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Quantum limit for driven linear non-Markovian open-quantum-systems

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Abstract
The interplay between non-Markovian dynamics and driving fields in the survival of entanglement between two non-degenerate oscillators is considered here. Based on exact analytical results for the non-Markovian dynamics of two parametrically coupled non-degenerate oscillators in contact with non-identical independent thermal baths, the out-of-equilibrium quantum limit derived in [Phys. Rev. Lett. 105 180501 (2010)] is generalised to the non-Markovian regime. Specifically, it is shown that non-Markovian dynamics, when compared to the Markovian case, allow for the survival of stationary entanglement at higher temperatures, with larger coupling strength to the baths and at smaller driving rates. The effect of the asymmetry of the (i) coupled oscillators, (ii) coupling strength to the baths at equal temperature, and (iii) temperature at equal coupling strength is discussed. In particular, it is shown that the non-Markovian character of the dynamics is capable of beating the resonant condition that states that the driving frequency equals the sum of the natural frequencies for the maximum rate of squeezing generation; hence, squeezing generation is more robust under non-Markovian dynamics.

1. Introduction
The survival of quantum features in hot environments is restricted by decoherence [1]. Quantum features such as entanglement survive in the presence of the environment if the typical energy scale of the system $\hbar \omega$ is larger than the energy scale $k_B T$ associated with thermal fluctuations [2]. For typical nano-mechanical resonators with oscillation frequencies between 1 MHz and 1 GHz, the quantum nature of these systems is revealed only if the temperature is well below 10 $\mu$K and 10 mK, respectively. However, current cryostats operate best at 15 mK [3].

Recently, it was shown that the presence of driving forces relaxes the traditional criterion $\hbar \omega / k_B T > 1$ discussed above [4]. To discuss the way in which this condition is relaxed, let $\gamma_\text{FB}$ be the time scale associated with the non-unitary dynamics induced by the thermal bath and $\gamma_p$ the pumping rate of the driving field. In terms of these time scales, the quantum limit for driven out-of-equilibrium quantum systems reads $\hbar \omega / k_B T_{\text{eff}} > 1$, where $T_{\text{eff}} = T_{\text{FB}} / \gamma_p$ is an effective temperature [4, 5]. For pumping rates larger than the decay rates, $\gamma_\text{FB} / \gamma_p < 1$, the effective temperature of the system turns out to be smaller than the thermal equilibrium temperature $T$. This allows for the survival of entanglement at higher equilibrium temperatures $T$. For two frequency-degenerate nano-mechanical resonators of frequency $\omega_0 = 2\pi \times 15$ MHz and mass $10^{-17}$ kg with a coupling amplitude $\sim 10^{-3} \text{ mrad}$ and a decay rate $\gamma_\text{FB} = 5 \times 10^{-3} \omega_0$, this condition implies that steady entanglement can be observed at tens of mK [4]. However, entanglement might survive in the range of Kelvin if the frequency or the coupling amplitude can be increased by a factor of 10 [4].

Very recently, a non-degenerate version of this physical model was implemented using two strongly inter-coupled doubly clamped beams [6]. The coupling was mediated by an exaggerated overhang between the clamped beams. The configuration presented in [6] sustains two spectrally closely-spaced vibration modes at $2\pi \times 246$ and $2\pi \times 262$ kHz with quality factors of 1300 and 2200, respectively. In combination with
piezoelectric transducers that are incorporated directly into the mechanical elements, it provides the key to realising efficient parametric down-conversion [6].

Moreover, there is now experimental evidence that the non-unitary dynamics of micro-resonators are driven by non-Markovian dynamics [7]. This, in turn, suggests that a similar behaviour may be encountered at the nano-mechanical level, a realm where the low temperature condition may already introduce non-Markovian correlations [8]. Motivated by the experimental results discussed above and by the recent interest and the important role of non-Markovian dynamics in, e.g., biological systems [9–14], quantum metrology [15, 16], foundations of quantum thermodynamics [8, 17, 18], nuclear reactions [19] and quantum computation based on organic π-conjugated copolymers [20], as well as by the intricate and delicate interplay between non-Markovian dynamics and driven fields in optimal control scenarios [21, 22], the quantum limit derived in [4] is extended here to the case of non-Markovian dynamics.

Specifically, in this article the non-Markovian entanglement dynamics of non-degenerate oscillators, parametrically coupled and in contact with independent non-identical thermal baths, are analytically solved by means of the influence functional approach of Feynman and Vernon [23]. In doing so, the main features of the model, as well as some of its physical implementations, are discussed in section 2. In that section, details of the analytic solution of the dynamics are also given whereas the concept of logarithmic negativity, as a measure of entanglement for continuous variable Gaussian states, is introduced in section 3. For the case of degenerate oscillators and identical thermal baths, a comprehensive analysis of the behaviour of entanglement as a function of the temperature and coupling constant to the baths is presented in section 4. The emergence of an stationary entanglement for continuous variable Gaussian states, is introduced in section 3. For the case of degenerate oscillators coupled to identical thermal baths considered in section 6.

The results discussed in these sections are valid for any parameter regime and allow for predicting that, when compared with the Markovian case, non-Markovian dynamics (i) decrease the time needed to generate entanglement, (ii) increase the temperature and the coupling-strength-to-the-environment limits at which steady state entanglement can be found, (iii) decrease the pumping rate at which a particular amount of entanglement is generated, and (iv) relax the resonant driving condition. These are the main findings of this article.

2. A paradigmatic model for several physical systems

The model described below is capable of describing a large variety of physical systems such as coupled trapped ions [24], coupled membranes or mechanical oscillators [25]. Specifically, two parametrically coupled harmonic oscillators with different masses \( m_α \) and frequencies \( ω_α \) are considered

\[
\hat{H}_S = \sum_{α=1}^{2} \frac{\hat{p}_α^2}{2m_α} + \frac{1}{2} m_α ω_α^2 \hat{q}_α^2 + c(t) \hat{q}_1 \hat{q}_2, \tag{1}
\]

where \( c(t) \) is an arbitrary time-dependent coupling function between the oscillators. This time dependence is critical for the generation and maintenance of entanglement [4, 21].

To describe the interaction with their surroundings (dissipative and decohering effects) in a rigorous way, the system-bath model [23, 26–29] in the context of the Caldeira–Leggett model is considered here. To avoid extra correlations between the oscillators from sharing a common thermal bath [30, 31], the two oscillators are coupled to independent thermal baths with different power noise [different spectral density \( J(ω) \) and different temperature \( T_B \); see below. The Hamiltonian of the baths \( \hat{H}_B \), including the interaction with the system of interest \( \hat{H}_I \), is then given by

\[
\hat{H}_B + \hat{H}_I = \sum_{α=1}^{2} \sum_{k=1}^{N} \frac{\hat{p}_{α,k}^2}{2m_{α,k}} + \frac{1}{2} m_{α,k} ω_{α,k}^2 \left( \hat{q}_{α,k} - \frac{c_{α,k}}{m_{α,k} ω_{α,k}^2} \hat{q}_α \right)^2, \tag{2}
\]

where the coefficients \( c_{α,k} \) are the coupling constants among the oscillators of the system of interest and each of the modes of their own thermal baths. It can be seen that the interaction is bilinear in the position operators of the systems and the thermal baths. This implies considering only the linear response of the thermal baths to the influence of the system. This consideration is valid only for the case of geometrically macroscopic thermal baths, which leads to a weak interaction among the oscillators in the system and each one of the oscillators comprising the baths [26, 28]. Note that in the interaction Hamiltonian (2), there are two terms that only depend on \( \hat{q}_α \) and on the coupling constants to the thermal baths. Those terms are included to compensate for the renormalisation of the harmonic potentials in the system of interest by the presence of the thermal baths [26, 28, 32]. By
considering these terms, it is ensured that the minimum of the global Hamiltonian with respect to the system-of-interest coordinates is determined only by the potentials in the system [32].

2.1. Analytic exact system dynamics

The dynamics of the coupled oscillators are solved by means of the influence functional approach [23]. Details can be found in appendix below. In solving the dynamics, the initial density operator of the coupled oscillators \( \hat{\rho}_S \) and their thermal baths \( \hat{\rho}_{TB1} \) and \( \hat{\rho}_{TB2} \) are assumed to factorise; i.e.,

\[
\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_{TB1}(0) \otimes \hat{\rho}_{TB2}(0);
\]

that is, it is supposed that at time \( t = 0 \) there are no initial correlations between the subsystems of the overall system. However, this assumption may not always be valid because in many applications, both the degrees of freedom of the system of interest and the environment to which it is attached form part of the same system. Thus, it is possible that the initial correlations are not available to the experimentalist [29]. Although initial conditions may be relevant in the generation of new control strategies in open quantum systems [33–36], they are not considered below for the sake of concreteness.

According to the influence functional approach, at time \( t \) the matrix elements of the density operator \( \hat{\rho}_S(t) \)

\[
\langle q_{s+}, q_{s-} | \hat{\rho}_S(t) | q_{s-}, q_{s+} \rangle = \int_{-\infty}^{\infty} ds \gamma_\alpha(s) \left[ q_{s+} - q_{s-} \right] \}
\]

where \( q_{s} = (q_{s+}, q_{s-}) \) and the propagating function of the reduced density matrix \( \Gamma(q_+, q_-, t; q_+, q_- ; 0) \)

\[
\Gamma(q_+, q_-, t; q_+, q_- ; 0) = \int D^2q_+ D^2q_- \exp \left\{ \frac{i}{\hbar} \left( \hat{S}_S[q_+] - \hat{S}_S[q_-] \right) \right\} \mathcal{F}[q_+, q_-].
\]

The dissipation kernel \( \gamma_\alpha(s) \) and noise kernel \( K_\alpha(s) \) are defined in terms of the spectral density \( J_\alpha(\omega) \)

\[
\gamma_\alpha(s) = \frac{1}{m_\alpha} \int_0^\infty \frac{d\omega_\alpha}{\pi} J_\alpha(\omega_\alpha) \cos(\omega_\alpha s),
\]

\[
K_\alpha(s) = \int_0^\infty \frac{d\omega_\alpha}{\pi} J_\alpha(\omega_\alpha) \coth\left( \frac{\hbar \omega_\alpha}{2k_B T_\alpha} \right) \cos(\omega_\alpha s),
\]

where \( k_B \) is the Boltzmann constant and \( T_\alpha \) the temperature of each of the thermal baths. The spectral density \( J_\alpha(\omega) = \pi \sum_{k=1}^N \frac{\delta(\omega_\alpha - \omega_{\alpha k})}{2m_\alpha} \) comprises all the information of the bath that is needed to account for its influence on the system.

In deriving an exact closed expression for the propagating function, it is necessary to evaluate the four-fold path integral in equation (5). The exact path integration is performed by taking advantage of the linearity of the system and by using standard techniques [26, 32]. Details on the derivation can be found in appendix. The propagating function is conveniently written as

\[
\Gamma(Q^+, Q^-, t; Q^+, Q^-; 0) = \frac{1}{N(t)} \exp \left\{ \frac{i}{\hbar} \sum_{\alpha=1}^N \left[ q_{s+}^{\ast} Q_\alpha(t) - q_{s-}^{\ast} Q_\alpha(0) \right] \right\}
\]

with \( Q_\alpha = \frac{1}{2} (q_{s+} + q_{s-}) \) and \( \text{tr} \hat{\rho}_S = 1 \).
\[
N(t) = \pi^2 \hbar^2 \left\{ |A_{16}(t)A_{38}(t) - A_{18}(t)A_{36}(t)| \right\}.
\]

The matrix elements \( A_{ij} \) are defined in equation (A.49). The new coordinates \( Q_{s} \) and \( q_{a} \) satisfy the following equation of motion
\[
\begin{align*}
Q_{1,2}(s) + \frac{c(s)}{m_{1,2}} Q_{2,3}(s) + \frac{d}{ds} \int_{0}^{s} du \gamma_{1,2}(s - u) Q_{1,2}(u) &= 0, \\
q_{1,2}(s) + \frac{c(s)}{m_{1,2}} q_{2,3}(s) - \frac{d}{ds} \int_{0}^{s} du \gamma_{1,2}(s - u) q_{1,2}(u) &= 0,
\end{align*}
\]

with the boundary conditions for \( Q_{a}(0) = Q_{a}, q_{a}(t) = q_{a} \), where this set of sixteen auxiliary functions \( \{U_{i}, V_{i}, u_{i}, v_{i}\} \) is obtained by numerical integration of the associated set of second-order non-local-in-time differential equations that arises for \( \{U_{i}, V_{i}, u_{i}, v_{i}\} \) after plugging (13) into equation (11) and special care must be exercised in the numerical integration. Note that under well-defined conditions, the spectral density can be experimentally reconstructed via linear absorption spectroscopy [38] and there is no need to assume a specific one.

For the degenerate Ohmic situation in Ref. [4], the solution to equation (11) can be expressed in terms of the solutions of Mathieu’s oscillator (see also [39]). The non-Ohmic character of the spectral density in equation (12) prevents the formulation of the solution of equation (11) in a similar fashion. However, their linear character allows for expressing the formal solution in the form
\[
\begin{align*}
Q_{1}(t, s) &= U_{1}(t, s) Q_{1}^{s} + U_{2}(t, s) Q_{2}^{s}, \\
Q_{2}(t, s) &= V_{1}(t, s) Q_{1}^{s} + V_{2}(t, s) Q_{2}^{s}, \\
q_{1}(t, s) &= u_{1}(t, s) q_{1}^{s} + u_{2}(t, s) q_{2}^{s}, \\
q_{2}(t, s) &= v_{1}(t, s) q_{1}^{s} + v_{2}(t, s) q_{2}^{s},
\end{align*}
\]

where this set of sixteen auxiliary functions \( \{U_{i}, V_{i}, u_{i}, v_{i}\} \) is obtained by numerical integration of the associated set of second-order non-local-in-time differential equations that arises for \( \{U_{i}, V_{i}, u_{i}, v_{i}\} \) after plugging (13) into equation (11). Because of the time-reversed character of the limits in the integral contribution to the equation of motion of the coordinate \( q_{a} \) in equation (11), special care must be exercised in the numerical integration. Note that a direct numerical integration of (11) would not allow for deriving the analytic result for the propagating function in equation (9).

Before proceeding, a note is in order. Strict non-Markovian dynamics are characterised by a flat \((\omega)\) independent power noise \( S(\omega) = \frac{1}{2} J(\omega) \coth \left( \frac{\omega}{2k_{B}T} \right) \). Although the analytic formulae derived here are valid at any temperature regime, interest in this article is in the high temperature regime. Thus, for the temperature regime relevant to this article, \( \coth \left( \frac{\omega}{2k_{B}T} \right) \sim 2k_{B}T \beta^{-1} \omega^{-1} \). Hence, for purely Ohmic dissipation \( J(\omega) \sim \omega \), the power noise becomes \(\omega\)-independent and the dynamics are Markovian. Thus, the case of purely Ohmic dissipation is referred to as Markovian dynamics whereas the case of Ohmic dissipation with a finite cut-off frequency is understood to account for non-Markovian dynamics. Further details on the power noise and its relation to the non-Markovian character of the dynamics can be found in [8, 36].

3. Entanglement quantification and covariance matrix elements

Due to the linearity of the system’s Hamiltonian (1) and the Gaussian character of the propagating function of the reduced density matrix in equation (9), every initial Gaussian state evolves into another Gaussian state. For the present kind of bipartite system of continuous variables in a Gaussian state, entanglement can be easily
quantified in terms of the logarithmic negativity [40]. This quantity gives a characterisation of the amount of entanglement which can be distilled into singlets.

To quantify entanglement in this case of Gaussian continuous variable states, only the covariance matrix $\sigma$ is needed. It reads $\sigma = \frac{1}{2} \left\{ \hat{\xi} \hat{\xi} + \hat{\xi}^\dagger \hat{\xi}^\dagger \right\} - \left\{ \hat{\xi} \right\} \left\{ \hat{\xi}^\dagger \right\}$, where $\hat{\xi} = (\hat{q}_1, \hat{q}_2, \hat{p}_1, \hat{p}_2)$ and $\hat{q}_1, \hat{q}_2, \hat{p}_1, \hat{p}_2$ are the position and momentum operators of the oscillators in the system of interest characterised by the Hamiltonian (1).

3.1. Entanglement quantification

The logarithmic negativity is defined as $E_N = -\frac{1}{2} \sum_{i=1}^4 \log_2 \left[ \min \left(1, 2 \left| |i| \right| \right) \right]$, where $i_1$'s are the symplectic eigenvalues of the covariance matrix. They are the normal eigenvalues of the matrix $-i\Omega \sigma$, with $\Omega = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$

the symplectic matrix and $I_2$ the identity matrix of dimension 2. For separable states, $\rho_S = \sum_j \rho_j^{(i)} \otimes \rho_j^{(j)}$, the logarithmic negativity of the system is zero and each oscillator can be described independently. For continuous variable systems, entanglement has as an upper limit the maximally entangled EPR wave function with $E_N \to \infty$. Hence, the amount of entanglement measured by the logarithmic negativity is unbounded from above.

3.2. Covariance matrix elements

To calculate any mean value of any operator associated with the observables of one of the oscillators in the system of interest, it is necessary to find the reduced density matrix associated with that oscillator. This is obtained by tracing out the reduced density matrix over the coordinates of the other oscillator; i.e., $\rho_{S_{12}} = tr_{S_2} \rho_S$. For instance, the matrix elements of the reduced density matrix associated with the first oscillator are given by

$$\left\langle Q_{\alpha}^* \right| \hat{\rho}_{S_1}(t) \left| q_{\alpha}^* \right\rangle = \int_{-\infty}^{\infty} dQ_{\alpha}^* \left\{ Q_{\alpha}^* \right| \hat{\rho}_{S_1}(t) \left| q_{\alpha}^* = 0 \right\rangle, \quad (14)$$

while for the second oscillator

$$\left\langle Q_{\alpha}^* \right| \hat{\rho}_{S_2}(t) \left| q_{\alpha}^* \right\rangle = \int_{-\infty}^{\infty} dQ_{\alpha}^* \left\{ Q_{\alpha}^* \right| \hat{\rho}_{S_2}(t) \left| q_{\alpha}^* = 0, \ q_{\alpha}^* = 0 \right\rangle.$$ 

To find the first and second moments of each oscillator, it is necessary to perform the following integrals:

$$\left\langle \hat{q}_{\alpha} \right\rangle(t) = \int_{-\infty}^{\infty} dQ_{\alpha}^* \ Q_{\alpha}^* \left\{ Q_{\alpha}^* \right| \hat{\rho}_{S_1}(t) \left| q_{\alpha}^* = 0 \right\rangle,$$

$$\left\langle \hat{p}_{\alpha} \right\rangle(t) = -i\hbar \int_{-\infty}^{\infty} dQ_{\alpha}^* \ \frac{d}{dq_{\alpha}^*} \left\{ Q_{\alpha}^* \right| \hat{\rho}_{S_1}(t) \left| q_{\alpha}^* = 0 \right\rangle,$$

$$\left\langle \hat{q}_{\alpha} \right\rangle(t) = \int_{-\infty}^{\infty} dQ_{\alpha}^* \ Q_{\alpha}^* \left\{ Q_{\alpha}^* \right| \hat{\rho}_{S_2}(t) \left| q_{\alpha}^* = 0 \right\rangle,$$

$$\left\langle \hat{p}_{\alpha} \right\rangle(t) = -\hbar^2 \int_{-\infty}^{\infty} dQ_{\alpha}^* \ \frac{d^2}{dq_{\alpha}^*} \left\{ Q_{\alpha}^* \right| \hat{\rho}_{S_2}(t) \left| q_{\alpha}^* = 0 \right\rangle.$$ 

To find the covariances between the position and/or momentum operators between the oscillators, it is necessary to integrate out

$$\left\langle \hat{q}_{\alpha} \hat{q}_{\beta} \right\rangle(t) = \int_{-\infty}^{\infty} dQ_{\alpha}^* \ dQ_{\beta}^* \ Q_{\alpha}^* Q_{\beta}^* \left\{ Q_{\alpha}^* Q_{\beta}^* \right| \hat{\rho}_{S_1}(t) \left| q_{\alpha}^* = 0, \ q_{\beta}^* = 0 \right\rangle,$$

$$\left\langle \hat{q}_{\alpha} \hat{p}_{\beta} \right\rangle(t) = -i\hbar \int_{-\infty}^{\infty} dQ_{\alpha}^* \ dQ_{\beta}^* \ Q_{\alpha}^* \ \frac{d}{dq_{\beta}^*} \left\{ Q_{\alpha}^* Q_{\beta}^* \right| \hat{\rho}_{S_1}(t) \left| q_{\alpha}^* = 0, \ q_{\beta}^* = 0 \right\rangle,$$

$$\left\langle \hat{q}_{\alpha} \hat{p}_{\beta} \right\rangle(t) = -i\hbar \int_{-\infty}^{\infty} dQ_{\alpha}^* \ dQ_{\beta}^* \ Q_{\alpha}^* \ \frac{d}{dq_{\alpha}^*} \left\{ Q_{\alpha}^* Q_{\beta}^* \right| \hat{\rho}_{S_1}(t) \left| q_{\alpha}^* = 0, \ q_{\beta}^* = 0 \right\rangle,$$

$$\left\langle \hat{p}_{\alpha} \hat{q}_{\beta} \right\rangle(t) = -\hbar^2 \int_{-\infty}^{\infty} dQ_{\alpha}^* \ dQ_{\beta}^* \ \frac{d^2}{dq_{\alpha}^* dq_{\beta}^*} \left\{ Q_{\alpha}^* Q_{\beta}^* \right| \hat{\rho}_{S_1}(t) \left| q_{\alpha}^* = 0, \ q_{\beta}^* = 0 \right\rangle.$$ 

For the case when no initial correlations exist between the oscillators in the system of interest, namely, $\hat{\rho}_S(0) = \hat{\rho}_{S_1}(0) \otimes \hat{\rho}_{S_2}(0)$, the exact analytic expression for the first and second moments as functions of the set of auxiliary functions $\{U_i, V_i, u_i, v_i\}$ can be found at http://gfam.udea.edu.co/lpachon/scripts/qosystems. Note that the expressions for the first and second moments (15) and the mixed moments (16) were calculated for arbitrary driving forces $c(t)$. This allows utilising the expressions (15) and (16) in the context, e.g., of optimal control of sideband cooling of nano-mechanical resonators [41].
3.3. Initial Gaussian states for the simulations

To obtain specific results, both oscillators are assumed to be in a general Gaussian state, and therefore the initial density matrix for each oscillator can be expressed in terms of the coordinates $\alpha_Q$ and $\alpha_q$ as

$$
\hat{\rho}^{(0)} = \frac{\exp\left\{-\frac{1}{2\sigma_Q^{(a)}} (Q_a - \langle Q_a \rangle)^2 - \frac{1}{2\sigma_q^{(a)}} (q_a - \langle q_a \rangle)^2 - \frac{1}{2\hbar} \left( \sigma_{pp}^{(a)} - \frac{\sigma_{qq}^{(a)}}{\sigma_{qq}^{(a)}} \right) q_a \right\}}{\sqrt{2\pi\sigma_Q^{(a)}\sigma_q^{(a)}}}
$$

where $\langle Q_a \rangle$, $\langle q_a \rangle$, $\sigma_{pp}^{(a)}$, $\sigma_{qq}^{(a)}$ and $\sigma_{qp}^{(a)}$ are the first moments of position and momentum, and the variance of the position, momentum and position-momentum, respectively, for the $a$th oscillator.

4. Entanglement dynamics for symmetric thermal baths

To characterise the influence of the non-Markovian dynamics in reaching a different quantum limit, symmetric thermal baths are considered at this point; i.e., $\gamma = \gamma_0/2$, $\Omega = \Omega_0/2$ and $T_1 = T_2 = T$. Moreover, to compare with results in [4], $c(t)$ is chosen here as

$$
c(t) = c_1 \cos \left( \omega_0 t \right),
$$

where $\omega_0$ denotes the frequency of the driving field and $c_1$ its constant amplitude. Although the results presented below are particular to the spectral density in equation (12), they encompass the most basic feature of non-Markovian dynamics; namely, a non-flat power noise [8].

Under the conditions established above, and for degenerate oscillators $\omega_1 = \omega_2 = \omega$ and $m_1 = m_2 = m$, entanglement is quantified below by means of the logarithmic negativity introduced in section 3.1. The undriven non-Markovian dynamics for this degenerate case were previously analysed in [42] and it was found that when $\omega$ falls inside the spectral density (non-Markovian case), entanglement persists for a longer time than in the Markovian case. For the present driven case, the dynamic features provided in the Markovian case [$\Omega = \omega_0$ in equation (12)] and the non-Markovian case [$\Omega = \omega$ in equation (12)] are presented in figures 1 and 2 for a variety of temperatures $T$ and coupling constants $\gamma$, respectively. The main feature depicted by figures 1 and 2 is that if a given amount of entanglement is reached under Markovian dynamics, the same amount can be found under non-Markovian dynamics, but for temperatures or coupling constants as high as double the value of the parameters under Markovian dynamics. In section 4.4, a quantitative discussion in this regard is provided.

4.1. Entanglement dynamics as a function of the bath temperature $T$

Figure 2 shows the time evolution of the logarithmic negativity for a variety of coupling temperatures. Dark thick curves correspond to the non-Markovian case [$\Omega = \omega$ in equation (12)] and light thin curves correspond to the...
Markovian case \( \Omega = 20\omega \) in equation (12). From there, it is clear that for the particular functional form of \( J(\omega) \) in equation (12), non-Markovian dynamics are able to support the same amount of steady state entanglement at twice the temperature of the corresponding Markovian case.

An additional dynamic feature present in figures 1 and 2 is that entanglement is generated at shorter times under non-Markovian dynamics than in the Markovian case. Since the rate of the incoherent processes decreases under non-Markovian dynamics (see section 4.4), it is then natural to expect that the driving force needs to perform less work to squeeze the normal modes of the oscillators when the dynamics are non-Markovian. Hence, if the power of the driving field is kept constant for the Markovian and the non-Markovian situations, entanglement will be generated faster in the non-Markovian case.

### 4.2. Entanglement dynamics as a function of the bath coupling constant \( \gamma \)

Figure 2 depicts the time dynamics of entanglement for a variety of coupling rates \( \gamma \) at fixed temperature. In analogy to the case discussed in figure 1, non-Markovian dynamics are able to support the same amount of steady state entanglement at twice the coupling rate \( \gamma \) of the corresponding Markovian case. Because of the non-Markovian character of the dynamics, the simulations over several periods of the driving force are very expensive in computational terms; it is therefore convenient to set the amplitude strength \( c_1 \) to a rather large value \( \omega = 0.2 \) so that the generation of entanglement occurs after a few periods of driving. However, the effects discussed above are clearly present for smaller values of the amplitude strength \( c_1 \) (see section 4.4).

### 4.3. Entanglement dynamics as a function of the initial state

One of the most attractive features of the generation of entanglement by driving forces in the presence of non-unitary dynamics is that the system reaches the same amount of stationary entanglement independently of its initial state [4]. Despite the non-Markovian character of the dynamics and the associated dependence on the history of the system evolution, this feature remains present here. Figure 3 shows the time evolution of the logarithmic negativity for a variety of initial states given by equation (17). There, it is clear that all of them reach the same amount of stationary entanglement.

### 4.4. Non-Markovian quantum limit

If the characteristic frequency of a given system is denoted by \( \omega \), the survival of quantum features such as entanglement can be predicted for parameters that satisfy the condition \( S(\omega) < \omega \mu \), with

\[
S(\omega) = \frac{1}{2\omega^2} J(\omega) \coth \left( \frac{\omega}{2\hbar \beta} \right) \text{ being the power noise and } \mu \text{ the pumping rate.}
\]

Specifically, at high temperatures of \( \hbar \beta \ll 1 \), entanglement survives in the steady state if

\[
k_B T J(\omega)/m < \hbar \omega \mu.
\]

Because of the various time/energy scales involved in this non-trivial out-of-equilibrium situation, the impact of non-Markovian dynamics can be understood in manifold ways. Specifically, it can be effectively ascribed to each time/energy scale independently. In doing so, the case of the spectral density in equation (12) is discussed below and three non-Markovian scaled parameters, \( T_{MB}, T_{MB} \text{ and } \mu_{MB} \), are introduced.
After plugging the spectral density (12) in the non-Markovian quantum limit given by equation (19), a new effective temperature can be defined

\[ T_{nM} = T \left( 1 - \frac{1}{1 + \Omega^2/\omega^2} \right), \] (20)

such that the quantum limit can be cast in the form found in [4]; namely, \( k_B T_{nM} / \hbar \omega \leq \mu / \gamma \), but with \( T_{nM} \) instead of \( T \). It is also possible to define an effective coupling constant

\[ \gamma_{nM} = \gamma \left( 1 - \frac{1}{1 + \Omega^2/\omega^2} \right). \] (21)

For \( \Omega \sim \omega \), \( T_{nM} \sim \frac{1}{2} T \) and \( \gamma_{nM} \sim \frac{1}{2} \gamma \). This scaling of the temperature or the coupling constant explains the results depicted in figures 1 and 2, respectively. Alternatively, the non-Markovian scaling factor \( 1 - \frac{1}{1 + \Omega^2/\omega^2} \) can be assigned to a third energy scale. We define

\[ \mu_{nM} = \mu \left( 1 - \frac{1}{1 + \Omega^2/\omega^2} \right)^{-1}, \] (22)

so that \( \mu_{nM} \geq \mu \) provided by the fact that \( 1 - \frac{1}{1 + \Omega^2/\omega^2} \leq 1 \). To be more concrete, note that the particular situation analysed in [4] corresponds to the case \( f/(\omega)/m = \gamma \omega \) and \( \mu \) corresponds to the imaginary part of the associated Mathieu coefficient. Assuming that to leading order in the coupling \( c_1 \) [see equation (18)] the imaginary part of the Mathieu coefficient can still be expressed by \( \mu \sim c_1/4\omega \), non-Markovian dynamics can be seen as effectively enhancing the coupling between oscillators. This implies that under non-Markovian dynamics, the amplitude of the driving force needed for the entanglement to survive is clearly smaller than in the Markovian case.

Although the non-Markovianly scaled parameters \( T_{nM}, \gamma_{nM} \) and \( \mu_{nM} \) are particular to the spectral density in equation (12), based on the extensively studied features of non-Markovian dynamics [8, 10–12, 14–16, 19, 21, 22], there is no apparent physical reason not to expect the same scaling scenario in general.

5. Entanglement dynamics for asymmetric thermal baths

The presence of driving forces above already placed the system into a non-trivial out-of-equilibrium situation. Another non-trivial out-of-equilibrium situation that this system may encounter is the case of environments at different temperatures \( T_1 \neq T_2 \) or resonators with different couplings constants \( \eta \neq \mu \). Motivated by the possible role that heat currents may play in the generation entanglement, these two situations are considered below.
5.1. Thermal baths at different temperatures: $T_1 \neq T_2$

From figure 1 it is clear that the lower the temperature, the larger the amount of steady state entanglement that is reached. However, because heat transfer between thermal baths at different temperatures is assisted here by the interaction between oscillators, the following should be considered: (i) the possible role of heat transfer in preserving/destroying quantum correlations between oscillators, and (ii) the classical/quantum nature of possible correlations established by heat fluxes at high/low temperatures among the oscillators. For concreteness of the present work, these concerns are analysed in a separated contribution and here our interest is restricted to the amount of entanglement that can be reached for different temperature ratios. Specifically, figure 4 depicts the time dynamics of entanglement for a variety of temperature ratios $T_2 / T_1$ at fixed $T_1$. The main features in this figure are (i) the strong dependence of the time at which entanglement is generated on the temperature ratio and (ii) the dependence of entanglement on the absolute value of the temperature of the baths and not only on the temperature difference. This, indeed, motivates a comprehensive analysis of entanglement dynamics and heat fluxes.

5.2. Thermal baths with different decay rates: $\gamma_1 \neq \gamma_2$

The effective temperature at which each oscillator thermalises is a function of the power noise of the bath, and therefore of the coupling constant [8]. Thus, another interesting situation from a thermodynamic point of view is the case when the coupling constants are different. Figure 5 depicts the time dynamics of entanglement for a variety of coupling constant ratios $\gamma_2 / \gamma_1$ at fixed $\gamma_1$. Up to the weak dependence of the time at which entanglement is generated on the coupling ratios, the behaviour is essentially the same as in figure 4. Below, the case of non-degenerate oscillators is analysed and the effect of non-Markovian dynamics in the resonance condition $\omega_3 = \omega_1 + \omega_2$ is discussed.

6. Entanglement dynamics for asymmetric oscillators

Since the case of degenerate oscillators lacks experimental relevance [43] and the generation of squeezing in mechanical setups was already achieved for non-degenerate oscillators [6], the dynamics of entanglement are considered next for different masses and frequencies. This last situation complements the analysis of the influence of asymmetries started in the previous section with the cases $T_1 \neq T_2$ and $\gamma_1 \neq \gamma_2$.

The undriven non-Markovian dynamics for the degenerate and non-degenerate cases were previously analysed in [43]. In particular, [43] addresses the effect of the resonance condition for degenerate oscillators and its relationship with the possibility of preserving entanglement at asymptotic times.

6.1. Entanglement dynamics for oscillators of different natural frequency $\omega_1 \neq \omega_2$

For small values of $\epsilon_1$ and in the rotating wave approximation, the maximum rate of generation of squeezing is obtained for $\omega_3 = \omega_1 + \omega_2$. For the degenerate case, it reads $\omega_3 = 2\omega_1$. Below, for this optimal condition, the dynamics of entanglement are analysed for a variety of frequency ratios $\omega_2 / \omega_1$ for fixed $\omega_1$. To isolate the effect of
non-Markovian dynamics, parameters are chosen so that no entanglement is found in the Markovian
degenerate case with $\omega = 2d_1$. Figure 6 not only shows that non-Markovian dynamics support the creation
and survival of steady entanglement for the degenerate case with 'resonant driving', but also over a broad range
of frequency detuning ($\sim 7\%$) and with 'non-resonant' driving. This feature adds to the known robustness of the
squeezing-entanglement generation against small detuning from the resonance frequency.
In the experimental work on the generation of two-mode squeezing reported in [6], $\omega \sim 0.93921$ with
$\omega = m_1$ and $\gamma \sim 10^{-3}$, so that for the non-Markovian scenario and pumping rate in figure 6,
entanglement can be reached at $\sim 37$ mK. For quality factors two orders of magnitude larger, entangled modes
could be found at $3.7$ K. Because entanglement is not found for this set of parameters in the Markovian case, note
that reaching these high temperatures is supported by the non-Markovian character of the dynamics.

6.2. Entanglement dynamics for oscillators of different mass $m_1 \neq m_2$
Figure 7 depicts the logarithmic negativity for a variety of mass ratios $m_2/m_1$ for fixed $m_1$. As expected, the
smaller the ratio $m_2/m_1$ is, the more effective the driving field is in generating entanglement out of the
modulation of the coupling strength. Note that the smaller the mass ratio is, not only the larger the value of $E_N$ is,
but also the shorter the time at which entanglement is generated. This is consistent with the intuitive idea of the effectiveness of the driving field in the presence of lighter masses.

7. Discussion

For highly symmetric cases, there is evidence that non-Markovian dynamics may allow for the survival of entanglement at temperatures higher than the corresponding Markovian case provided by the interaction with a common bath [31, 43]. For purely dephasing baths, non-Markovian dynamics may allow for larger values of entanglement [44]. Based on analytic exact results for the non-Markovian and out-of-equilibrium dynamics of two non-degenerate parametrically coupled harmonic oscillators, it is shown here that in the presence of time-dependent external fields, non-Markovian dynamics support the generation of out-of-equilibrium steady state entanglement at higher temperatures, larger coupling-to-the-environment constants and lower pumping rates than in the Markovian case. This delicate interplay between driving fields and non-Markovian dynamics sets a new quantum limit which incorporates the main time and energy scales of the physical systems at hand and that is capable of beating the resonant condition of the generation of normal mode squeezing.

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Appendix. Influence functional and propagating function of the system

The starting point in the influence functional theory is considering the density operator of the global system at time $t$, that in terms of the initial density operator $\hat{\rho}(0)$ is given by

$$\hat{\rho}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}_t\right)\hat{\rho}(0)\exp\left(\frac{i}{\hbar}\hat{H}_t\right).$$

(A.1)

where $\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_I$ is the Hamiltonian of the global system with $\hat{H}_S$, $\hat{H}_B$ and $\hat{H}_SB$ given by equations (1) and (2). Physically, the expression (A.1) implies that the global system is considered as a closed one, and therefore it is possible to evolve it in time by means of a unitary operator. In the coordinate representation, the matrix elements of the density operator $\hat{\rho}(t)$ read
\[
\{ q_{1+}^s, q_{2+}^s, q_{1-}^s, q_{2-}^s \} \rightarrow \begin{bmatrix} q_{1-}^s \quad q_{2-}^s \quad q_{1-}^s \quad q_{2-}^s \end{bmatrix} \]  
\]
\[
= \int_{-\infty}^{\infty} dq_{1+}^s dq_{2+}^s dq_{1-}^s dq_{2-}^s dq_{1+}^0 dq_{2+}^0 dq_{1-}^0 dq_{2-}^0 \exp \left( \frac{i}{\hbar} \left[ H_{t_{\infty}} \right] \right)
\]

where \( q_{ax} \) stands for the coordinates of the oscillators in the system of interest, and \( q_{ax} = (q_{a,1}, q_{a,2}, ..., q_{a,N_a}) \) the coordinates of the thermal baths. The matrix elements of the unitary time evolution operators are known as the propagating kernel \([23, 26, 29, 32]\) and are given by

\[
\left\{ q_{1+}^s, q_{2+}^s, q_{1-}^s, q_{2-}^s \right\} \rightarrow \begin{bmatrix} q_{1-}^s \quad q_{2-}^s \quad q_{1-}^s \quad q_{2-}^s \end{bmatrix} \]  
\]

This object evolves the overall system 'forward in time'; that is why the ' + ' subscript is used. An analogous expression stands for the matrix elements that evolve the overall system 'backward in time', and this is depicted by the ' - ' subscript in the coordinates. The path integrals in the propagating kernels must be evaluated over the paths \( q_{ax} \) and \( q_{ax} \) that satisfy the following boundary conditions

\[
q_{ax} = \begin{cases} q_{ax}^s, s = 0, \\ q_{ax}^s, s = t. \end{cases} \quad q_{ax} = \begin{cases} q_{ax}^s, s = 0, \\ q_{ax}^s, s = t. \end{cases}
\]

In equation (A.3), \( S \) stands for the action for the global system, defined as usual:

\[
S = \int_0^t ds L_s \left( \dot{q}_{1}(s), \dot{q}_{2}(s), q_{1}(s), q_{2}(s) \right) + \int_0^t ds L_B \left( \dot{q}_{1}(s), \dot{q}_{2}(s), q_{1}(s), q_{2}(s) \right)
\]

where \( L_s, L_B \) and \( L_I \) stand for the Lagrangian of the system of interest, the thermal baths and the interaction between these subsystems, respectively. The Lagrangian has the following form

\[
L_s = \sum_{a=1}^{2} \left( \frac{1}{2} m_a \dot{q}_a^2 - \frac{1}{2} m_a \omega_a^2 q_a^2 \right).
\]

By replacing the expression (A.3) into (A.2), the matrix elements of the density operator of the global system read

\[
\left\{ q_{1+}^s, q_{2+}^s, q_{1-}^s, q_{2-}^s \right\} \rightarrow \begin{bmatrix} q_{1-}^s \quad q_{2-}^s \quad q_{1-}^s \quad q_{2-}^s \end{bmatrix} \]  
\]

This expression describes the dynamics of the global system. However, not all of this information is of interest. The only information that is relevant for the present case is that of the system of interest under the
influence on the environment. Therefore, the relevant object here is the reduced density matrix, which is obtained by tracing out over the degrees of freedom of the thermal baths in equation (A.9); i.e.,

\[
\rho_S(q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow}, t) = \int_{-\infty}^{\infty} dq_{1\uparrow} \ dq_{2\uparrow} \ dq_{1\downarrow} \ dq_{2\downarrow} \ \left\{ q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow} \right\} \rho(t) \\
\times \left| q_{1\downarrow}, q_{2\downarrow}, q_{1\uparrow}, q_{2\uparrow} \right\rangle \\
= \int_{-\infty}^{\infty} dq_{1\uparrow} \ dq_{2\uparrow} \ dq_{1\downarrow} \ dq_{2\downarrow} \ dq_{1\uparrow}' \ dq_{2\uparrow}' \ dq_{1\downarrow}' \ dq_{2\downarrow}' \\
\times K \left( q_{1\uparrow}, q_{2\uparrow}, q_{1\uparrow}', q_{2\uparrow}', t; q_{1\downarrow}, q_{2\downarrow}, q_{1\downarrow}', q_{2\downarrow}', 0 \right) \\
\times K^* \left( q_{1\downarrow}, q_{2\downarrow}, q_{1\downarrow}', q_{2\downarrow}', t; q_{1\uparrow}, q_{2\uparrow}, q_{1\uparrow}', q_{2\uparrow}', 0 \right) \\
\times \left\{ q_{1\uparrow}', q_{2\uparrow}', q_{1\downarrow}', q_{2\downarrow}' \right\} \rho(0) \left| q_{1\uparrow}', q_{2\uparrow}', q_{1\downarrow}', q_{2\downarrow}' \right\rangle. \\
\text{(A.10)}
\]

At this point, it is assumed that the initial density operator of the global system factorises, \( \hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_A(0) \otimes \hat{\rho}_B(0) \). In the position representation, \( \hat{\rho}(0) \) reads

\[
\left\langle q_{1\uparrow}, q_{2\uparrow} \right| \hat{\rho}(0) \left| q_{1\downarrow}, q_{2\downarrow} \right\rangle \\
= \left\langle q_{1\uparrow}, q_{2\uparrow} \right| \hat{\rho}_S(0) \left| q_{1\downarrow}, q_{2\downarrow} \right\rangle \left\langle q_{1\uparrow} \right| \hat{\rho}_A(0) \left| q_{1\downarrow} \right\rangle \left\langle q_{2\uparrow} \right| \hat{\rho}_B(0) \left| q_{2\downarrow} \right\rangle \\
= \rho_S(q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow}, 0) \rho_A(q_{1\uparrow}, q_{1\downarrow}, 0) \rho_B(q_{2\uparrow}, q_{2\downarrow}, 0). \\
\text{(A.11)}
\]

By replacing this expression in equation (A.10), the reduced density matrix is found to read

\[
\rho_S(q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow}, t) \\
= \int_{-\infty}^{\infty} dq_{1\uparrow}' \ dq_{2\uparrow}' \ dq_{1\downarrow}' \ dq_{2\downarrow}' \ J \left( q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow}, t; q_{1\uparrow}', q_{2\uparrow}', q_{1\downarrow}', q_{2\downarrow}', 0 \right) \\
\times \rho_S(q_{1\uparrow}', q_{2\uparrow}', q_{1\downarrow}', q_{2\downarrow}') \left| q_{1\uparrow}', q_{2\uparrow}', q_{1\downarrow}', q_{2\downarrow}' \right\rangle, \\
\text{(A.12)}
\]

where

\[
J \left( q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow}, q_{1\uparrow}', q_{2\uparrow}', q_{1\downarrow}', q_{2\downarrow}', 0 \right) \\
= \int Dq_{1\uparrow} Dq_{2\uparrow} Dq_{1\downarrow} Dq_{2\downarrow} \exp \left\{ \frac{i}{\hbar} \left( S_A[q_{1\uparrow}, q_{2\uparrow}] - S_A[q_{1\downarrow}, q_{2\downarrow}] \right) \right\} \\
\times \mathcal{F} \left| q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow} \right\rangle. \\
\text{(A.13)}
\]

denotes the propagating function of the reduced density matrix and the object \( \mathcal{F} \left| q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow} \right\rangle \) is the influence functional \([23, 37]\) given by

\[
\mathcal{F} \left| q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow} \right\rangle = \int_{-\infty}^{\infty} dq_{1\uparrow}' \ dq_{2\uparrow}' \ dq_{1\downarrow}' \ dq_{2\downarrow}' \ \rho_B(q_{1\uparrow}', q_{1\downarrow}', 0) \rho_B(q_{2\uparrow}', q_{2\downarrow}', 0) \\
\times \left| q_{1\uparrow}', q_{2\uparrow}', q_{1\downarrow}', q_{2\downarrow}' \right\rangle \\
\times \int Dq_{1\uparrow} Dq_{2\uparrow} Dq_{1\downarrow} Dq_{2\downarrow} \exp \left\{ \frac{i}{\hbar} \left( S_B[q_{1\uparrow}, q_{2\uparrow}] - S_B[q_{1\downarrow}, q_{2\downarrow}] \\
+ S_1[q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow}] - S_1[q_{1\downarrow}, q_{2\downarrow}, q_{1\uparrow}, q_{2\uparrow}] \right) \right\}. \\
\text{(A.14)}
\]

A.1. Derivation of the influence functional

To derive the influence functional, consider the following property of the influence functional:

If a system is interacting simultaneously with two uncoupled and independent environments \( A \) and \( B \), with no initial correlations between them, then \([23]\)

\[
F = F_A \cdot F_B. \\
\text{(A.15)}
\]

Using the above property, it is possible to express the influence functional in (A.14) as the product of two influence functionals, one for each oscillator in the system of interest; namely,

\[
\mathcal{F} \left| q_{1\uparrow}, q_{2\uparrow}, q_{1\downarrow}, q_{2\downarrow} \right\rangle = \mathcal{F} \left| q_{1\uparrow}, q_{1\downarrow} \right\rangle \mathcal{F} \left| q_{2\uparrow}, q_{2\downarrow} \right\rangle, \\
\text{(A.16)}
\]
where

\[ F[q_{a+}, q_{a-}] = \int_{-\infty}^{\infty} dq_{a+}^* dq_{a-}^* \rho_{k_{a}}(q_{a+}^*, q_{a-}^*, 0) \]

\[ \times \int Dq_{a+} Dq_{a-} \exp \left\{ \frac{i}{\hbar} \left( S_{h_{a}}[q_{a+}] + S_{l_{a}}[q_{a+}, q_{a+}] - S_{h_{a}}[q_{a-}] - S_{l_{a}}[q_{a-}, q_{a-}] \right) \right\} , \quad \alpha = 1, 2. \]  

(A.17)

Having in mind that the oscillators encompassing each thermal bath are independent of them, the property of the influence functional stated above can be used once more so that each influence functional in equation (A.16) is given by

\[ F[q_{a+}, q_{a-}] = \prod_{k=1}^{N} F_k[q_{a+}, q_{a-}] , \]  

(A.18)

where each \( F_k[q_{a+}, q_{a-}] \) describes the influence of each oscillator in one thermal bath on the oscillator in the system of interest. Each of these influence functionals reads

\[ F_k[q_{a+}, q_{a-}] = \int_{-\infty}^{\infty} dq_{a,k+} dq_{a,k-} \int Dq_{a,k+} Dq_{a,k-} \exp \left\{ \frac{i}{\hbar} \left( S_{h_{a,k}}[q_{a,k+}] + S_{l_{a,k}}[q_{a+}, q_{a,k+}] \right) \right\} \]

\[ \times \int Dq_{a,k+} Dq_{a,k-} \exp \left\{ -\frac{i}{\hbar} \left( S_{h_{a,k}}[q_{a,k-}] + S_{l_{a,k}}[q_{a-, q_{a,k-}}] \right) \right\} . \]  

(A.19)

\( \langle q_{a,k+}^* | \rho_{h_{a,k}}^{(k)}(0) | q_{a,k-}^* \rangle \) denotes the initial density matrix for the \( k \)th oscillator in the thermal bath. For the case of a thermal bath at thermal equilibrium at a temperature \( T_{a,k} \), it reads (see, e.g., [32])

\[ \langle q_{a,k+}^* | \rho_{h_{a,k}}^{(k)}(0) | q_{a,k-}^* \rangle = 2 \sinh \left( \frac{\hbar \omega_{a,k}}{2} \right) \]  

\[ \left\{ \frac{m_{a,k} \hbar \omega_{a,k}}{2 \pi \hbar \sinh \left( \frac{\hbar \omega_{a,k}}{2} \right)} \right\} ^{1/2} \exp \left\{ -\frac{m_{a,k} \hbar \omega_{a,k}}{2 \hbar \sinh \left( \frac{\hbar \omega_{a,k}}{2} \right)} \left[ \frac{q_{a,k+}^2 + q_{a,k-}^2}{\hbar \omega_{a,k}} + \frac{q_{a,k+}^2 + q_{a,k-}^2}{\hbar \omega_{a,k}} \right] \right\} . \]  

(A.20)

The next step is to evaluate the path integrals in equation (A.19). In doing so, note that the global system under study is linear, and therefore it is only necessary to evaluate the actions \( S_{h_{a,k}}^{(k)} \) and \( S_{l_{a,k}}^{(k)} \) along the classical paths \( q_{a,k}(s) \) of each oscillator in the thermal baths. These classical paths are the solution to the equation of motion

\[ \ddot{q}_{a,k}(s) + \omega_{a,k}^2 q_{a,k}(s) = q_{a}(s) \frac{\epsilon_{a,k}}{m_{a,k}} , \]  

(A.21)

that can be obtained from the Lagrangians in (A.7) and (A.8). The trick for solving this differential equation lies in treating the system coordinate \( q_{a} \) as if it were a given function of time, so that a differential equation for a driven harmonic oscillator is obtained. Therefore, the path integrals in (A.19) correspond to the kernel for a driven harmonic oscillator (see, e.g., [32]), except for the term

\[ q_{a}^2 \frac{\epsilon_{a,k}}{2m_{a,k} \omega_{a,k}^2} . \]

This contribution can be taken out of the path integral because it does not contain any term that depends on the classical paths of the bath oscillators. Having in mind the boundary conditions in equation (A.4), taking \( q_{a-}(t) = q_{a+}^* \) for the tracing operation and using the propagation kernel for a driven harmonic oscillator, the path integrals in (A.19) are readily given by
\[ \int Dq_{a,k} \exp \left\{ \pm \frac{i}{\hbar} \left( S_{0}^{(k)} \left[ q_{a,k} \right] + S_{1}^{(k)} \left[ q_{a,k}, q_{a,k} \right] \right) \right\} \]

\[
= \sqrt{\frac{m_{a,k} \omega_{a,k}}{2 \pi (\pm i) \hbar \sin \left( \omega_{a,k} t \right)}} \exp \left\{ \pm \frac{i}{\hbar} \frac{m_{a,k} \omega_{a,k}}{2 \sin \left( \omega_{a,k} t \right)} \left[ \left( q_{a,k}^{'2} + q_{a,k}^{-2} \right) \cos \left( \omega_{a,k} t \right) \right. \right. \\
- 2q_{a,k}^{-2} q_{a,k}^{+2} \left. \left. \right] + \frac{c_{a,k} q_{a,k}^{+2}}{\sin \left( \omega_{a,k} t \right)} \int_{0}^{t} ds \sin \left( \omega_{a,k} (t - s) \right) q_{a,k}^{+2} \left( s \right) \right. \right. \\
+ \frac{c_{a,k} q_{a,k}^{-2}}{\sin \left( \omega_{a,k} t \right)} \int_{0}^{t} ds \sin \left( \omega_{a,k} (t - s) \right) q_{a,k}^{-2} \left. \left( s \right) \right) \} \cdot \left( \omega_{a,k} t \right) \}
\]

Using this expression and equation (A.20), the influence functional in equation (A.19) takes the form

\[ \mathcal{F} \left[ q_{a,+}, q_{a,-} \right] \]

\[
= 2 \sinh \left( \frac{\hbar \beta \omega_{a,k}}{2} \right) \sqrt{\frac{m_{a,k} \omega_{a,k}}{2 \pi (\pm i) \hbar \sin \left( \omega_{a,k} t \right)}} \sqrt{\frac{m_{a,k} \omega_{a,k}}{2 \sin \left( \omega_{a,k} t \right)}} \sqrt{\frac{m_{a,k} \omega_{a,k}}{2 \pi (\pm i) \hbar \sin \left( \omega_{a,k} t \right)}} \]

\[
\times \int_{-\infty}^{\infty} dq_{a,k}^{+} dq_{a,k}^{-} dq_{a,k}^{+2} + dq_{a,k}^{-2} \exp \left\{ - \frac{m_{a,k} \omega_{a,k}}{2 \sin \left( \omega_{a,k} t \right)} \left[ \left( q_{a,k}^{'2} + q_{a,k}^{-2} \right) \cos \left( \omega_{a,k} t \right) \right. \right. \\
- 2q_{a,k}^{-2} q_{a,k}^{+2} \left. \left. \right] + \frac{c_{a,k} q_{a,k}^{+2}}{\sin \left( \omega_{a,k} t \right)} \int_{0}^{t} ds \sin \left( \omega_{a,k} (t - s) \right) q_{a,k}^{+2} \left( s \right) \right. \right. \\
+ \frac{c_{a,k} q_{a,k}^{-2}}{\sin \left( \omega_{a,k} t \right)} \int_{0}^{t} ds \sin \left( \omega_{a,k} (t - s) \right) q_{a,k}^{-2} \left. \left( s \right) \right) \} \cdot \left( \omega_{a,k} t \right) \}
\]

After performing the integrations over \( q_{a,k}^{+2}, q_{a,k}^{+2}, q_{a,k}^{-2} \), the influence functional reads

\[ \mathcal{F} \left[ q_{a,+}, q_{a,-} \right] \]

\[
= \exp \left\{ - \frac{i}{\hbar} \left( q_{a,+}^{+} + q_{a,-}^{-} \right) \int_{0}^{t} ds \frac{c_{a,k}^{2}}{2m_{a,k} \omega_{a,k}^{2}} \cos \left( \omega_{a,k} s \right) \left[ q_{a,+}^{+} \left( s \right) - q_{a,-}^{-} \left( s \right) \right] \right. \right. \\
+ \int_{0}^{t} ds \int_{0}^{t} du \frac{c_{a,k}^{2}}{2m_{a,k} \omega_{a,k}^{2}} \cos \left( \omega_{a,k} (s - u) \right) \left[ q_{a,+}^{+} \left( u \right) + q_{a,-}^{-} \left( u \right) \right] \left[ q_{a,+}^{+} \left( s \right) - q_{a,-}^{-} \left( s \right) \right] \right. \right. \\
\times \exp \left\{ - \frac{i}{\hbar} \frac{c_{a,k}^{2}}{2m_{a,k} \omega_{a,k}} \int_{0}^{t} ds \int_{0}^{t} du \coth \left( \frac{\hbar \beta \omega_{a,k}}{2} \right) \cos \left( \omega_{a,k} \left( u - s \right) \right) \right. \right. \\
\times \left[ q_{a,+}^{+} \left( s \right) - q_{a,-}^{-} \left( s \right) \right] \left[ q_{a,+}^{+} \left( u \right) - q_{a,-}^{-} \left( u \right) \right] \right. \right. \]

\[ \} \cdot \left( \omega_{a,k} t \right) \}
\]

The above expression describes the influence of the kth oscillator in the \( \alpha \)th thermal bath on the \( \alpha \)th oscillator in the system of interest. By replacing this expression in (A.18), the complete expression for the influence functional is obtained for one thermal bath. Specifically,
By replacing the above expression in (A.16), the total influence functional for the system of interest reads

$$F \left[ q_{a^+}, q_{a^-} \right]$$

$$= \exp \left\{ -\frac{i}{\hbar} \left( q_{a^+} + q_{a^-} \right) \int_0^t ds \sum_{k=1}^N \frac{\varepsilon_{a,k}^2}{2m_{a,k} \omega_{a,k}^2} \cos \left( \omega_{a,k} s - q_{a^-}(s) \right) \right\}$$

$$+ \int_0^t ds \int_0^t du \sum_{k=1}^N \frac{\varepsilon_{a,k}^2}{2m_{a,k} \omega_{a,k}^2} \cos \left( \omega_{a,k} (s - u) \right) \left[ \dot{q}_{a^+}(u) + \dot{q}_{a^-}(u) \right] \left[ q_{a^+}(s) - q_{a^-}(s) \right] \left[ q_{a^+}(u) - q_{a^-}(u) \right] \} \right\} \times \exp \left\{ -\frac{i}{\hbar} \int_0^t ds \int_0^t du \sum_{k=1}^N \frac{\varepsilon_{a,k}^2}{2m_{a,k} \omega_{a,k}^2} \coth \left( \frac{\hbar \omega_{a,k}}{2} \right) \right\} \times \cos \left( \omega_{a,k} (u - s) \right) \left[ q_{a^+}(s) - q_{a^-}(s) \right] \left[ q_{a^+}(u) - q_{a^-}(u) \right] \right\}. \quad (A.25)$$

As it is customary in the literature, the limit to the continuum in the spectrum of the bath is performed by means of the spectral density $J_\alpha (\omega_\alpha)$, which is defined as

$$J_\alpha (\omega_\alpha) = \pi \sum_{k=1}^N \frac{\varepsilon_{a,k}^2}{2m_{a,k} \omega_{a,k}} \delta (\omega_\alpha - \omega_{a,k}). \quad (A.26)$$

Therefore, for a discrete set of modes in the thermal baths, the above spectral density is made up of Dirac deltas. However, for the set of oscillators in the thermal baths to behave as a formal thermal bath, it is assumed that the frequencies $\omega_{a,k}$ are so dense as to form a continuous spectrum. Thus, in the continuous limit, the spectral density can be represented as a continuous and smooth function on the frequency $\omega_\alpha$ [26]. This allows for defining the dissipation $\gamma_\alpha (s)$ and noise $K_\alpha (s)$ kernels in terms of the spectral density as

$$\gamma_\alpha (s) = \frac{2}{m_\alpha} \int_0^\infty \frac{d\omega_\alpha}{\pi} J_\alpha (\omega_\alpha) \cos (\omega_\alpha s), \quad (A.27)$$

$$K_\alpha (s) = \int_0^\infty \frac{d\omega_\alpha}{\pi} J_\alpha (\omega_\alpha) \coth \left( \frac{\hbar \omega_\alpha}{2} \right) \cos (\omega_\alpha s). \quad (A.28)$$

By replacing (A.26) in the last expressions,

$$\gamma_\alpha (s) = \frac{1}{m_\alpha} \sum_{k=1}^N \frac{\varepsilon_{a,k}^2}{2m_{a,k} \omega_{a,k}^2} \cos (\omega_{a,k} s), \quad (A.29)$$

$$K_\alpha (s) = \sum_{k=1}^N \frac{\varepsilon_{a,k}^2}{2m_{a,k} \omega_{a,k}^2} \coth \left( \frac{\hbar \omega_{a,k}}{2} \right) \cos (\omega_{a,k} s), \quad (A.30)$$

which can be identified with the summations in the argument of the exponentials in equation (A.25). Having in mind these considerations, and using the expressions (A.26), (A.27) and (A.28), equation (A.25) can be expressed in terms of the dissipation and noise kernels as

$$F \left[ q_{a^+}, q_{a^-} \right]$$

$$= \exp \left\{ -\frac{i m_\alpha}{\hbar} \left( q_{a^+} + q_{a^-} \right) \right\} \int_0^t ds \gamma_\alpha (s) \left[ q_{a^+}(s) - q_{a^-}(s) \right]$$

$$+ \int_0^t ds \int_0^t du \gamma_\alpha (s - u) \left[ \dot{q}_{a^+}(u) + \dot{q}_{a^-}(u) \right] \left[ q_{a^+}(s) - q_{a^-}(s) \right] \left[ q_{a^+}(u) - q_{a^-}(u) \right] \right\} \times \exp \left\{ -\frac{i m_\alpha}{\hbar} \int_0^t ds \int_0^t du \gamma_\alpha (u - s) \left[ q_{a^+}(s) - q_{a^-}(s) \right] \left[ q_{a^+}(u) - q_{a^-}(u) \right] \right\}. \quad (A.31)$$

By replacing the above expression in (A.16), the total influence functional for the system of interest reads

$$F \left[ q_{a^+}, q_{a^-}, q_{a^+}, q_{a^-} \right]$$

$$= \prod_{a=1}^2 \exp \left\{ -\frac{i m_\alpha}{\hbar} \left( q_{a^+} + q_{a^-} \right) \right\} \int_0^t ds \gamma_\alpha (s) \left[ q_{a^+}(s) - q_{a^-}(s) \right]$$

$$+ \int_0^t ds \int_0^t du \gamma_\alpha (s - u) \left[ \dot{q}_{a^+}(u) + \dot{q}_{a^-}(u) \right] \left[ q_{a^+}(s) - q_{a^-}(s) \right] \left[ q_{a^+}(u) - q_{a^-}(u) \right] \right\} \times \exp \left\{ -\frac{i m_\alpha}{\hbar} \int_0^t ds \int_0^t du \gamma_\alpha (u - s) \left[ q_{a^+}(s) - q_{a^-}(s) \right] \left[ q_{a^+}(u) - q_{a^-}(u) \right] \right\}. \quad (A.32)$$
A.2. Explicit expression for the propagating function

To provide an explicit expression for the propagating function associated with the system of interest, consider the expression (A.13)

\[
\eta = \int \exp \left( \frac{i}{\hbar} \left( S_\eta \{ q_1^+, q_2^+ \} - S_\eta \{ q_1^-, q_2^- \} \right) \right) F \left[ q_1^+, q_2^+, q_1^-, q_2^- \right],
\]

with \( F \left[ q_1^+, q_2^+, q_1^-, q_2^- \right] \) given by the expression (A.32). Since the path integrals in (A.33) are quadratic, they can be performed exactly\(^{37}\). This leaves for the propagating function \(^{4, 29, 30, 39}\) the following expression

\[
\eta = \int \exp \left( \frac{i}{\hbar} \left( S_\eta \{ q_1^+, q_2^+ \} - S_\eta \{ q_1^-, q_2^- \} \right) \right) F \left[ q_1^+, q_2^+, q_1^-, q_2^- \right],
\]

\[(A.34)\]

To simplify the last expression, consider a new set of coordinates\(^{4, 29, 39}\)

\[Q_a = \frac{1}{2} \left( q_{a+} + q_{a-} \right), \quad q_a = q_{a+} - q_{a-},\]

\[(A.36)\]

where the Jacobian of the transformation is equal to one. Hence, equation (A.35) takes the form

\[
J \left( Q', q'; t; Q', q', 0 \right) = \frac{1}{N(t)} \exp \left\{ \frac{i}{\hbar} \int_0^t ds \left\{ \sum_{a=1}^2 \left[ \frac{1}{2} m_0 \dot{q}_{a+}^2(s) - \frac{1}{2} m_0 \omega_0^2 \dot{q}_{a+}^2(s) - c(s) q_{a+}(s) q_{a+}(s) \right] \right\} \right\}
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \int_0^t ds \left\{ \sum_{a=1}^2 \left[ \frac{1}{2} m_0 \dot{q}_{a-}^2(s) - \frac{1}{2} m_0 \omega_0^2 \dot{q}_{a-}^2(s) - c(s) q_{a-}(s) q_{a-}(s) \right] \right\} \right\}
\]

\[
\times F \left[ q_1^+, q_2^+, q_1^-, q_2^- \right].
\]

\[(A.35)\]

To simplify the last expression, consider a new set of coordinates\(^{4, 29, 39}\)

\[
J \left( Q', q'; t; Q', q', 0 \right) = \frac{1}{N(t)} \exp \left\{ \frac{i}{\hbar} \int_0^t ds \left\{ \sum_{a=1}^2 \left[ m_0 \dot{Q}_a(s) \dot{q}_a(s) - m_0 \omega_0^2 Q_a(s) q_a(s) - m_0 Q'_a r_a(s) q_a(s) \right] \right\}
\]

\[
- \int_0^t du \gamma_a(s - u) Q_a(u) q_a(s) \right\} \right\}
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \int_0^t ds \left\{ \sum_{a=1}^2 \left[ \frac{1}{2} m_0 \dot{Q}_a^2(s) - m_0 \omega_0^2 \dot{Q}_a^2(s) - c(s) Q_a(s) Q_a(s) \right] \right\} \right\}
\]

\[
\times F \left[ q_1^+, q_2^+, q_1^-, q_2^- \right].
\]

\[(A.37)\]

where \( Q = (Q_1, Q_2) \) and \( q = (q_1, q_2) \). The next step is to determine and solve the equations of motion for the coordinates \( Q_a \) and \( q_a \). For this, it is necessary to use the Euler–Lagrange equations for the Lagrangian \( L \) that appears as the integrand in the imaginary phase of the exponential in equation (A.37). For the coordinates \( Q_a \),

the equations of motion have the following form

\[
\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{Q}_a} \right) - \frac{\partial L}{\partial Q_a} = 0,
\]

\[(A.38)\]

whereas for the coordinates \( q_a \), the Euler–Lagrange equations are

\[
\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{q}_a} \right) - \frac{\partial L}{\partial q_a} = 0.
\]

\[(A.39)\]

By using equation (A.38) in (A.37), the equation of motion for the coordinate \( Q_a \) can be written as

\[
\dot{Q}_a(s) + \omega_0^2 Q_a(s) + \frac{c(s)}{m_1} Q_a^2(s) + \frac{d}{ds} \int_0^s du \gamma_a(s - u) Q_a(u) = 0,
\]
\[ Q_2(s) + \omega_s^2 Q_2(s) + \frac{c(s)}{m_2} Q_2(s) + \frac{d}{ds} \int_0^s du \gamma_2(s - u) Q_2(u) = 0. \]  
(A.40)

In order to find the equations of motion for the coordinates \( q_a \), consider the following equality

\[ Q_a' \int_0^s ds \gamma_a(s) q_a(s) + \int_0^s ds \int_0^s du \gamma_a(u - s) Q_a(u) q_a(s) \]

\[ = q_a' \int_0^s ds \gamma_a(t - s) Q_a(s) - \int_0^s ds \int_0^s du \gamma_a(u - s) q_a(u) Q_a(s). \]  
(A.41)

Thus, the propagating function in equation (A.37) can be written as

\[
J(Q^*, q^*, t; Q', q', 0) = \frac{1}{N(t)} \exp \left\{ \frac{i}{\hbar} \int_0^t ds \left\{ \sum_{a=1}^2 \left[ m_a \dot{Q}_a(s) q_a(s) - m_a \omega_s^2 Q_a(s) q_a(s) \right.ight.ight.

\[ - q_a' \gamma_a(t - s) Q_a(s) + \int_0^s du \gamma_a(u - s) q_a(u) Q_a(s) \left.ight] \left.ight.ight.

\[ - c(s) \left[ \dot{Q}_1(s) q_2(s) + Q_2(s) q_1(s) \right] \right\} \right.

\[ \times \exp \left\{ \frac{-1}{\hbar} \int_0^t ds \int_0^s du \sum_{a=1}^2 K_a(u - s) q_a(s) q_a(u) \right\}. \]  
(A.42)

By virtue of the expression (A.39), the equations of motion for the coordinates \( q_a \) are given by

\[
\ddot{q}_1(s) + \omega_s^2 q_1(s) + \frac{c(s)}{m_1} q_2(s) - \frac{d}{ds} \int_0^s du \gamma_1(u - s) q_1(u) = 0,
\]

\[
\ddot{q}_2(s) + \omega_s^2 q_2(s) + \frac{c(s)}{m_2} q_1(s) - \frac{d}{ds} \int_0^s du \gamma_2(u - s) q_2(u) = 0. \]  
(A.43)

The coordinates \( Q_a \) and \( q_a \) satisfy the boundary conditions

\[
Q_a(s) = \begin{cases} Q'_a, & s = 0, \\ Q''_a, & s = t, \end{cases} \quad q_a(s) = \begin{cases} q'_a, & s = 0, \\ q''_a, & s = t. \end{cases} \]  
(A.44)

Since the equations of motion (A.41) and (A.43) are linear, the solution to them can be written as in equation (13). Once the solutions for the equations of motion are calculated, only one step is left to find the propagating function. The partial integration of the first term in the imaginary phase of the exponential is

\[
J(Q^*, q^*, t; Q', q', 0) = \frac{1}{N(t)} \exp \left\{ \frac{i}{\hbar} \sum_{a=1}^2 \dot{Q}_a(t) q_a(0) - q'_a Q_a(0) \right\} \]  
(A.45)
By replacing equation (13) into the last expression, the propagating function reads

\[
J(Q^q, q^q, t; Q', q', 0) = \frac{1}{N(t)} \exp \left\{ \frac{i}{\hbar} \sum_{\alpha=1}^{n} \left[ \frac{1}{2} \left( \dot{Q}^\alpha(t) Q^\alpha + \dot{q}^\alpha(t) q^\alpha + \frac{1}{2} \left( \frac{\partial^2 \phi_1}{\partial x_1^\alpha \partial x_1^\alpha} + \frac{\partial^2 \phi_2}{\partial x_2^\alpha \partial x_2^\alpha} \right) \right) \right] \right\},
\]

(A.46)

or alternatively,

\[
J(Q^q, q^q, t; Q', q', 0) = \frac{1}{N(t)} \exp \left\{ \frac{i}{\hbar} \left( x_1 \cdot A x_1 - \frac{1}{\hbar} x_2 \cdot B x_2 \right) \right\},
\]

(A.48)

where \( x_1 = (Q^q, q^q, q^q, q^q) \), \( x_2 = (q^q, q^q) \) and the matrix elements \( A_{ij} \) and \( B_{ij} \) of the matrices \( A \) and \( B \) are given by

\[
A_{ij} = \frac{1}{2} \frac{\partial \phi_1}{\partial x_1^i \partial x_1^j}, \quad B_{ij} = \frac{1}{2} \frac{\partial \phi_2}{\partial x_2^i \partial x_2^j},
\]

(A.49)

respectively. Here \( \phi_1 \) and \( \phi_2 \) are the imaginary and real components, respectively, of the phase in the propagating function. In terms of the matrix elements \( A_{ij} \) and \( B_{ij} \), the propagating function can be written as

\[
J(Q^q, q^q, t; Q', q', 0) = \frac{1}{N(t)} \exp \left\{ \frac{i}{\hbar} \left( q^q \left[ A_{12}(t) Q^q + A_{23}(t) Q^q + A_{34}(t) Q^q + A_{45}(t) Q^q \right] \right.ight.

\[
+ q^q \left[ A_{16}(t) Q^q + A_{27}(t) Q^q + A_{38}(t) Q^q + A_{49}(t) Q^q \right] \right.

\[
\left. - q^q \left[ A_{17}(t) Q^q + A_{28}(t) Q^q + A_{39}(t) Q^q + A_{49}(t) Q^q \right] \right) \right\}

\times \exp \left\{ \frac{1}{\hbar} \left[ q^q B_{12}(t) + q^q B_{23}(t) + q^q B_{34}(t) + q^q B_{45}(t) \right.ight.

\[
\left. + 2 q^q B_{12}(t) + 2 q^q B_{23}(t) + 2 q^q B_{34}(t) \right) \right\},
\]

(A.50)

References

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