antisymmetry. The Leibniz property (56) and the fundamental identity (59) are not satisfied, in general. To some extent, the loss of the latter two properties is a subjective shortcoming and dependent on the

specific application context. But, objectively, this approach is in agreement with the $\star$-product quantization of the examples given above.

For example, by virtue of the MBs for the 2-sphere, expressed in gnomonic coordinates,

$$
\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= \{\{X, H_{QM}\}\} = (1 + Q^2)(P_X + (P \cdot Q)X) \\
\frac{\mathrm{d}P_X}{\mathrm{d}t} &= \{\{P_X, H_{QM}\}\} = -P_X(P \cdot Q)(1 + Q^2) - X(P^2 + (P \cdot Q)^2) - \frac{5}{4}\hbar^2 X.
\end{aligned}
\tag{71}
$$

The first of these is classical in form, while the second contains a quantum correction that is a hallmark of the method. By comparison, we also find exactly the same results using Nambu's approach:

$$
\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{-1}{2\hbar^2}[X, L_X, L_Y, L_Z]_\star, \qquad \frac{\mathrm{d}P_X}{\mathrm{d}t} = \frac{-1}{2\hbar^2}[P_X, L_X, L_Y, L_Z]_\star,
\tag{72}
$$

where the second of these includes the quantum correction $-5\hbar^2 X/4$ as above.

In fact, this result generalizes to arbitrary functions of phase space with no explicit time dependence, for all coordinate frames. Specifically, for S^2, it follows directly from (71) and (9) (with MBs supplanting PBs, $\{\{L_x, L_y\}\} = L_z$, $\{\{L_y, L_z\}\} = L_x$, $\{\{L_z, L_x\}\} = L_y$) that the MB with the Hamiltonian (14) equals Nambu's QNB, for an arbitrary function k of phase space:

$$
[k, L_x, L_y, L_z]_\star = \mathrm{i}\hbar[k, \boldsymbol{L} \cdot \star \boldsymbol{L}]_\star = -2\hbar^2\{\{k, H_{qm}\}\},
\tag{73}
$$

so that

$$
\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{-1}{2\hbar^2}[k, L_x, L_y, L_z]_\star.
\tag{74}
$$

For $\hbar \to 0$, it naturally goes to (54).

As a derivation, this ensures that consistency requirements (56) and (59) *are* satisfied, with the suitable insertion of $\star$-multiplication in the proper locations to ensure full combinatoric analogy:

$$
[A \star B, L_x, L_y, L_z]_\star = A \star [B, L_x, L_y, L_z]_\star + [A, L_x, L_y, L_z]_\star \star B,
\tag{75}
$$

and

$$
\begin{aligned}
[[L_x, L_y, L_z, D]_\star, E, F, G]_\star &+ [D, [L_x, L_y, L_z, E]_\star, F, G]_\star \\
&+ [D, E, [L_x, L_y, L_z, F]_\star, G]_\star + [D, E, F, [L_x, L_y, L_z, G]_\star]_\star \\
&= [L_x, L_y, L_z, [D, E, F, G]_\star]_\star.
\end{aligned}
\tag{76}
$$

The reader might also wish to note from (71) that, for any phase-space constant A,

$$
[A, B, C, D]_\star = 0
\tag{77}
$$

holds identically, in contrast to the 3-argument QNB [13]. Thus, $\mathrm{d}A/\mathrm{d}t = 0$ is consistent and *no* debilitating constraint among the arguments B, C, D is imposed; the inconsistency identified in [13] is a feature of odd-argument QNBs and does not restrict the even-argument QNBs of phase space considered here.

By contrast, one might try to define a quantized NB $\{\{,,,\}\}_\star$ simply by taking $\star$-products of the phase-space gradients that appear in the classical NB and applying Jordan's trick of symmetrizing all such products at the expense of making the algebra non-associative. This also fails to grant all three mathematical desiderata (antisymmetry, Leibniz property and FI).

But, more importantly, it does not give the same equations of motion. Although, in gnomonic coordinates, $\mathrm{d}X/\mathrm{d}t$ is as given above, the other equation of motion would now become

$$\frac{\mathrm{d}P_X}{\mathrm{d}t} = \{\{P_X, L_X, L_Y, L_Z\}\}_\star = \{\{P_X, H_{QM}\}\}_\star + \frac{3}{2}\hbar^2 X. \tag{78}$$

Thus, in general, quantum corrections differ in these various methods.

In practice, however, given the simple energy spectrum and other features of the usual Moyal $\star$-product quantization (essentially, its equivalence to standard Hilbert-space quantum mechanics), it is clearly the preferred method for conventional problems such as the ones solved in this paper. In any case, quantum deformations of the NB should not only link (16) up with (69) and (60), as above, but also provide equivalents to the $\star$-genvalue equation (17) for static WFs, needed to support the spectral theory in such a formalism.

As indicated, in general, the QNB (which provide the correct quantization rule for the systems considered) need not satisfy the Leibniz property and FI for consistency, as they are not necessarily derivations. For example, for S^3, to quantize (55) for $N = 3$, note that

$$[k, P_1, L_{12}, P_2, L_{23}, P_3]_\star = 3\mathrm{i}\hbar^3(P_2 \star \{\{k, H_{qm}\}\} + \{\{k, H_{qm}\}\} \star P_2) + \mathcal{Q}, \tag{79}$$

where $\mathcal{Q}$ is an $\mathrm{O}(\hbar^5)$ sum of triple commutators of k with invariants. Consequently, the proper quantization of (55) is

$$[k, P_1, L_{12}, P_2, L_{23}, P_3]_\star = 3\mathrm{i}\hbar^3 \frac{\mathrm{d}}{\mathrm{d}t}(P_2 \star k + k \star P_2) + \mathcal{Q}, \tag{80}$$

and again reduces to (55) in the $\hbar \to 0$ limit, as $\mathcal{Q}$ is subdominant in $\hbar$ to the time derivative term. The right-hand side not being an unadorned derivation on k, it does not impose a Leibniz rule analogous to (75) on the left-hand side, so it fails the mathematical desiderata mentioned, at no compromise to its validity, however. The $N > 3$ case parallels the above through use of fully symmetrized products.

6. Conclusions

The first aim of this paper has been to illustrate the power and simplicity of phase-space quantization of superintegrable systems which would suffer from operator ordering ambiguities in conventional quantization. Many of these σ-models quantized here, such as the S^N models, have already been quantized conventionally [2]–[5] through elaborate operator algebra preserving the maximal symmetries of these systems (see especially the second reference of [1]). But not all, such as the chiral models, whose geometrical complication has so far only partially yielded to indirect methods [21].

Here, the procedure of determining the proper symmetric quantum Hamiltonian has yielded remarkably compact and elegant expressions, since a survey of all alternate operator orderings in a problem with such ambiguities amounts, in deformation quantization, to a survey of the 'quantum correction' $\mathrm{O}(\hbar)$ pieces of the respective kernel functions, i.e. the inverse Weyl transforms of those operators, and their study, is greatly systematized and expedited. The choice-of-ordering problems then reduce to purely $\star$-product algebraic ones, as the resulting preferred orderings are specified through particular deformations in the c-number kernel expressions resulting from the particular solution in phase space. For the N-spheres, our results agree with the results of [1]–[5], while quantum Hamiltonians for chiral models such as (38) are new. With functional methods confined to phase space, we have also illustrated how the spectra of such Hamiltonians may be

obtained. One might wish to contrast the quantum correction found here in (15) to the free-space angular-momentum quantum correction [25], which is also $O(\hbar^2)$, although a constant, reflecting the vanishing curvature of that underlying manifold. Predictably, on the north pole of (15), $u = 1$, and these expressions coincide. This difference and pole coincidence carries over for all dimensions, as is evident in the quantum correction (42) for S^N.

More elaborate isometries of general manifolds in such models are expected to yield to analysis similar to what has been illustrated for the prototypes considered here.

The second main conclusion of this paper has been a surprising application: quantization of maximally superintegrable systems in phase space has facilitated explicit testing of NB quantization proposals through direct comparison of the conventional quantum answers thus found. The classical evolution of all functions in phase space for such systems is alternatively specified through NBs. However, quantization of NBs has been considered problematic ever since their inception. Nevertheless, it was demonstrated that Nambu's early quantization prescription [13] can, indeed, succeed, despite widespread expectations to the contrary. Comparison to the standard Moyal deformation quantization utilized in this work vindicates Nambu's early quantization prescription (and invalidates other prescriptions) for systems such as S^N. We thus stress the utility of phase-space quantization as a comparison testing tool for NB quantization proposals.

Acknowledgments

Appendix

Equations (36) and (37) are implicit in [22] and throughout the literature, but it may be useful here to provide a direct geometrical proof.

For any Vielbein satisfying $g_{ab} = V_a^j V_b^j$ and the Maurer–Cartan equation

$$\partial_{[a} V_{b]}^j + f_{jmn} V_a^m V_b^n = 0, \tag{81}$$

we have ((3.16) of [22])

$$D_a V^{bi} = -f_{imn} V_a^m V^{bn}, \tag{82}$$

or

$$\partial_a V^{bi} = -\Gamma_{ac}^b V^{ci} - f_{imn} V_a^m V^{bn}, \tag{83}$$

where the Christoffel connection is the usual functional of just the metric g_{ab}. These hold for both groups of Vielbeine for the same metric and hence Christoffel connection; and so, for both groups of charges $\pm V^{aj} p_a$, it follows that

$$
\begin{aligned}
\{V^{aj} p_a, V^{bk} p_b\} &= p_a V^{bk} (\partial_b V^{aj}) - p_b V^{aj} (\partial_a V^{bk}) \\
&= p_b V^{ak} (-\Gamma_{ac}^b V^{cj} - f_{jmn} V_a^m V^{bn}) - p_b V^{aj} (-\Gamma_{ac}^b V^{ck} - f_{kmn} V_a^m V^{bn}) \\
&= -2 f_{jkn} V^{bn} p_b.
\end{aligned}
\tag{84}
$$

Actually, for any two Vielbeine, V and $\tilde{V}$, producing the same metric (and hence Christoffel connection) and obeying their own Maurer–Cartan equations, both satisfy equations like (83), and hence algebras like (84).

The cross-PBs, however, need not automatically vanish in the general case:

$$
\begin{aligned}
\{V^{aj}p_a, \tilde{V}^{bk}p_b\} &= (\tilde{V}^{ak}\partial_a V^{bj} - V^{aj}\partial_a \tilde{V}^{bk})p_b \\
&= \tilde{V}^{ak}(-\Gamma^b_{ac}V^{cj} - f_{jmn}V^m_a V^{bn})p_b - V^{aj}(-\Gamma^b_{ac}\tilde{V}^{ck} - \tilde{f}_{kmn}\tilde{V}^m_a \tilde{V}^{bn})p_b \\
&= (-f_{lmn}V^m_a V^n_c \delta^{lj}\tilde{V}^{ak} + \tilde{f}_{lmn}\tilde{V}^m_d \tilde{V}^n_c \delta^{lk}V^{dj})g^{cb}p_b \\
&= (-f_{lmn}V^m_a V^n_c V^l_d V^{dj}\tilde{V}^{ak} + \tilde{f}_{lmn}\tilde{V}^m_d \tilde{V}^n_c \tilde{V}^l_a \tilde{V}^{ak}V^{dj})g^{cb}p_b \\
&= (-f_{lmn}V^l_d V^m_a V^n_c + \tilde{f}_{lmn}\tilde{V}^m_d \tilde{V}^l_a \tilde{V}^n_c)V^{dj}\tilde{V}^{ak}g^{cb}p_b \\
&= -(f_{lmn}V^l_d V^m_a V^n_c + \tilde{f}_{lmn}\tilde{V}^l_d \tilde{V}^m_a \tilde{V}^n_c)V^{dj}\tilde{V}^{ak}g^{cb}p_b.
\end{aligned}
\tag{85}
$$

The terms in parentheses in the final line are actually the torsions on the respective manifolds ((3.10) of [22]), induced by the corresponding Vielbeine, up to a normalization:

$$
S_{dac} = f_{lmn}V^l_d V^m_a V^n_c, \qquad \tilde{S}_{dac} = \tilde{f}_{lmn}\tilde{V}^l_d \tilde{V}^m_a \tilde{V}^n_c.
\tag{86}
$$

Hence

$$
\{V^{aj}p_a, \tilde{V}^{bk}p_b\} = -(S_{dac} + \tilde{S}_{dac})V^{dj}\tilde{V}^{ak}g^{cb}p_b.
\tag{87}
$$

However, for the specific chirally enantiomorphic Vielbeine defined above, $V^m_a = {}^{(+)}V^m_a$ and $\tilde{V}^m_a = {}^{(-)}V^m_a$, it further follows from equation (3.10) of [22] that, in fact,

$$
\tilde{S}_{dac} = -\tfrac{1}{2}\mathrm{Tr}(\partial_{[d}U^{-1}\,U\partial_a U^{-1}\,U\partial_{c]}U^{-1}\,U) = \tfrac{1}{2}\mathrm{Tr}(U^{-1}\partial_{[d}U\,U^{-1}\partial_a U\,U^{-1}\partial_{c]}U) = -S_{dac}, \tag{88}
$$

and thus, indeed, (37) holds.

References

[1] Tempesta P, Turbiner A and Winternitz P 2001 *J. Math. Phys.* **42** 4248–57 (*Preprint* hep-th/0011209)
     Kalnins E, Kress J and Winternitz P 2002 *J. Math. Phys.* **43** 970–83 (*Preprint* math-ph/0108015)
     Rodriguez M and Winternitz P 2002 *J. Math. Phys.* **43** 1309–22 (*Preprint* math-ph/0110018)
     Wojciechowski S 1983 *Phys. Lett.* A **95** 279–81
     Caseiro R, Françoise J-P and Sasaki R 2001 *J. Math. Phys.* **42** 5329–40 (*Preprint* hep-th/0102153)
[2] Velo G and Wess J 1971 *Nuovo Cimento* A **1** 177
     Lin H, Lin W and Sugano R 1970 *Nucl. Phys.* B **16** 431
[3] Lakshmanan M and Eswaran K 1975 *J. Phys. A: Math. Gen.* **8** 1658
     also see Leznov A 2002 *Nucl. Phys.* B **640** 469–80 (*Preprint* hep-th/0203225)
[4] Higgs P 1979 *J. Phys. A: Math. Gen.* **12** 309
[5] Leemon H 1979 *J. Phys. A: Math. Gen.* **12** 489
[6] Moyal J 1949 *Proc. Camb. Phil. Soc.* **45** 99–124
[7] Curtright T, Fairlie D and Zachos C 1998 *Phys. Rev.* D **58** 025002
[8] Zachos C 2002 *Int. J. Mod. Phys.* A **17** 297–316 (*Preprint* hep-th/0110114)
[9] Weyl H 1927 *Z. Phys.* **46** 1–33
     Also reviewed in Weyl H 1931 *The Theory of Groups and Quantum Mechanics* (New York: Dover)
[10] Groenewold H 1946 *Physica* **12** 405–60
     Also see Leaf B 1968 *J. Math. Phys.* **9** 65–72
[11] Hietarinta J 1984 *J. Math. Phys.* **25** 1833–40
[12] Berry M 1977 *Phil. Trans. R. Soc.* A **287** 237–71
[13] Nambu Y 1973 *Phys. Rev.* D **7** 2405–12

Institute *of* **Physics** Φ DEUTSCHE PHYSIKALISCHE GESELLSCHAFT

[14] Hietarinta J 1997 *J. Phys. A: Math. Gen.* **30** L27–33
Mukunda N and Sudarshan E 1976 *Phys. Rev.* D **13** 2846–50
Chatterjee R 1996 *Lett. Math. Phys.* **36** 117–26
[15] Takhtajan L 1994 *Commun. Math. Phys.* **160** 295–316 (*Preprint* hep-th/9301111)
Filippov V T 1985 *Sib. Math. J.* **26** 879–91
[16] Nutku Y Quantization with Nambu–Poisson brackets: the harmonic oscillator! *Preprint*
Gonera C and Nutku Y 2001 *Phys. Lett.* A **285** 301–6
[17] Fairlie D 1964 *Proc. Camb. Phil. Soc.* **60** 581–6
[18] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1977 *Lett. Math. Phys.* **1** 521–30
Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys., NY* **111** 61–110
Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys., NY* **111** 111–51
[19] Curtright T and Zachos C 2001 *Mod. Phys. Lett.* A **16** 2381–5 (*Preprint* hep-th/0105226)
[20] Curtright T, Uematsu T and Zachos C 1996 *Nucl. Phys.* B **469** 488–512 (*Preprint* hep-th/0011137)
[21] Chu M and Goddard P 1994 *Phys. Lett.* B **337** 285–93 (*Preprint* hep-th/9407116)
[22] Braaten E, Curtright T and Zachos C 1985 *Nucl. Phys.* B **260** 630–88
[23] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science)
[24] Dito G, Flato M, Sternheimer D and Takhtajan L 1997 *Commun. Math. Phys.* **183** 1–22 (*Preprint* hep-th/9602016)
Dito G and Flato M 1997 *Lett. Math. Phys.* **39** 107–25 (*Preprint* hep-th/9609114)
Awata H, Li M, Minic D and Yoneya T 2001 *J. High Energy Phys.* JHEP02(2001)013 (*Preprint* hep-th/9906248)
Minic D and Tze C-H 2002 *Phys. Lett.* B **536** 305–14 (*Preprint* hep-th/0202173)
[25] Dahl J-P and Springborg M 1982 *Mol. Phys.* **47** 1001–19
Shewell J 1959 *Am. J. Phys.* **27** 16–21