New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG IOP Institute of Physics

PAPER • OPEN ACCESS

Time, entropy generation, and optimization in low-dissipation heat devices

To cite this article: A Calvo Hernández *et al* 2015 *New J. Phys.* **17** 075011

New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

CrossMark

OPEN ACCESS

RECEIVED
6 April 2015

ACCEPTED FOR PUBLICATION
18 May 2015

PUBLISHED
13 July 2015

PAPER

Time, entropy generation, and optimization in low-dissipation heat devices

A Calvo Hernández[1], **A Medina**[2] **and J M M Roco**[1]

[1] Departamento de Física Aplicada and Instituto Universitario de Física Fundamental y Matemáticas (IUFFyM), Universidad de Salamanca, E-37008 Salamanca, Spain
[2] Departamento de Física Aplicada, Universidad de Salamanca, E-37008 Salamanca, Spain

E-mail: anca@usal.es

Keywords: irreversibility, entropy generation, constraints, time

Abstract

We present new results obtained from the Carnot-like low-dissipation model of heat devices when size- and time-constraints are taken into account, in particular those obtained from the total cycle time and the contact times of the working system with the external heat reservoirs. The influence of these constraints and of the characteristic time scale of the model on the entropy generation allows for a clear and unified interpretation of different energetic properties for both heat engines and refrigerators (REs). Some conceptual subtleties with regard to different optimization criteria, especially for REs, are discussed. So, the different status of power input, cooling power, and the unified figure of merit χ are analyzed on the basis of their absolute or local role as optimization criteria.

1. Introduction

Carnot in his famous work 'Réflexions Sur la Puissance Motrice du Feu' ('Reflections on the motive power of fire') [1], presented the results for the first systematic study of the physical processes governing steam engines. Carnot showed that the efficiency of a cyclic heat engine (HE), working between two heat reservoirs at temperatures T_h and T_c ($T_h > T_c$), which transforms an amount of heat $|Q_h|$ extracted from a heat reservoir at T_h into an amount of work $|W|$, is at most $\eta = |W|/|Q_h| \leqslant 1 - T_c/T_h \equiv \eta_C$ (Carnot efficiency). This result can be extended to any heat device, such as refrigerators (REs), operating between two heat reservoirs at temperatures T_h and T_c, so that they must have an efficiency lower than that known as the Carnot coefficient of performance (COP) $\epsilon_C \equiv T_c/(T_h - T_c)$.

The theoretical implications of this Carnot's result are crucial in the development of equilibrium thermodynamics, which provides a complete description of reversible processes, i.e., quasi-static processes that have an infinite duration. But, on the other hand, its practical implications are more limited, since the upper limit, η_C or ϵ_C, is only reached by heat devices that operate reversibly, which implies that the processes should have an infinite duration, and therefore their output power is zero. In order to obtain realistic bounds for the performance of real heat devices, two ingredients have been revealed to be fundamental: (1) the use of suitable models accounting for the intrinsic irreversibilities of different finite-rate processes; (2) the choice of a suitable functional to be optimized with respect to the characteristic parameters of the model. These two ingredients constitute the very core of finite time thermodynamics, whose seed is the famous efficiency at maximum power $1 - \sqrt{\tau}$ ($\tau \equiv T_c/T_h$), attained in different contexts by Curzon–Ahlborn [2], Novikov [3], Chambadal [4], Yvon [5] and Reitlinger [6] (an update on the historical roots of this equation can be found in a recent publication [7]). Besides these, a great variety of different criteria based on thermo-economic, compromise, and sustainability considerations have been reported [8–18].

In the present work we deal with Carnot-like heat devices working between two heat reservoirs at constant temperatures T_h and T_c. For these devices we shall address the issue of the choice of the above-mentioned ingredients from the same starting point: the cornerstone Clausius theorem of thermodynamics [19, 20] and the introduction of entropy as a state function. The Clausius theorem establishes that for irreversible Carnot-like

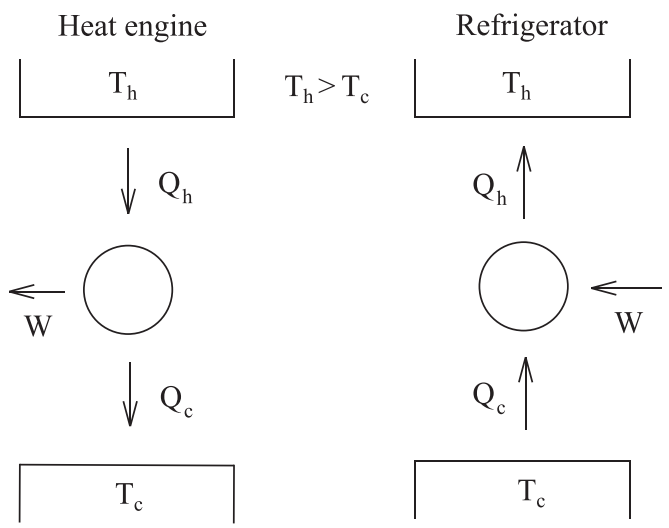

Figure 1. Carnot heat devices.

heat devices $(Q_h/T_h) + (Q_c/T_c) < 0$, and then from thermodynamics, there is no mathematical relationship that links directly the heat exchanges Q_h/T_h and Q_c/T_c, as occurs in the reversible case, for which this inequality becomes an equality. The unavoidable irreversibilities are the reason why in a real device the above inequality takes place. Or seen from another point of view, the deviation of the performance of the device with respect to the corresponding Carnot limit is the cause of the inequality, provided that T_c/T_h is directly related with the Carnot value and, taking into account the first law of thermodynamics, Q_c/Q_h is directly related with the real performance.

For the concrete low-dissipation Carnot model proposed by Esposito *et al* [21], and considered throughout this work, the deviations from reversibility yield an entropy generation in each heat exchange process. This entropy generation is assumed as inversely proportional to the time duration of the process, so that the reversible regime is recovered in the limit of infinite times.

On the other hand, the time- and size-constraints must be included in any realistic model of heat devices in order to further optimize their design and performance. As noted by Uzdin and Kosloff [22], optimization should be done after the heat device becomes capable of performing the task it is designed for. In this line, we paid special emphasis on the irreversibilities associated not only to the total cycle time but also to the contact times of the working system with the external heat reservoirs. The influence of these constraints and of the characteristic time scale of the model on the entropy generation allows for a clear and unified interpretation of some known bounds, together with new time-dependent results for different energetic properties and optimization criteria.

The main particular objectives of the present work are the following: (1) to introduce a characteristic time scale for the low-dissipation model, defined by the ratio between the change of entropy of the heat exchanges in the reversible limit and the dissipation constants; (2) to study the behavior of the energetic properties in this characteristic time scale; and (3) to analyze the suitability of different optimization criteria, whether they are considered for engines or REs. The paper is structured as follows: in section 2 we introduce the influence of the constraints on the original low-dissipation model and the relevant energetic magnitudes for both HEs and REs; in section 3 numerical results for HEs and REs are presented with special emphasis on the different optimization criteria and their role as absolute or local figure of merit; finally, in section 4 we summarize some results and conclusions.

2. Theoretical model

We deal with Carnot-like heat devices for which a cyclic working system exchanges heat with two heat reservoirs (see figure 1). For these devices Clausius' theorem states that:

$$\frac{Q_h}{T_h} + \frac{Q_c}{T_c} \leqslant 0 \qquad \Leftrightarrow \qquad \Delta S_{tot} = \Delta S_{T_h} + \Delta S_{T_c} \geqslant 0, \tag{1}$$

where ΔS_{tot} is the total entropy change, and ΔS_{T_h} and ΔS_{T_c} are the entropy changes of the hot and cold reservoirs, respectively. In equation (1) equality holds if and only if all the processes are reversible.

For Carnot heat devices all processes are reversible and therefore have an infinite time duration. Thus the equality in equation (1) must hold:

$$0 = \Delta S_{T_h}^{(C)} + \Delta S_{T_c}^{(C)} \Rightarrow \Delta S \equiv |\Delta S_{T_h}^{(C)}| = |\Delta S_{T_c}^{(C)}|, \tag{2}$$

with $\Delta S_{T_h}^{(C)}$ and $\Delta S_{T_c}^{(C)}$ being the entropy changes of the hot and cold reservoirs, respectively, for this reversible case.

Moving away from reversibility, for Carnot-like heat devices the times t_h and t_c, associated to the heat exchanges between the working system and the hot and cold reservoirs, respectively, will be finite. Thus, in order to establish a relationship between this finite time of the processes and the entropy generation (the cause of inequality in equation (1)) we assume the low-dissipation model [21]. This model considers that the entropy generation in the processes of heat exchanges between working system and the hot and cold reservoirs are inversely proportional to the time of the process. Then, we get:

$$\Delta S_{T_h} = \mp \Delta S + \frac{\Sigma_h}{t_h}, \tag{3}$$

$$\Delta S_{T_c} = \pm \Delta S + \frac{\Sigma_c}{t_c}, \tag{4}$$

where the parameters Σ_h and Σ_c contain the information about how dissipation increases as one moves from the reversibility limit. Then, the reversible regime is approached in the limits $t_h \to \infty$ and $t_c \to \infty$. The signs $\mp$ ($\pm$) in equations (3) and (4) account for the opposite sense of the heat fluxes exchanged with the hot (cold) reservoir for HEs and REs, respectively; in equations (3) and (4) we have assumed the criterion that the energies absorbed by the working fluid are positive while the energies released by the working fluid are negative.

Two comments are in order here. First, the low dissipation model does not assume that the temperature dependence between T_h and T_c is small, i.e., equations (3) and (4) are not limited to the linear response regime. Second, additional contributions to the entropy production in equations (3) and (4) with complicated time-dependence could arise due to non-equilibrium external couplings but, in agreement with [21], we consider Carnot-like cycles with a reversible limit where these additional contributions are negligible or absent.

Due to the fact that the working system suffers a cyclic process, the total entropy ΔS_{tot} will be determined by the sum of ΔS_{T_h} and ΔS_{T_c} given from equations (3) and (4):

$$\Delta S_{tot} = \frac{\Sigma_h}{t_h} + \frac{\Sigma_c}{t_c}. \tag{5}$$

Assuming that the rest of the processes which take place in the low dissipation heat devices are instantaneous, the total time of the cycle must be given by $t = t_h + t_c$. We define a fractional contact time with the cold reservoir $\alpha \equiv t_c/t$, the total dissipation parameter $\Sigma_T \equiv \Sigma_h + \Sigma_c$, and the relative dissipation parameters $\tilde{\Sigma}_h \equiv \Sigma_h/\Sigma_T$ and $\tilde{\Sigma}_c \equiv \Sigma_c/\Sigma_T$. Then, provided that $\tilde{\Sigma}_h = 1 - \tilde{\Sigma}_c$, equation (5) can be expressed in terms of t, α, Σ_T and $\tilde{\Sigma}_c$ in the following form:

$$\Delta S_{tot} = \Sigma_T \left[\frac{1 - \tilde{\Sigma}_c}{(1-\alpha)t} + \frac{\tilde{\Sigma}_c}{\alpha t} \right]. \tag{6}$$

It is interesting to express the relation between the total entropy ΔS_{tot} and the entropy associated to the heat exchanges with the reservoirs in the baseline reversible machine ΔS. Then we obtain

$$\widetilde{\Delta S}_{tot} \equiv \frac{\Delta S_{tot}}{\Delta S} = \frac{\Sigma_T}{\Delta S\, t} \left[\frac{1 - \tilde{\Sigma}_c}{1-\alpha} + \frac{\tilde{\Sigma}_c}{\alpha} \right]. \tag{7}$$

The quantity $\Sigma_T/\Delta S$ has time units and it defines a characteristic time scale as the ratio between the dissipation parameter, Σ_T, and the entropy exchange, ΔS, associated to the heat fluxes. The magnitude of the heat exchangers is unavoidably related with the size of the working system and the surface through which heat fluxes take place. Provided that for a reversible (Carnot) heat device the heat exchanges are given by $Q_h = T_h(\Delta S)$ and $Q_c = T_c(\Delta S)$, ΔS is somewhat a measure of the size of the system and its contact surface with the external reservoirs.

The above fact suggests that to define a dimensionless time $\tilde{t} \equiv (t\ \Delta S)/\Sigma_T$. On the basis of the value of $\tilde{t}$, one can expect that cycle total times $\tilde{t} \leqslant 1$ should correspond to working regimes for which dissipations are very important and, even, could determine that the device does not work more as an energy converter (see later for particular examples). Conversely, for devices working in regimes with $\tilde{t} \gg 1$ the role of dissipations should not be not so drastic and the model makes sense in its purpose to describe a real energy converter.

The total entropy generated per unit time $\Delta \dot{S}_{tot} \equiv \Delta S_{tot}/t$ can be expressed in a dimensionless way as

$$\widetilde{\Delta \dot{S}}_{tot} \equiv \frac{\Delta S_{tot}}{t} \frac{\Sigma_T}{\Delta S^2} = \frac{1}{\tilde{t}} \left[\frac{1 - \tilde{\Sigma}_c}{(1-\alpha)\tilde{t}} + \frac{\tilde{\Sigma}_c}{\alpha \tilde{t}} \right], \tag{8}$$

so that $\widetilde{\dot{\Delta S}}_{\rm tot}$ accounts for a balance between entropy generation due to time irreversibilities and entropy changes associated to the external heat fluxes (size irreversibilities): $\widetilde{\dot{\Delta S}}_{\rm tot}$-values close or greater than one mean that entropy generation due to time irreversibilities is of the same order or greater than the entropy changes associated to device size. Just the opposite occurs for working regimes where $\widetilde{\dot{\Delta S}}_{\rm tot} \ll 1$.

From equilibrium thermodynamics, the relationships between the entropy changes of the hot $\Delta S_{T_{\rm h}}$ and cold $\Delta S_{T_{\rm c}}$ reservoirs and the amounts of heat $Q_{\rm h}$ and $Q_{\rm c}$ exchanged with working system are given by

$$\Delta S_{T_{\rm h}} = -\frac{Q_{\rm h}}{T_{\rm h}}, \tag{9}$$

$$\Delta S_{T_{\rm c}} = -\frac{Q_{\rm c}}{T_{\rm c}}. \tag{10}$$

Thus, from equations (3), (4), (9) and (10), we obtain for $Q_{\rm h}$ and $Q_{\rm c}$ in the finite-time case:

$$Q_{\rm h} = T_{\rm h}\left(\pm\Delta S - \frac{\Sigma_{\rm h}}{t_{\rm h}}\right), \tag{11}$$

$$Q_{\rm c} = T_{\rm c}\left(\mp\Delta S - \frac{\Sigma_{\rm c}}{t_{\rm c}}\right). \tag{12}$$

Using the first law of thermodynamics the amount of work W produced by the HE or needed by the RE in every cycle is given by

$$W = -Q_{\rm h} - Q_{\rm c}, \tag{13}$$

which can be evaluated from the low dissipation model by simply substituting equations (11) and (12) into equation (13).

Thus, equations (3)–(13) provide a unified theoretical framework for low dissipation HEs and REs. Now, and because of the different purpose of HE and RE, we continue to develop the low dissipation model separately for each type of device, in order to assess the most relevant specific magnitudes in terms of the characteristic parameters of the model.

2.1. Low dissipation HEs

Just taking the correct signs in equations (11) and (12), one obtains the heat exchanges between the heat reservoirs and the working system for HE:

$$Q_{\rm h} = T_{\rm h}\left(\Delta S - \frac{\Sigma_{\rm h}}{t_{\rm h}}\right), \tag{14}$$

$$Q_{\rm c} = T_{\rm c}\left(-\Delta S - \frac{\Sigma_{\rm c}}{t_{\rm c}}\right). \tag{15}$$

Following the procedure to obtain equations (6)–(8), from equations (14) and (15), we obtain the dimensionless heat exchanges per unit time:

$$\dot{\tilde{Q}}_{\rm h} \equiv \frac{Q_{\rm h}}{t}\frac{\Sigma_{\rm T}}{T_{\rm c}\Delta S^2} = \frac{1}{\tau}\left(1 - \frac{1-\tilde{\Sigma}_{\rm c}}{(1-\alpha)\tilde{t}}\right)\frac{1}{\tilde{t}}, \tag{16}$$

$$\dot{\tilde{Q}}_{\rm c} \equiv \frac{Q_{\rm c}}{t}\frac{\Sigma_{\rm T}}{T_{\rm c}\Delta S^2} = \left(-1 - \frac{\tilde{\Sigma}_{\rm c}}{\alpha\tilde{t}}\right)\frac{1}{\tilde{t}}. \tag{17}$$

Substituting equations (14) and (15) in equation (13) one obtains the (dimensionless) generated power output, $\tilde{P}$, as:

$$\tilde{P} \equiv -\frac{W}{t}\frac{\Sigma_{\rm T}}{T_{\rm c}\Delta S^2} = \left[\frac{1}{\tau} - 1 - \frac{1}{\tau}\left(\frac{1-\tilde{\Sigma}_{\rm c}}{(1-\alpha)\tilde{t}}\right) - \frac{\tilde{\Sigma}_{\rm c}}{\alpha\tilde{t}}\right]\frac{1}{\tilde{t}}, \tag{18}$$

and from equations (16) and (18) the efficiency η of the HE can be obtained easily considering that

$$\eta \equiv \frac{-W}{Q_{\rm h}} = \frac{\tilde{P}}{\dot{\tilde{Q}}_{\rm h}}. \tag{19}$$

2.2. Low dissipation REs

Considering the appropriate signs for REs in equations (11) and (12) one obtains:

$$Q_h = T_h\left(-\Delta S - \frac{\Sigma_h}{t_h}\right), \tag{20}$$

$$Q_c = T_c\left(\Delta S - \frac{\Sigma_c}{t_c}\right). \tag{21}$$

Then, using equations (20) and (21) the dimensionless heating rate, $\dot{\tilde{Q}}_h$, and cooling rate, $\tilde{R} \equiv \dot{\tilde{Q}}_c$, are given by

$$\dot{\tilde{Q}}_h \equiv \frac{Q_h}{t}\frac{\Sigma_T}{T_h \Delta S^2} = \left(-1 - \frac{1-\tilde{\Sigma}_c}{(1-\alpha)\tilde{t}}\right)\frac{1}{\tilde{t}}, \tag{22}$$

$$\tilde{R} \equiv \dot{\tilde{Q}}_c \equiv \frac{Q_c}{t}\frac{\Sigma_T}{T_h \Delta S^2} = \tau\left(1 - \frac{\tilde{\Sigma}_c}{\alpha\tilde{t}}\right)\frac{1}{\tilde{t}}, \tag{23}$$

and the dimensionless power input, $\dot{\tilde{W}}$, needed by the low dissipation RE as

$$\tilde{P}_{in} \equiv \dot{\tilde{W}} \equiv \frac{W}{t}\frac{\Sigma_T}{T_h \Delta S^2} = \left[1 - \tau + \left(\frac{1-\tilde{\Sigma}_c}{(1-\alpha)\tilde{t}}\right) + \frac{\tilde{\Sigma}_c}{\tau\alpha\tilde{t}}\right]\frac{1}{\tilde{t}} \tag{24}$$

and from equations (23) and (24) the efficiency ϵ of the RE can be expressed as

$$\epsilon \equiv \frac{Q_c}{W} = \frac{\tilde{R}}{\tilde{P}_{in}}. \tag{25}$$

2.3. The unified figure of merit χ

In [23], some of the authors of the present work addressed the problem of finding a unified optimization criterion for both HE and RE focused on the common characteristics of every energy converter (the working cyclic system) instead of any specific coupling to external heat reservoirs which can vary according to a particular arrangement. Then, we introduced a figure of merit, χ, defined in terms of the working system and avoiding external coupling characteristics. Physically, this unified figure of merit is based on the heat rate input in the cyclic working system, $\dot{Q}_{in}$, and on the efficiency of the energy converter, z. For HE, $\dot{Q}_{in} \equiv \dot{Q}_h$, $z \equiv \eta$ and for RE, $\dot{Q}_{in} \equiv \dot{Q}_c$, $z \equiv \epsilon$. Mathematically it reads as

$$\chi = \frac{z\, Q_{in}}{t}. \tag{26}$$

From the low dissipation model we can obtain some physical insights of this criterion in regards with the optimization of the total entropy generation. Coming back to Clausius theorem in equation (1) and to the evaluation of the total entropy of the heat devices from equations (9) and (10), we obtain that

$$\Delta S_{tot} = -\frac{Q_h}{T_h} - \frac{Q_c}{T_c} \geqslant 0. \tag{27}$$

Using equations (13) and (19), from equation (27) the total entropy per unit time $\Delta\dot{S}_{tot}$ for a HE can be expressed as:

$$\Delta\dot{S}_{tot} = \frac{\Delta S_{tot}}{t} = \frac{\dot{Q}_h}{T_c}(\eta_c - \eta), \tag{28}$$

where η_c is the Carnot efficiency and $\dot{Q}_h \equiv Q_h/t$ is the amount of heat per unit time absorbed by the working system in a HE.

In the same way, using equations (13) and (25), from equation (27) the total entropy per unit time $\Delta\dot{S}_{tot}$ for a RE is

$$\Delta\dot{S}_{tot} = \frac{\Delta S_{tot}}{t} = \frac{\dot{Q}_c}{T_h}\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_c}\right), \tag{29}$$

where ϵ_c is the Carnot COP and $\dot{Q}_c \equiv Q_c/t$ is the amount of heat per unit time absorbed by the cyclic working system. Up to first order in ϵ_c, this equation can be re-written as:

$$\Delta\dot{S}_{tot} = \frac{\Delta S_{tot}}{t} = \frac{\dot{Q}_c}{T_h \epsilon_c^2}(\epsilon_c - \epsilon). \tag{30}$$

Below, in section 3.3 we numerically justify this approximation. The formal symmetry of equations (28) and (30) in regard to the differences with the Carnot values η_c and ϵ_c supports the definition of the unified

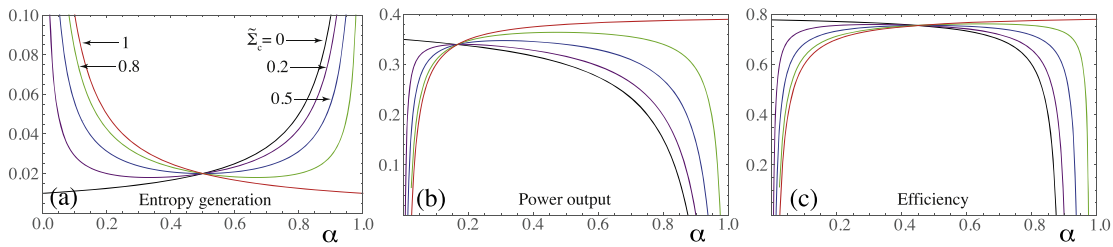

Figure 2. Dimensionless functions $\widetilde{\dot{\Delta S}}_{\rm tot}(\tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$, $\tilde{P}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t}) \equiv \tilde{\chi}^{\rm (HE)}$, and $\eta(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ in terms of the fractional contact time $\alpha = t_{\rm c}/t$ for $\tau = 0.2$, $\tilde{t} = 10$, and the labeled $\tilde{\Sigma}_{\rm c}$-values: $\tilde{\Sigma}_{\rm c} = 0$ (black); $\tilde{\Sigma}_{\rm c} = 0.2$ (purple); $\tilde{\Sigma}_{\rm c} = 0.5$ (blue); $\tilde{\Sigma}_{\rm c} = 0.8$ (green); and $\tilde{\Sigma}_{\rm c} = 1$ (red).

optimization criterion χ for both HE and RE, based on the heat rate absorbed by the working system, $\dot{Q}_{\rm in}$, and the efficiency of the device, z, in order to minimize the total entropy generation under some constraints.

From equations (19) and (26) it is straightforward to obtain that the figure merit χ for HE is given by

$$\chi^{\rm (HE)} = \frac{\eta\, Q_{\rm h}}{t_{\rm cycle}} = -\frac{W}{t_{\rm cycle}}, \tag{31}$$

which exactly coincides with the power output. In the dimensionless form it reads as $\chi^{\rm (HE)}$

$$\tilde{\chi}^{\rm (HE)} \equiv \chi^{\rm (HE)} \frac{\Sigma_{\rm T}}{T_{\rm c}\Delta S^2} = \tilde{P}. \tag{32}$$

From equations (25) and (26) it is obtained that χ for RE is given by

$$\chi^{\rm (RE)} = \frac{\epsilon\, Q_{\rm c}}{t}, \tag{33}$$

and finally, the dimensionless expression for $\chi^{\rm (RE)}$ is

$$\tilde{\chi}^{\rm (RE)} \equiv \chi^{\rm (RE)} \frac{\Sigma_{\rm T}}{T_{\rm h}\Delta S^2} = \epsilon\tilde{R}. \tag{34}$$

In section 3.1 we will present numerical results in order to check the suitability of χ, whether it is considered for engines or REs.

3. Results

3.1. Heat Engines

We start showing in figure 2 the behavior of the dimensionless functions $\widetilde{\dot{\Delta S}}_{\rm tot}(\tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$, $\tilde{P}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t}) \equiv \tilde{\chi}^{\rm (HE)}$, and $\eta(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ in terms of the fractional contact time $\alpha = t_{\rm c}/t$ for some representative values of τ, $\tilde{\Sigma}_{\rm c}$, and $\tilde{t}$.

When the dissipation is fully due to the hot thermal bath $\tilde{\Sigma}_{\rm h} = 1$ ($\tilde{\Sigma}_{\rm c} = 0$), the entropy generation rate presents, see figure 2(a), an intrinsic value for any fixed total time, $1/(1-\alpha)\tilde{t}^2$, according to equation (8). From this baseline value, $\widetilde{\dot{\Delta S}}_{\rm tot}(\tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ monotonically increases as α does, i.e., irreversibilities increase as heat dissipated in the cold reservoir is long lasting for a fixed total cycle time. On the contrary, for $\tilde{\Sigma}_{\rm h} = 0$ ($\tilde{\Sigma}_{\rm c} = 1$) $\widetilde{\dot{\Delta S}}_{\rm tot}(\tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ monotonically increases from the baseline value $1/\alpha\tilde{t}^2$ as α decreases, i.e., irreversibilities increase as heat dissipated in the hot reservoir is long lasting for a total fixed cycle time. In between these two extreme cases we observe in figure 2(a) how $\widetilde{\dot{\Delta S}}$ shows a well defined minimum value on α for each value of the dissipation parameters. For $\tilde{\Sigma}_{\rm c}$ smaller than 0.5 these minima values increase with α while the opposite happens for $\tilde{\Sigma}_{\rm c}$ higher than 0.5. The fully symmetric case $\alpha = 0.5$ and $\tilde{\Sigma}_{\rm c} = 0.5$ defines a peculiar state which highlights the change of trend of the minima values of the entropy generation rate.

The behaviors of $\tilde{P}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ and $\eta(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ are, obviously, conditioned by the one of the entropy generation, as it can be checked in figures 2(b) and (c), respectively. So, high values of $\tilde{\Sigma}_{\rm h}$ ($\tilde{\Sigma}_{\rm c} \to 0$) give high values of power and efficiency for longer contact times with the hot reservoir ($\alpha \to 0$), while longer contact times with the cold reservoir ($\alpha \to 1$) favor the existence of states with high efficiency and power for high values of $\tilde{\Sigma}_{\rm c}$. We stress the existence of α-values for which the heat device gives power output (and then it behaves as a true HE) as well as of α-values not allowed for a low dissipation HE depending of the particular values of the dissipation parameters $\tilde{\Sigma}_{\rm c}$ and $\tilde{\Sigma}_{\rm h}$ for a fixed total cycle time.

From the results in figure 2 it is evident that $\eta(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$, $\tilde{P}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$, and $\widetilde{\dot{\Delta S}}_{\rm tot}(\tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ present well-defined optimal values (maxima for power and efficiency and minima for entropy generation) in terms of the

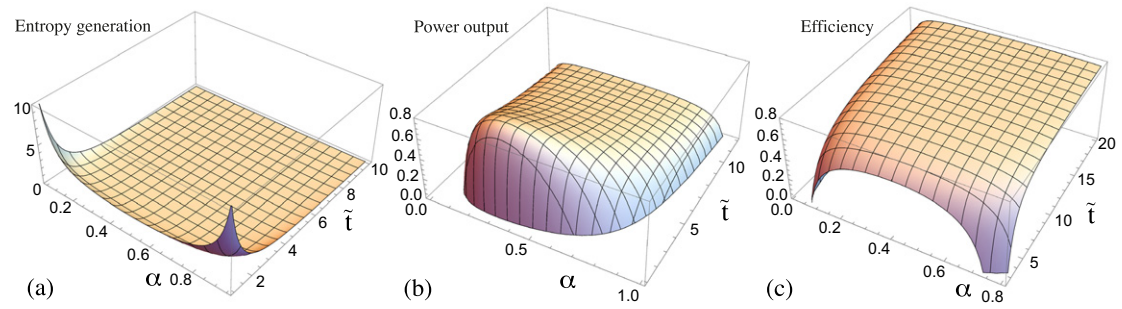

Figure 3. 3D-plots of the dimensionless functions $\widetilde{\dot{\Delta S}}_{\rm tot}(\tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$, $\tilde{P}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t}) \equiv \tilde{\chi}^{\rm (HE)}$, and $\eta(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ in terms of $\alpha = t_{\rm c}/t$ and $\tilde{t}$ for $\tau = 0.2$, $\tilde{\Sigma}_{\rm c} = 0.5$.

fractional contact times α. However, in general this is not true in terms of the reduced total cycle time $\tilde{t}$ or in terms of the other variables involved in the model. Only the power output (i.e., the figure of merit $\chi^{\rm (HE)}$) shows an absolute maximum in terms of both $\tilde{t}$ and α. This is best viewed in figure 3, where we show the 3D-plots of these functions in terms of α and $\tilde{t}$: the local maxima (minima) of $\eta(\tau, \tilde{\Sigma}_{\rm c}, \alpha, t)$ and $\widetilde{\Delta S}_{\rm tot}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ in function of α for an fixed total time are clearly visible for the three functions, besides of the commented absolute $\tilde{t}$-maximum of power output. The influence of $\tilde{t}$ is, as expected, to increase (decrease) the efficiency (entropy generation) value. Indeed in the long time limit $\tilde{t} \to \infty$ the reversible Carnot value of the efficiency and the zero value of the entropy generation are eventually obtained. Accordingly, $\tilde{P}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, t)$ shows a continuous decay with $\tilde{t}$ towards a zero-value in the reversible limit after its absolute maximum value. On the other hand, as $\tilde{t}$ decreases and becomes close to one, $\widetilde{\dot{\Delta S}}_{\rm tot}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, \tilde{t})$ increases abruptly, taking values greater than one, and power output $\tilde{P}(\tau, \tilde{\Sigma}_{\rm c}, \alpha, t)$ becomes negative. Thus, as expected, for these operating regimes the role of irreversibilities is so important as to determine that the device can not work more as a HE.

It is easy to show from equation (18) by setting the derivatives of $\tilde{P}$ with respect to α and $\tilde{t}$ that the absolute maximum power holds at values of $\alpha_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c})$ and $\tilde{t}_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c})$ given, respectively, by

$$\alpha_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c}) = \frac{1}{1 + \sqrt{\dfrac{1 - \tilde{\Sigma}_{\rm c}}{\tau \tilde{\Sigma}_{\rm c}}}}, \tag{35}$$

$$\tilde{t}_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c}) = \frac{2}{1 - \tau}\left(\sqrt{\tau \tilde{\Sigma}_{\rm c}} + \sqrt{1 - \tilde{\Sigma}_{\rm c}}\right)^2. \tag{36}$$

From these values, the efficiency at maximum power is

$$\eta_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c}) \equiv \eta(\tau, \tilde{\Sigma}_{\rm c}, \alpha_{\max P}, \tilde{t}_{\max P}) = \frac{(1 - \tau)\left[1 + \sqrt{\dfrac{\tau \tilde{\Sigma}_{\rm c}}{1 - \tilde{\Sigma}_{\rm c}}}\right]}{\left[1 + \sqrt{\dfrac{\tau \tilde{\Sigma}_{\rm c}}{1 - \tilde{\Sigma}_{\rm c}}}\right]^2 + \tau\left(1 - \dfrac{\tilde{\Sigma}_{\rm c}}{1 - \tilde{\Sigma}_{\rm c}}\right)}, \tag{37}$$

which is exactly the result reported by Esposito *et al* (see equation (8) in [21]). Indeed, the well-known upper, lower, and symmetric bounds in equation (11) of [21] follow straightforwardly under the conditions $\tilde{\Sigma}_{\rm c} \to 0$, $\tilde{\Sigma}_{\rm c} = 1/2$, and $\tilde{\Sigma}_{\rm c} \to 1$: $\eta_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c} \to 0) = \frac{\eta_{\rm c}}{2 - \eta_{\rm c}}$, $\eta_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c} = \frac{1}{2}) = 1 - \sqrt{\tau} = 1 - \sqrt{1 - \eta_{\rm c}}$ (i.e., the Curzon–Ahlborn efficiency), and $\eta_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c} \to 1) = \frac{\eta_{\rm c}}{2}$. Also, from equation (35) we obtain that

$$\frac{\alpha_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c})}{1 - \alpha_{\max \tilde{P}}(\tau, \tilde{\Sigma}_{\rm c})} \equiv \frac{\tilde{t}_{\rm c}}{\tilde{t}_{\rm h}} = \sqrt{\tau \frac{\tilde{\Sigma}_{\rm c}}{1 - \tilde{\Sigma}_{\rm c}}}, \tag{38}$$

which is the contact time ratio at maximum power reported in equation (12) of [21].

In order to clarify the different role played by the considered figures of merit and the optimization space allowed by the constraints, it is also interesting to pay attention to the behavior of the optimized contact times under minimum entropy generation and maximum efficiency conditions, a point rarely treated in the literature. From figures 3(a) and (c) we note that, opposite to power output in figure 3(b), these functions do not show any absolute maximum value in terms on both α and $\tilde{t}$. However they show a local optimization on α.

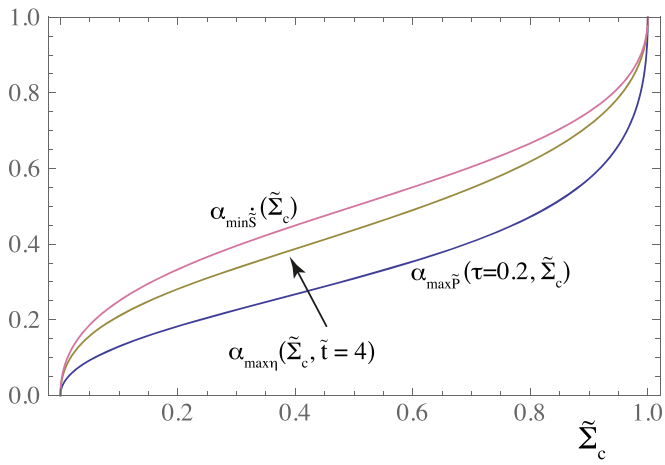

Figure 4. Behavior of the fractional contact times at minimun entropy generation, $\alpha_{\min\dot{\tilde{S}}}(\tilde{\Sigma}_c)$, maximum power output, $\alpha_{\max\tilde{P}}(\tau = 0.2, \tilde{\Sigma}_c)$, and maximum efficiency, $\alpha_{\max\eta}(\tilde{\Sigma}_c, \tilde{t} = 4)$.

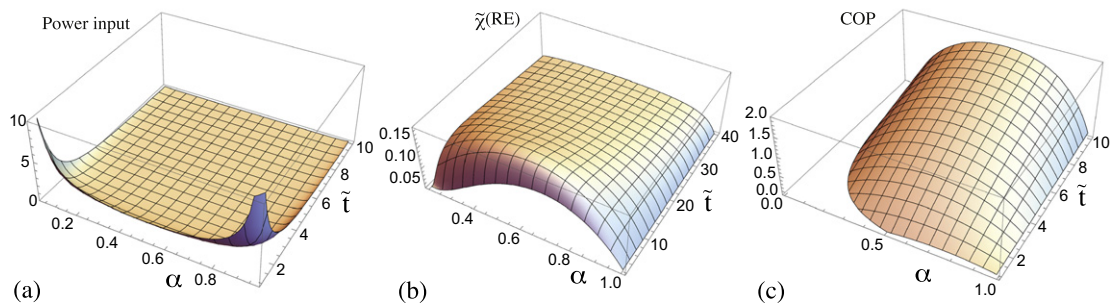

Figure 5. Dimensionless power input $\tilde{P}_{\text{in}}(\tau, \tilde{\Sigma}_c, \alpha, \tilde{t})$, $\tilde{\chi}^{(\text{RE})}(\tau, \tilde{\Sigma}_c, \alpha, \tilde{t})$, and COP $\epsilon(\tau, \tilde{\Sigma}_c, \alpha, \tilde{t})$ in terms of $\alpha = t_c/t$ and $\tilde{t}$ for $\tau = 0.8$, $\tilde{\Sigma}_c = 0.5$.

By setting the derivatives of η and $\widetilde{\dot{\Delta S}}_{\text{tot}}$ with respect to α we obtain the following expressions:

$$\alpha_{\max\eta}(\tilde{\Sigma}_c, \tilde{t}) = \frac{\tilde{\Sigma}_c\tilde{t} - \sqrt{(1-\tilde{\Sigma}_c)(2\tilde{\Sigma}_c + \tilde{t} - 1)\tilde{\Sigma}_c\tilde{t}}}{(2\tilde{\Sigma}_c - 1)\tilde{t}}, \tag{39}$$

$$\alpha_{\min\dot{\tilde{S}}}(\tilde{\Sigma}_c) = \frac{1}{1 + \sqrt{\dfrac{1-\tilde{\Sigma}_c}{\tilde{\Sigma}_c}}}. \tag{40}$$

Due to the different dependence of these local fractional optimal times, a comparison between them and with $\alpha_{\max\tilde{P}}(\tau, \tilde{\Sigma}_c)$ should be done carefully, and must take into account at fixed cycle time for particular values of τ and $\tilde{\Sigma}_c$. As an illustration we show in figure 4 the behavior of these three functions in terms on $\tilde{\Sigma}_c$. It is easy to show from equations (35) and (39) that $\alpha_{\max\eta}(\tilde{\Sigma}_c, \tilde{t} \to \infty) = \alpha_{\min\dot{\tilde{S}}}(\tilde{\Sigma}_c)$ and that $\alpha_{\max\tilde{P}}(\tau \to 1, \tilde{\Sigma}_c) = \alpha_{\min\dot{\tilde{S}}}(\tilde{\Sigma}_c)$. In words, optimized contact times become equivalent to the one predicted by the minimum entropy generation in long time cycles and also when the temperatures of the external heat baths become equals (i.e. when the cycle is performed between very closer temperatures). The implication of this fact on the unified figure of merit χ will be analyzed below in section 3.3.

3.2. Refrigerators

For low dissipation RE some of the thermodynamic functions behave as expected for the counterpart ones of a HE, but other specific functions show important subtleties in regards to its optimization. To begin with, in figure 5 we show 3D-plots of the power input, the figure of merit χ, and the COP in terms of α and $\tilde{t}$ for a characteristic $\tau = 0.8$ and $\tilde{\Sigma}_c = 0.5$. The election of these particular values does not change the main results. Note that the entropy generation in RE is exactly the same as for HE. From these figures three relevant points can be raised out:

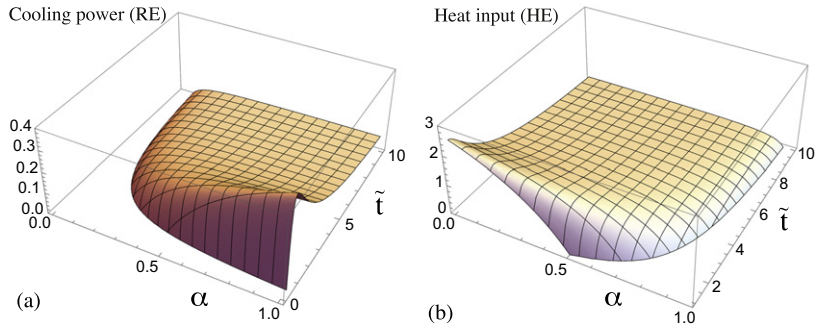

Figure 6. Dimensionless cooling power $\tilde{R}(\tau = 0.8, \tilde{\Sigma}_c, \alpha, \tilde{t})$ for a RE and dimensionless heat absorbed by a HE $\dot{\tilde{Q}}_h(\tau = 0.2, \tilde{\Sigma}_c, \alpha, \tilde{t})$ in terms of $\alpha = t_c/t$ and $\tilde{t}$ for $\tilde{\Sigma}_c = 0.5$.

- The COP, $\epsilon(\tau, \tilde{\Sigma}_c, \alpha, \tilde{t})$, as the efficiency for HE, shows (see figure 5(c)) well defined local maxima on the fractional contact times α and a monotonic increase with $\tilde{t}$ up to get the corresponding Carnot COP under quasi static conditions ($\tilde{t} \to \infty$).
- $\tilde{\chi}^{(RE)}(\tau, \tilde{\Sigma}_c, \alpha, \tilde{t})$, as the power output in HE, shows (figure 5(b)) an absolute maxima on both α and $\tilde{t}$ with a progressive decay as $\tilde{t}$ increases after its maximum value. Indeed, this behavior can be considered a first consequence of the unified definition of the figure of merit χ in terms of the heat input in the heat device, section 2.3. The results of this optimization criterion have been analyzed by de Tomás *et al* [23] under symmetric conditions, by Wang *et al* [24] under asymmetric conditions, and the extension to a low-dissipation model with finite time effects for the adiabatic steps was reported by Hu *et al* [25]. The minimally nonlinear model by Izumida *et al* [26] handled the same issue under the perspective of the nonlinear effects in the framework of fluxes and thermodynamic forces and Long and Liu [27] analyzed the influence of the non isothermal processes in the heat exchanges.
- The power input $\tilde{P}_{in}(\tau, \tilde{\Sigma}_c, \alpha, \tilde{t})$ in RE, equation (24), shows (figure 5(a)) well defined minima values on α but a monotonic decaying on $\tilde{t}$, in opposition with the power output behavior in a HE. Thus, (minimum) power input in RE could be considered as a $\tilde{t}$-dependent figure of merit in this kind of models. Its optimized contact times are the same that the obtained under minimum entropy generation ($\alpha_{\min \tilde{P}_{in}}(\tilde{\Sigma}_c) \equiv \alpha_{\min \dot{\tilde{S}}}(\tilde{\Sigma}_c)$). In this regard, we see that minimum entropy generation and minimum power input are equivalent figures of merit in RE. Although the resulting COP at minimum power $\epsilon_{\min \tilde{P}_{in}}(\tau, \tilde{\Sigma}_c, \tilde{t})$ is t-dependent, we give here some limit results under high asymmetric and symmetric conditions:

$$\epsilon_{\min \tilde{P}_{in}}(\tau, 0, \tilde{t}) = \frac{\tilde{t}\tau}{1 + \tilde{t}(1 - \tau)}, \tag{41}$$

$$\epsilon_{\min \tilde{P}_{in}}(\tau, 1, \tilde{t}) = \frac{\tau(1 - \tilde{t})}{1 + \tilde{t}(1 - \tau)}, \tag{42}$$

and

$$\epsilon_{\min \tilde{P}_{in}}\left(\tau, \frac{1}{2}, \tilde{t}\right) = \frac{\tau(\tilde{t} - 1)}{2 + \tilde{t}(1 - \tau)}. \tag{43}$$

Another peculiarity in RE stems from the behavior of the dimensionless cooling power $\tilde{R}(\tau, \tilde{\Sigma}_c, \alpha, \tilde{t})$ given by equation (23) and shown in figure 6(a). The distinctive feature is, besides the absence of local maxima on α, the existence of a well defined local maxima on $\tilde{t}$. In other words, the cooling power is a figure of merit in regards to the total cycle time, independently of the contact times with the external reservoirs. It can be found by setting the derivative of $\tilde{R}$ with respect to $\tilde{t}$ that

$$\tilde{t}_{\max \tilde{R}}(\tilde{\Sigma}_c, \alpha) = \frac{2\tilde{\Sigma}_c}{\alpha}, \tag{44}$$

and that the optima COP-value is given by:

$$\epsilon_{\max \tilde{R}}(\tau, \tilde{\Sigma}_c, \alpha) = \frac{\tau}{2\left(\frac{3}{2} + \frac{\alpha(1 - \tilde{\Sigma}_c)}{2(1 - \alpha)\tilde{\Sigma}_c} - \tau\right)}. \tag{45}$$

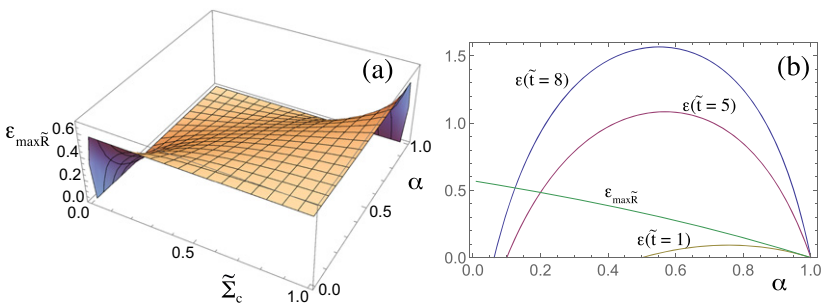

Figure 7. (a) 3D-plot of the COP at maximum cooling power $\epsilon_{\max\tilde{R}}(\tau = 0.8, \tilde{\Sigma}_c, \alpha)$; (b) comparison between $\epsilon_{\max\tilde{R}}(\tau = 0.8, \tilde{\Sigma}_c = 0.5, \alpha)$, and the COP $\epsilon(\tau = 0.8, \tilde{\Sigma}_c = 0.5, \alpha, \tilde{t})$ for $\tilde{t} = 1$, $\tilde{t} = 5$, and $\tilde{t} = 8$.

The equation above is plotted in figure 7(a) versus α and $\tilde{\Sigma}_c$ for an usual value of $\tau = 0.8$. We also plot in figure 7(b) its behavior on α in order to compare with those obtained from figure 5(c) at different times. We stress the monotonic decay of $\epsilon_{\max\tilde{R}}(\tau, \tilde{\Sigma}_c, \alpha)$ versus the parabolic behavior showed by $\epsilon(\tau, \tilde{\Sigma}_c, \alpha, \tilde{t})$, in agreement with the $1/\alpha$ decay of $\tilde{t}_{\max\tilde{R}}(\tilde{\Sigma}_c, \alpha)$ in equation (44): greater α-values imply shorter cycle times with greater dissipations.

The dissipative high asymmetric limits are α-independent and they are given by:

$$\epsilon_{\max\tilde{R}}(\tau, 0, \alpha) = 0, \tag{46}$$

$$\epsilon_{\max\tilde{R}}(\tau, 1, \alpha) = \frac{\tau}{3 - 2\tau} = \frac{\epsilon_c}{3 + \epsilon_c}, \tag{47}$$

while the symmetric limit is α-dependent

$$\epsilon_{\max\tilde{R}}\left(\tau, \tilde{\Sigma}_c = \frac{1}{2}, \alpha\right) = \frac{\tau}{2\left(\frac{3}{2} + \frac{\alpha}{2(1-\alpha)} - \tau\right)}, \tag{48}$$

with a particular value at $\alpha = 1/2$ given by:

$$\epsilon_{\max\tilde{R}}\left(\tau, \frac{1}{2}, \frac{1}{2}\right) = \frac{\tau}{2(2-\tau)} = \frac{\epsilon_c}{2(2+\epsilon_c)}. \tag{49}$$

The above particular values of the COP at maximum cooling power are easily amenable to compare with similar results obtained under tight-coupling condition in minimally nonlinear models by Izumida *et al* [28], and for an autonomous thermoelectric device modeled assuming the exoreversible hypothesis by Apertet *et al* [29]. The slight differences could be explained as a consequence of the different assumptions taken into account in each kind of model.

3.3. Physical meaning of χ as unified figure of merit

In section 2.3 we stressed the formal symmetry of the entropy generation in order to introduce the figure of merit χ as an appropriate tool in the optimization of both HE and RE. The results above show that χ has an absolute maximum value (see figures 3(b) and 5(b)), while other thermodynamic magnitudes (as efficieny, COP, power input, cooling power) only show local optimal values. Here we try to go further by analyzing why χ behaves so well as a unified criterion for both HE and RE and its connection with the entropy generation, which is ultimately the consequence of the overall dissipations.

For RE devices $\tilde{\chi}^{(\mathrm{RE})} = \tilde{R}\epsilon$. We saw in figure 6(a) that the cooling power, $\tilde{R}$, shows a local maximum on $\tilde{t}$, while the COP, ϵ, shows local maxima on the fractional contact times α, see figure 5(c). For a HE device $\tilde{\chi}^{(\mathrm{HE})} = \dot{\tilde{Q}}_h\eta \equiv \tilde{P}$ and we saw in figures 2(c) and 3(c) that the local maxima on α of the efficiency η. In figure 6(b) we show now a view of the dimensionless heat input rate, $\dot{\tilde{Q}}_h$, for a HE. It shows a clear $\tilde{t}$-maximum value at $\tilde{t}_{\max\dot{\tilde{Q}}_h} = \frac{2(\tilde{\Sigma}_c - 1)}{\alpha - 1}$. Then, we could conclude that for a generic low dissipation heat device χ, defined as the product of the converter efficiency z times the heat absorbed $\dot{Q}_{in}$ by the working system, means a compromise between two functions each one with a different optimization space (z in the space of fractional contact times α and $\dot{Q}_{in}$ in the time domain $\tilde{t}$) which together allow for an absolute maximum value of χ.

In figure 8(a) we depicted the times $\tilde{t}_{\max\chi}$ at maximum $\tilde{\chi}$ for HEs and REs and in figure 8(b) the corresponding optimized entropies, as functions of $\tilde{\Sigma}_c$. It shows that both times present a non monotonic behavior on $\tilde{\Sigma}_c$, being $\tilde{t}_{\max\chi}$ for REs greater than for HEs, except for small values of $\tilde{\Sigma}_c$ ($\leqslant 0.15$), with a maximum

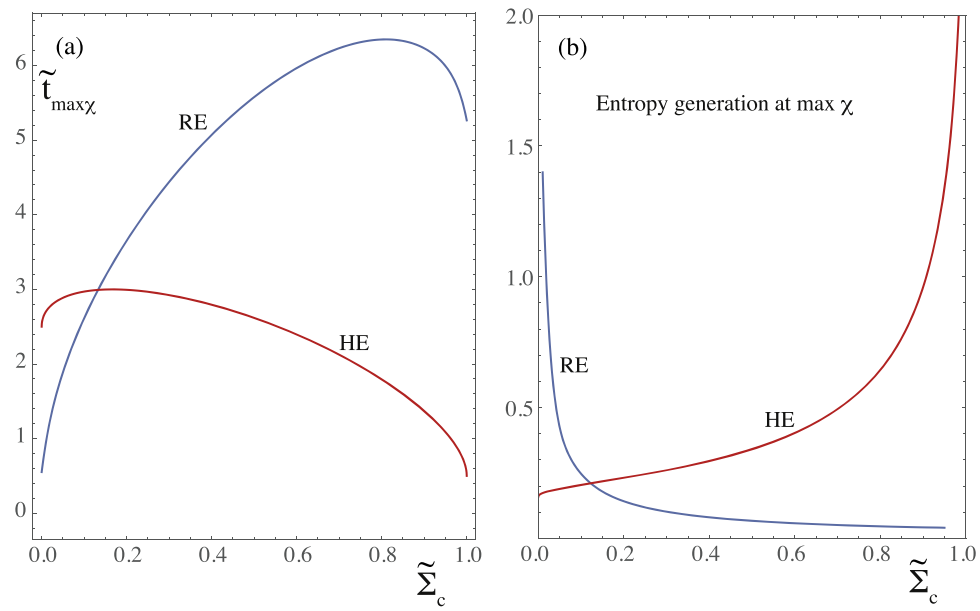

Figure 8. (a) Optimized total times at maximum χ for refrigerators $\tilde{t}_{\max\tilde{\chi}^{(\mathrm{RE})}}(\tau = 0.8,\ \tilde{\Sigma}_c)$ and heat engines $\tilde{t}_{\max\tilde{\chi}^{(\mathrm{HE})}}(\tau = 0.2,\ \tilde{\Sigma}_c)$; (b) behavior of the corresponding optimized entropies $\tilde{\Delta\dot{S}}_{\mathrm{tot},\max\tilde{\chi}^{(\mathrm{RE})}}(\tau = 0.8,\ \tilde{\Sigma}_c)$ and $\tilde{\Delta\dot{S}}_{\mathrm{tot},\max\tilde{\chi}^{(\mathrm{HE})}}(\tau = 0.2,\ \tilde{\Sigma}_c)$.

value that is more than two times greater than for HEs. Provided that the total entropy generation is a monotonous decreasing function on time, this yields the optimized entropy generation for REs to be much lower than for HEs, except for small values of $\tilde{\Sigma}_c$. This result can also be understood on the basis of equation (29). For usual real REs, the difference of temperatures between the hot and cold external reservoirs is much less than for common HEs (in our choice $\tau = 0.8$ for REs, while $\tau = 0.2$ for HEs). This fact implies that real REs work under a situation of non-thermal equilibrium which is not so drastic as for HEs. Then, one can expect that the COP, ϵ, of a low dissipation RE should be closer to Carnot COP, ϵ_C. This is the reason why the approximation $\Delta\dot{S}_{\mathrm{tot}} \simeq \frac{\dot{Q}_c}{\epsilon_C^2\, T_h}(\epsilon_c - \epsilon)$ we made in section 2.3 is justified, and ultimately why $\chi = \dot{Q}_c\ \epsilon$ works as a true optimization criteria for REs, not only from a practical point of view (as a compromise between the two functions $\dot{Q}_c$ and ϵ), but also, and much more important, from the perspective of the second law of thermodynamics.

4. Summary and conclusions

Unified results for both Carnot-like low-dissipation models of HE and RE have been reported by analyzing the influence of the time- and size-constraints on the main energetic properties of each heat device. The dependence on both the total cycle time and on the contact times with the external heat reservoirs have been analyzed for the main thermodynamic magnitudes of each device, in terms of the entropy generation and the characteristic time scale provided by the model. The role played by different optimization criteria has been clarified and some previous bounds for efficiency and COP have been also recovered. In particular, the suitability of the figure of merit χ as a unified optimization criterion for HE and RE has been analyzed and its relation with the entropy generation stated out.

As a main conclusion we note the importance of the time-constraints on the performance and optimization of heat devices owing their interplay with the entropy generation, which ultimately is the fundamental thermodynamic magnitude accounting for the unavoidable irreversibilities of real heat devices. The different nature of these irreversibilities, their treatment under the possible constraints, and the connection with entropy generation are some elements which deserve future studies in order to account for more complete, realistic, and unified heat devices models [30, 31].

Acknowledgments

We acknowledge Ministerio de Economía y Competitividad of Spain under Grant ENE2013-40644-R.

References

[1] Carnot S 1824 *Réflexions Sur la Puissance Motrice du Feu* (Paris: Bachelier)
[2] Curzon F L and Ahlborn B 1975 *Am. J. Phys.* **43** 22
[3] Novikov I 1957 *Atommaya Energ.* **3** 409
[4] Chambadal P 1957 *Les Centrales Nucleaires* (Paris: Armand Colin)
[5] Yvon J 1955 *Proc. Int. Conf. on Peaceful Uses of Atomic Energy (Geneva, United Nations)* p 387
[6] Reitlinger H B 1929 *Sur l'utilisation de la Chaleur Dans Les Machines a Feu* (Liege: Vaillant-Carmanne)
[7] Vaudrey A, Lanzetta F and Feidt M 2014 arXiv:1406.5853
[8] Berry R S, Kazakov V A, Sieniutycz S, Szwast Z and Tsirlin A M 2000 *Thermodynamics Optimization of Finite-Time Processes* (Chichester: Wiley)
[9] Wu C, Chen L and Chen J 2004 *Advances in Finite-Time Thermodynamics: Analysis and Optimization* (New York: Nova Science)
[10] Durmayaz A, Sogut O S, Sahin B and Yavuz H 2004 *Prog. Energy Combust. Sci.* **30** 175
[11] Bejan A 1996 *Entropy Generation Minimization* (Boca Raton, FL: CRC Press)
[12] Yan Z and Chen J 1990 *J. Phys. D: Appl. Phys.* **23** 136–40
[13] Angulo-Brown F 1991 *J. Appl. Phys.* **69** 7465–71
[14] Velasco S, Roco J M M, Medina A and Calvo Hernández A 1997 *Appl. Phys. Lett.* **71** 1130
Velasco S, Roco J M M, Medina A and Calvo Hernández A 1997 *Phys. Rev. Lett.* **78** 3241
[15] Jiménez de Cisneros B and Calvo Hernández A 2008 *Phys. Rev.* E **77** 041127
[16] Schmiedl T and Seifert U 2007 *Phys. Rev. Lett.* **98** 108301
Schmiedl T and Seifert U 2008 *Europhys. Lett.* **81** 20003
[17] Gaveau B, Moreau M and Schulman L S 2010 *Phys. Rev. Lett.* **105** 060601
[18] Moreau M, Gaveau B and Schulman L S 2011 *Eur. Phys. J.* D **62** 67–71
[19] Clausius R 1888 *Theorie Mecanique de la Chaleur* (Paris: Gauthier-Villars)
[20] Kestin J 1976 *The Second Law of Thermodynamics* (Stroudsburg, PA: Dowden, Hutchinson and Ross, Inc.) pp 87–98, 133–61, 162–93
[21] Esposito M, Kawai R, Lindenberg K and Van den Broeck C 2010 *Phys. Rev. Lett.* **105** 150603
[22] Uzdin R and Kosloff R 2014 *Europhys. Lett.* **108** 40001
[23] de Tomás C, Calvo Hernández A and Roco J M M 2012 *Phys. Rev.* E **85** 010104(R)
[24] Wang Y, Li M, Tu Z C, Calvo Hernández A and Roco J M M 2012 *Phys. Rev.* E **86** 011127
[25] Hu Y, Wu F, Ma Y, He J, Wang J, Calvo Hernández A and Roco J M M 2013 *Phys. Rev.* E **88** 062115
[26] Izumida Y, Okuda K, Calvo Hernández A and Roco J M M 2013 *Europhys. Lett.* **101** 10005
[27] Long R and Liu W 2014 *J. Phys. A: Math. Theor.* **47** 325002
[28] Izumida Y, Okuda K, Roco J M M and Calvo Hernández A 2015 *Phys. Rev.* E **91** 052140
[29] Apertet Y, Ouerdane H, Michot A, Goupil C and Lecoeur P 2013 *Europhys. Lett.* **103** 40001
[30] Sheng S Q and Tu Z C 2014 *Phys. Rev* E **89** 012129
Sheng S Q and Tu Z C 2015 *New J. Phys.* **17** 045013
[31] Izumida Y and Okuda K 2015 arXiv: 1501.03987v1