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Discrete Lorentz covariance for quantum walks and quantum cellular automata

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Abstract
We formalize a notion of discrete Lorentz transforms for quantum walks (QW) and quantum cellular automata (QCA), in \((1 + 1)\)-dimensional discrete space-time. The theory admits a diagrammatic representation in terms of a few local, circuit equivalence rules. Within this framework, we show the first-order-only covariance of the Dirac QW. We then introduce the clock QW and the clock QCA, and prove that they are exactly discrete Lorentz covariant. The theory also allows for non-homogeneous Lorentz transforms, between non-inertial frames.

Keywords: discrete Lorentz transformation, local Lorentz covariance, circuit transformation

1. Introduction

Symmetries in quantum walks (QW). For the purpose of quantum simulation (on a quantum device) as envisioned by Feynman [1], or for the purpose of exploring the power and limits of discrete models of physics, a great deal of effort has gone into discretizing quantum physical phenomena. Most of these lead to QW models of the phenomena. QWs are dynamics having the following features.

- The underlying spacetime is a discrete grid.
- The evolution is unitary.
• It is causal, i.e. information propagates strictly at a bounded speed.
• It is homogeneous, i.e. translation-invariant and time-independent.

By definition, therefore, they have several of the fundamental symmetries of physics, built-in. But can they also have Lorentz covariance? The purpose of this paper is to address this question.

Summary of results. Lorentz covariance states that the laws of physics remain the same in all inertial frames. Lorentz transforms relate spacetimes as seen by different inertial frames. This paper formalizes a notion of discrete Lorentz transforms, acting upon wavefunctions over discrete spacetime. It formalizes the notion of discrete Lorentz covariance of a QW, by demanding that a solution of the QW be Lorentz transformed into another solution, of the same QW.

Before the formalism is introduced, the paper investigates a concrete example: the Dirac QW [2–5]. The Dirac QW is a natural candidate for being Lorentz covariant, because its continuum limit is the covariant, free-particle Dirac equation [5–7]. This example helps us build our definitions. However, the Dirac QW turns out to be first-order covariant only. In order to obtain exact Lorentz covariance, we introduce a new model, the clock QW, which arises as the quantum version of a covariant classical random walk [8]. However, the clock QW requires an observer-dependent dimension for the internal state space. In order to overcome this problem, the formalism is extended to multiple-walkers QWs, i.e. quantum cellular automata (QCA).

Indeed, the clock QCA provides a first finite-dimensional model of an exactly covariant QCA. We use numerous figures to help our intuition. In fact, the theory admits a simple diagrammatic representation, in terms of a few local, circuit equivalence rules. The theory also allows for non-homogeneous Lorentz transforms, a specific class of general coordinate transformations, and yet expressive enough to switch between non-inertial frames.

Related works. Researchers have tried to reconcile discreteness and Lorentz covariance in several ways.

In the causal set approach, only the causal relations between the spacetime events is given. Without a background spacetime Lorentz covariance is vacuous. If, however, the events are generated from a Poissonian distribution over a flat spacetime, then covariance is recovered in a statistical sense [9].

Researchers working on lattice Boltzmann methods for relativistic hydrodynamics also take a statistical approach: the underlying model breaks Lorentz covariance, but the statistical distributions generated are covariant [10].

Loop quantum gravity offers a deep justification for the statistical approach. By interpreting spacetime intervals as the outcome of measurements of quantum mechanical operators, one can obtain covariance for the mean values, while keeping to a discrete spectrum [11, 12].

The idea of interpreting space and time as operators with a Lorentz invariant discrete spectrum goes back to Snyder [13]. This line of research goes under the name of doubly special relativity (DSR). Relations between DSR and QWs are discussed in [14]. In the DSR approach, a deformation of the translational sector of the Poincaré algebra is required.

Instead of deforming the translation operator algebra, one could look at dropping translational invariance of the QW evolution. Along these lines, models have been constructed for QWs in external fields, including specific cases of gravitational fields [15, 16].
Another non-statistical, early approach is to restrict the class of allowed Lorentz transforms, to a subgroup of the Lorentz group whose matrices are over the integers numbers [17]. Unluckily, there are no non-trivial integral Lorentz transforms in (1+1)-dimensions. Moreover, interaction rules that are covariant under this subgroup are difficult to find [18, 19].

Approach. The approach of the present paper is non-statistical: we look for exact Lorentz covariance. Spacetime remains undeformed, always assumed to be a regular lattice, and the QW remains homogeneous. While keeping to 1 + 1 dimensions and integral transforms, we allow for a global rescaling, so that we can represent all Lorentz transforms with rational velocity. The basic idea is to map each point of the lattice to a lightlike rectangular spacetime patch, as illustrated in figures 1 and 10.

Plan of the paper. The remainder of this paper is organized as follows. In section 2 we set notations by recalling the Dirac QW and the proof of covariance of the Dirac equation. In section 3 we discuss the first-order-only covariance of the Dirac QW. In section 4 we formalize discrete Lorentz transforms, covariance, and discuss non-homogeneous Lorentz transforms. In sections 5 and 6 apply this theory to the clock QW and the clock QCA respectively. We finish with a discussion in section 8.

2. Preliminaries

2.1. Finite difference Dirac equation and the Dirac QW

The Dirac equation. The (1+1)-dimensional free particle Dirac equation is (with Planck’s constant and the velocity of light set to one)

\[ \partial_0 \psi = -i m \sigma_3 \psi - \partial_x \sigma_3 \psi, \]  

(1)
where \( m \) is the mass of the particle, \( \psi = \psi(t, x) \) is a spacetime wavefunction from \( \mathbb{R}^{1+1} \) to \( \mathbb{C}^2 \) and \( \sigma_j \) (\( j = 0, \ldots, 3 \)) are the Pauli spin matrices, with \( \sigma_0 \) the identity. Equation (1) corresponds to the Weyl (or spinor) representation [20].

**Lightlike coordinates.** In order to study covariance, it is always a good idea to switch to lightlike coordinates \( r = (t + x)/2 \) and \( l = (t - x)/2 \), in which a Lorentz transform is just a rescaling of the coordinates. Redefine the wavefunction via \( \psi(r + l, r - l) \rightarrow \psi(r, l) \), then equation (1) becomes

\[
\left( \begin{array}{cc}
\partial_r & 0 \\
0 & \partial_l
\end{array} \right) \psi = -im\sigma_1 \psi.
\] (2)

**Finite-difference Dirac equation.** In this paper \( e^{\varepsilon \partial_\mu} \) will be used as a notation for the translation by \( \varepsilon \) along the \( \mu \)-axis (with \( \mu = 0, 1 \)), i.e. \( (e^{\varepsilon \partial_\mu} \psi)(x_\mu) = \psi(x_\mu + \varepsilon) \).

Using the first order expansion of the the exponential, the spacetime wavefunction \( \psi \) is a solution of the Dirac equation if and only if, as \( \varepsilon \rightarrow 0 \),

\[
\left( \begin{array}{cc}
e^{\varepsilon \partial_\mu} & 0 \\
0 & e^{\varepsilon \partial_\mu}
\end{array} \right) \psi = \left( \begin{array}{cc}
\text{Id} + \left( \frac{\varepsilon \partial_r}{\varepsilon \partial_l} \right) & 0 \\
0 & \frac{\varepsilon \partial_l}{\varepsilon \partial_r}
\end{array} \right) \psi + O(\varepsilon^2)
\] = (\text{Id} - i\varepsilon \sigma_1) \psi + O(\varepsilon^2). \] (3)

Equivalently, if we denote \( \psi = (\psi_+, \psi_-)^T \), then \( \psi \) is a solution of the Dirac equation if and only if, to first order in \( \varepsilon \) and as \( \varepsilon \rightarrow 0 \),

\[
\psi_+(r + \varepsilon, l) = \psi_+(r, l) - i\varepsilon \psi_- (r, l)
\] and \( \psi_- (r, l + \varepsilon) = \psi_- (r, l) - i\varepsilon \psi_+ (r, l) \). (4)

If we now suppose that \( \varepsilon \) is fixed, and consider that \( \psi \) is a spacetime wavefunction from \((\varepsilon\mathbb{Z})^2\) to \( \mathbb{C}^2 \), then equation (4) defines a finite-difference scheme for the Dirac equation (FD Dirac). As a dynamical system, this FD Dirac is illustrated in figure 2 with

\[
C = \begin{pmatrix}
1 & -i\varepsilon m \\
-i\varepsilon m & 1
\end{pmatrix}.
\] (5)

**The Dirac QW.** We could have gone a little further with equation (3). Indeed, by recognizing in the right-hand side (rhs) of the equation the first order expansion of an exponential, we get

\[
\left( \begin{array}{cc}
e^{\varepsilon \partial_\mu} & 0 \\
0 & e^{\varepsilon \partial_\mu}
\end{array} \right) \psi = e^{-im\sigma_1} \psi + O(\varepsilon^2).
\] (6)

In fact, \( \psi \) is a solution of the Dirac equation if and only if, as \( \varepsilon \rightarrow 0 \), equation (6) is satisfied. See [5, 6] for a rigorous, quantified proof of convergence.

If we again say that \( \varepsilon \) is fixed, and so that \( \psi \) is a discrete spacetime wavefunction, then equation (6) defines a QW for the Dirac equation (Dirac QW) [2–5, 7]. Indeed, as a dynamical system, this Dirac QW is again illustrated in figures 2 but this time taking

\[
C = e^{-im\sigma_1} = \begin{pmatrix}
cos(\varepsilon m) & -\sin(\varepsilon m) \\
-\sin(\varepsilon m) & \cos(\varepsilon m)
\end{pmatrix}.
\] (7)

which is exactly unitary, i.e. to all orders in \( \varepsilon. \)
In the original \((t, x)\) coordinates, both the FD Dirac and the Dirac QW evolutions are given by \(\psi(x) = T \psi(x, 0)\), where \(T = e^{-i\sigma_3}\) is the shift operator and \(C\) is the matrix appearing in equation (5) or (7) respectively (see [5] for details). In the case of the Dirac QW, \(W = TC\) is referred to as the walk operator: it is shift-invariant and unitary. \(C\) is referred to as the coin operator, acting over the 'coin space', which is \(\mathcal{H} \cong \mathbb{C}^2\) for the Dirac QW. Equation (6) reads as follows: the top and bottom components of the coin space get mixed up by the coin operator, and then the top component moves at lightspeed towards the right, whereas the bottom component goes in the opposite direction.

**Remark 1.** Let \(\alpha, \beta\) be arbitrary positive integers. Notice that knowing the value of the scalars \(\psi_+(r \cdot l), \ldots, \psi_+(r + (\alpha - 1)\epsilon, l)\) carried by the right-incoming wires, together with the scalars \(\psi_-(r \cdot l), \ldots, \psi_-(r + (\beta - 1)\epsilon, l)\) carried by the left-incoming wires, fully determines \(\psi(r + i\epsilon, l + j\epsilon)\) for \(0 \leq i \leq (\alpha - 1)\) and \(0 \leq j \leq (\beta - 1)\), as made apparent in figure 3. We denote by \(\bar{C}(i, j)\) the operator which, given the vectors...
Let us review the covariance of the Dirac equation in a simple manner, that will be useful for us later. Consider a change of coordinates $\alpha' = r, \beta' = ll$. This transformation is proportional by a factor of $\alpha\beta$ to the Lorentz transform shown in figure 3, i.e.

$$C \psi(r, l) = \left( C_+ \oplus C_- \right) \psi(r, l).$$

We write $C_m$ for the operator, instead of $C$, when we want to make explicit its dependency upon the parameter $m$.

### 2.2. Scaled Lorentz transforms and covariance

Let us review the covariance of the Dirac equation in a simple manner, that will be useful for us later. Consider a change of coordinates $r' = ar, l' = \beta l$. This transformation is proportional by a factor of $\sqrt{\alpha\beta}$ to the Lorentz transform.
whose velocity parameter is \( \alpha = \frac{\alpha - \beta}{\alpha + \beta} \). Let us define \( \tilde{\psi}(r', l') = \tilde{u}(\alpha r, \beta l) = \psi(r, l) \). A translation by \( \alpha \) along \( r \) (respectively \( \beta \) along \( l \)) becomes a translation by \( \alpha \epsilon \) along \( r' \) (respectively \( \beta \epsilon \) along \( l' \)). Hence the Dirac equation now demands that as \( \epsilon \to 0 \),

\[
\left( e^{\alpha \epsilon \partial_r} 0 \right) \tilde{\psi} = \left( 1 - i \epsilon m \frac{-i \epsilon m}{1} \right) \tilde{\psi} + O(\epsilon^2).
\]

Equivalently, to first order in \( \epsilon \) and as \( \epsilon \to 0 \),

\[
\tilde{\psi}_+(r' + \alpha \epsilon, l') = \tilde{\psi}_+(r', l') - i \epsilon \alpha \tilde{\psi}_-(r', l'),
\]

\[
\tilde{\psi}_-(r' + \beta \epsilon, l' + \beta \epsilon) = \tilde{\psi}_-(r', l') - i \epsilon \beta \tilde{\psi}_+(r', l').
\]

Unfortunately, whenever \( \alpha \neq \beta \), this is not in the form of a Dirac equation. In other words the coordinate change alone does not take the Dirac equation into the Dirac equation.

**Remark 2.** In section 5 we will study the clock QW, inspired by

\[
\left( e^{\alpha \epsilon \partial_r} 0 \right) \psi = e^{-i \epsilon m \alpha \psi}.
\]

Meanwhile, notice that in the first order, the top and bottom \( \epsilon \) can be taken to be different, leading to

\[
\left( e^{\epsilon \partial_r} 0 \right) \tilde{\psi} = \left( 1 - i \epsilon m / \sqrt{\alpha} \frac{-i \epsilon m / \alpha}{1} \right) \tilde{\psi} + O(\epsilon^2),
\]

\[
\left( e^{\epsilon \partial_r} 0 \right) \left( \tilde{\psi}_+/\sqrt{\beta} \right) = \left( \frac{1}{-i \epsilon m / \sqrt{\alpha \beta}} \right) \left( \tilde{\psi}_+/\sqrt{\alpha} \right) + O(\epsilon^2).
\]

Let us define

\[
S = \left( \frac{1}{\sqrt{\beta}} 0 \right) \frac{1}{\sqrt{\alpha}}
\]

and \( \psi' = S \tilde{\psi} \).

Call this \( \psi' \) the Lorentz transformed of \( \psi \), instead of \( \tilde{\psi} \). Now we have

\[
\left( e^{\epsilon \partial_r} 0 \right) \psi' = \left( 1 - i \epsilon m / \sqrt{\alpha \beta} \frac{-i \epsilon m / \sqrt{\alpha \beta}}{1} \right) \psi'.
\]
i.e. the Dirac equation just for a different mass $m' = m/\sqrt{\alpha \beta}$. This different mass is due to the fact that the transformation to primed coordinates that we considered was a scaled Lorentz transform. In the special case where $\alpha \beta = 1$, the above is just the proof of Lorentz covariance of the Dirac equation.

3. A discrete Lorentz transform for the Dirac QW

3.1. Normalization problem and its solution

Normalization problem in the discrete case. Take $\psi (r, l)$ a solution of the Dirac QW such that the initial condition is normalized and localized at single point e.g. $\psi (0, 0) = (1, 0)^T$ and $\psi (r, l) = (0, 0)^T$ for $t = r + l = 0$. Then, after applying the Lorentz transform described in section 2.2, the initial condition is $\psi' (0, 0) = (1/\sqrt{\beta}, 0)^T$ and $\psi' (r, l) = (0, 0)^T$ for $t = r + l = 0$ which is not normalized for any non-trivial Lorentz transform, see figure 4(a). Hence, we see that the Lorentz transform described in section 2.2, i.e. that used for the covariance of the continuous Dirac equation, is problematic in the discrete case: the transformed observer sees a wavefunction which is not normalized. This seems a paradoxical situation since in the limit when $\epsilon \to 0$, the discrete case tends towards the continuous case, which does not have such a normalization issue. In order to fix this problem, let us look more closely at how normalization is preserved in the continuous case.

Normalization in the continuous case. Now take $\psi (r, l)$ a solution of the massless Dirac equation such that the initial condition is the normalized, right-moving rectangular function, i.e. $\psi (r, l) = (1/\sqrt{2}, 0)^T$, for $0 \leq l < 1$ and $\psi (r, l) = (0, 0)^T$ elsewhere. The Lorentz transform of $\psi$ is

$$
\psi' (r', l') = S \psi (r'/\alpha, l' / \beta) = \begin{cases} 
\left( \frac{1}{\sqrt{2} \beta} \right) & 0 \leq l' < \beta, \\
0 & \text{elsewhere}, 
\end{cases}
$$

which is normalized. We see that the $S$ matrix is no longer a problem for normalization, but rather it is needed to compensate for the larger spread of the wavefunction, see figure 4(b). This suggests that the normalization problem for the localized initial condition in the discrete case could be fixed, by allowing the discrete Lorentz transform to spread out the initial condition.

From the continuous to the discrete. Intuitively, we could think of defining the discrete Lorentz transform as the missing arrow ‘Discrete $\Lambda$’ that would make the following diagram commute

$$
\begin{array}{c}
\text{Dirac} \\
\downarrow \text{Discretize} \\
\text{QW}
\end{array} \xrightarrow{\Lambda} \begin{array}{c}
\text{Dirac}' \\
\downarrow \text{Discretize} \\
\text{QW}'
\end{array}
$$

In other words,

Discrete $\Lambda \circ \text{Discretize} := \text{Discretize} \circ \Lambda$
and hence, 

\[ \Lambda \] := Discretize \( \bigcirc \) \( \Lambda \bigcirc \) Interpolate.

For instance, if the localized walker was to be interpolated as a rectangular function instead of a Dirac peak, that rectangular function will be spread out by the continuous \( \Lambda \), and may discretize as a more spread out walker. The discrete Lorentz transform that we propose next does just that. However, it will be phrased directly in the discrete setting. Later in section 4 we provide a more general and diagrammatic definition of discrete Lorentz transform and discrete Lorentz covariance.

### 3.2. A discrete Lorentz transform

In the continuous case we had \( \psi' (r', l') = S \psi (r, l) \). Hence \( \psi' (r', l') = S \psi (r' / \alpha, l' / \beta) \). In the discrete case, however \( \psi' \) is a spacetime wavefunction from \( (e \mathbb{Z})^2 \) to \( \mathbb{C}^2 \), as in figure 2(b). Hence, demanding, for instance, that \( \psi' (e, 0) = S \psi (e / \alpha, 0) \) becomes meaningless, because

---

**Figure 4.** The normalization problem and solution in the \( m = 0, \alpha = 1, \beta = 2 \) case. (a) If \( \psi_\epsilon (0, 0) \) gets interpreted as a right-traveling Dirac peak, then its transformed version is \( \psi_\epsilon' (0, 0) = \psi_\epsilon (0, 0) / \sqrt{2} \), which is not normalized. (b) If \( \psi_\epsilon (0, 0) \) gets interpreted as a right-moving rectangular function, then its transformed version spreads out as \( \psi_\epsilon' (0, 0) = \psi_\epsilon' (0, 1) = \psi_\epsilon (0, 0) / \sqrt{2} \), which is normalized.
(ψ/α, 0) is undefined. The normalization issues and the related discussion of section 3.1 suggests setting ψ′(ε, 0) to Sψ(0, 0), and not to 0. More generally, we will take

\[ \forall r' \in \varepsilon \alpha \mathbb{Z}, \quad \psi_+(r', l') = \frac{\psi_+(r'/\alpha, \lfloor l'/\beta \rfloor)}{\sqrt{\beta}} \]

and

\[ \forall l' \in \varepsilon \beta \mathbb{Z}, \quad \psi_-(r', l') = \frac{\psi_-(\lfloor r'/\alpha \rfloor, l'/\beta)}{\sqrt{\alpha}}, \]

where \([. \rfloor\) takes the closest multiple of ε that is less or equal to the number. Notice that this implies that for all \(r' \in \varepsilon \alpha \mathbb{Z}\) and \(l' \in \varepsilon \beta \mathbb{Z}\), we have \(\psi'(r', l') = S\psi(r'/\alpha, l'/\beta)\), as in the continuous case. However, what if we have neither \(r' \in \varepsilon \alpha \mathbb{Z}\) nor \(l' \in \varepsilon \beta \mathbb{Z}\)? As was illustrated in figure 3, this spacetime region is now fully determined, i.e. we set

\[ \forall r', l' \in \varepsilon \mathbb{Z}, \quad \psi'(r', l') = \overline{C}_{m}(i, j) \overline{\psi}(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor) \]

(9)

with \(m' = m/\sqrt{\alpha \beta}\), i.e. \(r' = r' - \lfloor r'/\alpha \rfloor \varepsilon\), \(j \varepsilon = l' - \lfloor l'/\beta \rfloor \varepsilon\), \(\overline{C}_{m}(i, j)\) as defined in remark 1, and \(\overline{\psi}(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor)\) again as defined in remark 1, namely

\[
\overline{\psi}_+\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor\right) = \begin{pmatrix}
\psi'_+\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor\right) \\
\vdots \\
\psi'_+\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor + (\beta - 1)\varepsilon\right)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{\psi_+\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor\right)}{\sqrt{\beta}} \\
\vdots
\end{pmatrix}
\]

and similarly

\[
\overline{\psi}_-\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor\right) = \begin{pmatrix}
\psi_-\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor\right) \\
\vdots
\end{pmatrix}
\]

and finally

\[
\overline{\psi}\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor\right) = \begin{pmatrix}
\overline{\psi}_+\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor\right) \\
\overline{\psi}_-\left(\lfloor r'/\alpha \rfloor, \lfloor l'/\beta \rfloor\right)
\end{pmatrix}.
\]

This finishes to define a discrete Lorentz transform \(L_{\alpha, \beta}\), which is illustrated in figure 5.

An equivalent, more concise way of specifying this discrete Lorentz transform \(L_{\alpha, \beta}\) is as follows. First, consider the isometry \(E_{\beta}\) (respectively \(E_{\alpha}\)) which codes \(\psi_+(r, l)\) (respectively \(\psi_-(r, l)\)) into the more spread out \(\tilde{\psi}_+(r, l) = E_{\beta} \psi_+(r, l)\) (respectively \(\tilde{\psi}_-(r, l) = E_{\alpha} \psi_-(r, l) = \overline{\psi}_-(\alpha r, \beta l)\)), and let \(\tilde{\psi}(r, l) = \tilde{\psi}_-(r, l) \oplus \tilde{\psi}_+(r, l)\), and \(m' = m/\sqrt{\alpha \beta}\). Second, construct \(\psi' = L_{\alpha, \beta} \psi\) by replacing every spacetime point \(\psi(r, l)\) with the lightlike rectangular spacetime patch \(\overline{C}_{m}(i, j) \tilde{\psi}(r, l)\) for \(i = 0 \ldots (\alpha - 1), j = 0 \ldots (\beta - 1)\).
Does this discrete Lorentz transform fix the normalization problem of section 3.1? Let us evaluate this question.

3.3. From continuous to discrete current and norm

3.3.1. Continuous current and norm. In order to evaluate the norm of a spacetime wavefunction $\psi$ in the continuous setting, we need the following definition. We say that a surface $\sigma$ is a Cauchy surface if it intersects every causal curve exactly once (a causal curve being a curve whose tangent vector is always timelike or lightlike). The relativistic current $j^\mu = (j^0, j^1, j^2, j^3)$ is equal to $j^\mu = (\psi^+ \psi^+)^{1/2}$, and in lightlike coordinates becomes

Figure 5. A discrete Lorentz transform, with parameters $\alpha = 3$, $\beta = 2$.

Does this discrete Lorentz transform fix the normalization problem of section 3.1? Let us evaluate this question.
\( j^s = (|\psi_+|^2, |\psi_-|^2), s = \pm \). The norm of \( \psi \) along a Cauchy surface \( \sigma \) is defined by integrating the current \( j^s \) along \( \sigma \)

\[
\| \psi \|^2_\sigma = \int_\sigma j^s n_s d\sigma,
\]

where \( n_s \) is the unit normal vector to \( \sigma \) in \( r, l \) coordinates.

If \( \psi \) is a solution of the Dirac equation, then this definition does not actually depend on the surface \( \sigma \) (for a proof see for instance [21], chapter 4), and so in this case we can write \( \|\psi\|^2_\sigma = \|\psi\|^2 \).

This definition of norm is Lorentz invariant, indeed

\[
\| \psi \|^2_\sigma = \int_\sigma j^s n_s d\sigma \\
= \int_\sigma \left( \frac{|\psi_+|^2}{\beta} \beta dl + \frac{|\psi_-|^2}{\alpha} \alpha dr \right) \\
= \int_{\sigma'} \left( |\psi' -|^2 d{l'} + |\psi' -|\beta^2 d{r'} \right) \\
= \int_{\sigma'} j'^s n'_s d\sigma' \\
= \| \psi' \|^2_{\psi'}.
\]

### 3.3.2. Discrete Cauchy surfaces

We now provide discrete counterparts to the above notions, beginning with discrete Cauchy surfaces. Let us consider a function \( \sigma: \mathbb{Z} \to \{R, L\} \), and an origin \((n_0, l_0)\). Together, they describe a piecewise linear curve made up of segments of the following form (in red)

\[ R \quad L \]

i.e. this curve intersects the spacetime lattice in two ways, labeled \( R \) and \( L \) (right, left). The centering on the origin is done as in figure 6(a). We say that such a curve is a discrete Cauchy surface if it does not contain infinite sequences of contiguous \( R \) or \( L \). One can easily see that such a surface must intersect every lightlike curve exactly once. For concreteness, notice that the discrete equivalent to the continuous constant-time \( t = 0 \) Cauchy surface, is described by

\[
\sigma(n) = \begin{cases} 
L & \text{for even } n, \\
R & \text{for odd } n 
\end{cases}
\]

with origin \((0, 0)\).

### 3.3.3. Discrete current and discrete norm

Similarly, let us define the discrete current carried by a wavefunction \( \psi \). At each wire connecting two points of the discrete lattice, the current is given by
In analogy with the continuous case, we can evaluate the norm of \( \psi \) along a surface \( \sigma \) as follows

\[
\sum_{\sigma(\tau) \in \mathcal{J}} \psi = \mathcal{S}(i), \quad (12)
\]

where \( \mathcal{S}(i) \) is the current of the wire at intersection \( i \). For instance, for the discrete constant-time surface the above expression evaluates to the usual \( L^2 \)-norm of a spacelike wavefunction

\[
\sum_{\tau} \sum_{\tau'} \psi \psi = \int_{0}^{t} \mathcal{S}(i) = \int_{0}^{t} \mathcal{S}(i)
\]

3.3.4. Cauchy surface independence of the discrete norm. If \( \psi \) is a solution of a QW, then just like the continuous case, the discrete norm does not depend on the discrete Cauchy surface chosen for evaluating it. The proof outline is as follows. First, two Cauchy surfaces \( \sigma \) and \( \sigma' \) can be made to coincide on an arbitrary large region using only a finite sequence of swap moves.

Figure 6. Discrete Cauchy surfaces and their transformations. (a) Centering on the origin \((n_0, l_0)\) of the discrete Cauchy surface. (b) Lorentz transform of the same piece of surface for \( \alpha = 3 \), \( \beta = 2 \).

\[
\mathcal{S}(i) = |\psi_-(r, l + \epsilon)|^2
\]

\[
\mathcal{S}(i) = |\psi_+(r + \epsilon, l)|^2
\]

\[
\mathcal{S}(i) = |\psi_+(r, l + \epsilon)|^2
\]

\[
\mathcal{S}(i) = |\psi_-(r + \epsilon, l)|^2
\]

In analogy with the continuous case, we can evaluate the norm of \( \psi \) along a surface \( \sigma \) as follows

\[
\| \psi \|_{2}^{2} = \sum_{i \in \mathcal{J}} \mathcal{S}(i), \quad (12)
\]

where \( \mathcal{S}(i) \) is the current of the wire at intersection \( i \). For instance, for the discrete constant-time surface the above expression evaluates to the usual \( L^2 \)-norm of a spacelike wavefunction

\[
\| \psi \|_{2}^{2} = \sum_{i \in \mathcal{J}} \mathcal{S}(i)
\]

\[
= \sum_{i \in 2\mathcal{J} + 1} \| \psi_+(i, - i) \|^2 + \sum_{i \in 2\mathcal{J}} \| \psi_-(i, - i) \|^2
\]

\[
= \sum_{i} \| \psi(i, - i) \|^2 = \| \psi \|^2.
\]
Second, swap moves leave the norm invariant, because of unitarity of the \( C \) gate (see figure 2).

\[
\psi_+ (r + \varepsilon, l)^2 + \psi_- (r, l + \varepsilon)^2 = |\psi_+ (r, l)|^2 + |\psi_- (r, l)|^2.
\]

Third, take a positive \( \delta \). By having \( \sigma \) to coincide with \( \sigma' \) on a large enough region, we obtain that \( l(\|\psi\|_\sigma - \|\psi\|_{\sigma'}) \leq \delta \). Lastly, since \( \delta \) is arbitrary, \( \|\psi\|_\sigma = \|\psi\|_{\sigma'} \).

### 3.3.5. Lorentz invariance of the discrete norm

Finally, we will prove the analogue of equation (11) in the discrete setting. First of all we define how a discrete Cauchy surface \( \sigma \) transforms under a discrete Lorentz transform with parameters \( \alpha, \beta \). The sequence \( \sigma' \) is constructed from \( \sigma \) by replacing each \( L \) by \( \alpha L \) and each \( R \) with \( \beta R \), starting from the center. The origin \( (r_0, l_0) \) is mapped to the point \( (r'_0, l'_0) = (\alpha r_0, \beta l_0) \). For instance, the piece of surface in figure 6(a) is transformed as in figure 6(b). We obtain (where \( \mathcal{R}_\sigma \) and \( \mathcal{L}_\sigma \) are the sets of right and left intersections respectively)

\[
\|\psi\|^2 = \sum_{i \in \mathcal{R}_\sigma} |\psi_+ (r_i, l_i)|^2 + \sum_{i \in \mathcal{L}_\sigma} |\psi_- (r_i, l_i)|^2
= \sum_{i \in \mathcal{R}_\sigma} \beta \left| \psi_+ (r_i, l_i) \right|^2 + \sum_{i \in \mathcal{L}_\sigma} \alpha \left| \psi_- (r_i, l_i) \right|^2
= \sum_{i' \in \mathcal{R}_{\sigma'}} |\psi'_+ (r_i', l_i')|^2 + \sum_{i' \in \mathcal{L}_{\sigma'}} |\psi'_- (r_i', l_i')|^2
= \sum_{i'} j' (i') = \|\psi'\|^2.
\]

### 3.4. The first-order-only Lorentz covariance of the Dirac QW

In section 2.1 we defined the Dirac QW, and explained when a spacetime wavefunction \( \psi \) is a solution for it. In section 3.2 we defined a discrete Lorentz transform, taking a spacetime wavefunction \( \psi \) into another spacetime wavefunction \( \psi' \). In section 3.3 we showed that this transformation preserves the norm, i.e. \( \|\psi\|_\sigma^2 = \|\psi'\|_{\sigma'}^2 \). The question that remains is whether the Dirac QW is Lorentz covariant with respect to this discrete Lorentz transform. In other words, is it the case that \( \psi' \) is itself a solution of the Dirac QW, for some \( m' \)? This demand is concrete.
translation of the main principle of special relativity, stating that the laws of physics (here, the Dirac QW) remain the same in all inertial reference frames (here, those of $\psi$ and $\psi'$).

Recall that the discrete Lorentz transform works by replacing each point of the spacetime lattice by a lightlike rectangular patch of spacetime, which can be understood as a ‘biased, zoomed in version’ of that point, see figure 5. Internally, each patch is a piece of spacetime solution of the Dirac QW by construction, see equation (9). But is it the case that the patches match up, to form the entire spacetime wavefunction of a solution? After all, there could be inconsistencies in between patches: values carried by the incoming wires to the next patches, e.g. $\psi_+(r + \epsilon, l)$ (respectively $\psi_-(r, l + \epsilon)$) could be different from those carried by the wires coming out of the preceding patch, i.e. $\psi_+(r, l) = C_+\psi_+(r, l)$ (respectively $\psi_-(r, l) = C_-\psi_-(r, l)$). More precisely, we need both $\psi_+(r, l) = \psi_+(r + \epsilon, l)$ and $\psi_-(r, l) = \psi_-(r, l + \epsilon)$ for every $r, l$. This potential mismatch is represented by the discontinuations of the wires of figure 5(b). Clearly, the patches making up $\psi'$ match up to form the spacetime wavefunction of a solution if and only if there are no such inconsistencies. We now evaluate these inconsistencies.

In the first order, the Dirac QW and the finite-difference Dirac equation are equivalent, as shown in section 2.1. This makes it easier to compute the outcoming values of the patches, which should match the corresponding incoming wires (see figure A.1 in appendix A). Let $m' = m/\sqrt{\alpha\beta}$. In general, we obtain (to first order in $\epsilon$, for $i = 0\ldots\beta - 1$, $j = 0\ldots\alpha - 1$)

\[
\psi_+(r, l)_i = \left( C_+\psi_+(r, l) \right)_i = \psi_+(r, l)_i - \alpha\epsilon\psi_+(r, l)_i - \beta\epsilon\psi_+(r, l)_i
\]

and

\[
\psi_-(r, l)_j = \left( C_-\psi_-(r, l) \right)_j = \psi_-(r, l)_j - \beta\epsilon\psi_+(r, l)_i - \alpha\epsilon\psi_-(r, l)_j
\]

Hence, the wires do match up in the first order. However, the second order cannot be fixed, even if we allow for arbitrary encodings. The proof of this statement is left for appendix A.

The lack of second order covariance of the Dirac QW can be interpreted in several ways. First, as saying that the Dirac QW is not a realistic model. This interpretation motivates us to explore, in the next sections, the question whether other discrete models (QWs or QCA) could not suffer this downside, and be exactly covariant. Second, as an indication that Lorentz covariance breaks down at Planck scale. Third, as saying that we have no choice but to view $\epsilon$ as an infinitesimal, so that we can ignore its second order. In this picture, the Dirac QW would be understood as describing an infinitesimal time evolution, but in the same formalism as that of...
discrete time evolutions, i.e. in an alternative language to the Hamiltonian formalism. Formulating an infinitesimal time quantum evolution in such a way has an advantage: it sticks to the language of unitary, causal operators [22] and readily provides a quantum simulation algorithm.

3.5. Transformation of velocities

In section 3.2 we defined a discrete Lorentz transform, which takes a spacetime wavefunction \( \psi \) into a Lorentz transformed wavefunction \( \psi' \). In section 3.4 we proved that the Dirac QW is first-order covariant. Is it the case that the velocity of \( \psi \) is related to the velocity of \( \psi' \) according to the transformation of velocity rule of special relativity? We will show that it is indeed the case, so long as we transform the ‘local velocity field’ \( v(r, l) \), defined as

\[
v(r, l) = \frac{|\psi_+(r, l)|^2 - |\psi_-(r, l)|^2}{||\psi(r, l)||^2}.
\]

(13)

In order to see how we arrive at this formula, let us first recall the definition of velocity in the continuous case.

For the Dirac equation, the velocity operator is obtained via the Heisenberg formula, \( \frac{d\hat{x}}{dt} = i[H, \hat{x}] = \sigma_3 \) (see equation (1)). Thus, in the discrete setting it is natural to define the velocity operator as \( \Delta X = X - WXW^\dagger \), where \( X \) is the position operator, \( X = \sum_x x P_x = \sum_x |x \rangle \langle x| \) and \( W = TC \) is the walk operator. We have

\[
\Delta X = X - WXW^\dagger = X - TCXC^\dagger T^\dagger = X - TXT^\dagger
\]

\[
= \left( \sum_x x P_x \begin{pmatrix} 0 & 0 \\ 0 & \sum_x x P_x \end{pmatrix} \right) - \left( \sum_x x P_{x+1} \begin{pmatrix} 0 & 0 \\ 0 & \sum_x x P_{x-1} \end{pmatrix} \right)
\]

\[
= \sigma_3.
\]

Thus the expected value of \( \Delta X \) at the time slice \( t = 0 \) is, as in the continuous case,

\[
\langle \sigma_3 \rangle_\psi = \sum_{i \in \mathbb{Z}} |\psi_+(i, -i)|^2 - |\psi_-(i, -i)|^2
\]

\[
= \sum_{i \in \mathbb{Z}} p(i, -i) v(i, -i),
\]

(14)

where \( p(r, l) = ||\psi(r, l)||^2 \). It should be noted that it is not a constant of motion. This is in fact a manifestation of the Zitterbewegung, whose connection with the continuous case was well studied by several authors [6, 7, 23]. Equation (14) justifies our definition of local velocity.

Let us now consider the case of a walker which at \( t = 0, x = 0 \), has internal degree of freedom \( \psi = (\psi_+, \psi_-)^T \). We will relate \( v = v(0, 0) \) and \( v' = v'(0, 0) \) as calculated from a Lorentz transformed observer with parameters \( \alpha, \beta \). We have \( v = (\|\psi_+\|^2 - \|\psi_-\|^2)/\|\psi\|^2 \) and \( v' = (\|\psi_+\|^2 (1 + v)/2 + \|\psi_-\|^2 (1 - v)/2 \). Now, let us apply a discrete Lorentz transform. At point \( (0, 0) \), it takes \( \psi \) into \( \psi' = S\psi \), whose corresponding velocity is
\[ v' = \frac{|\psi_+'|^2 - |\psi_-|^2}{\|\psi'\|^2} = \frac{\alpha |\psi_+|^2 - \beta |\psi_-|^2}{\alpha |\psi_+|^2 + \beta |\psi_-|^2} \]
\[ = \frac{\alpha \|\psi\|^2 (1 + v) - \beta \|\psi\|^2 (1 - v)}{\alpha \|\psi\|^2 (1 + v) + \beta \|\psi\|^2 (1 - v)} \]
\[ = \frac{\alpha - \beta}{\alpha + \beta} \frac{v + \frac{\alpha - \beta}{\alpha + \beta}}{1 + \nu \frac{\alpha - \beta}{\alpha + \beta}} \]
\[ = \frac{v + u}{1 + vu}, \]

where \( u = (\alpha - \beta)/(\alpha + \beta) \) is the velocity that corresponds to the discrete Lorentz transform with parameters \( \alpha, \beta \). Thus the local velocity associated to a spacetime wavefunction \( \psi \) is related to the local velocity of the corresponding Lorentz transformed \( \psi' \) by the rule of addition of velocities of special relativity.

4. Formalization of discrete Lorentz covariance in general

We will now provide a formal, general notion of discrete Lorentz transform and Lorentz covariance for QW and QCA.

4.1. Over QW

Beforehand, we need to explain which general form we assume for QW.

4.1.1. General form of QWs. Intuitively speaking, a QW is a single particle or walker moving in discrete-time steps on a lattice. Axiomatically speaking, QWs are shift-invariant, causal, unitary evolutions over the space \( \mathcal{H}_c \), where \( c \) is the dimension of the internal degrees of freedom of the walker. Constructively speaking, in turns out [24–26] that, at the cost of some simple recordings, any QW can be put in a form which is similar to that of the circuit for the Dirac QW shown figure 2(c). In general, however, \( c \) may be larger than 2 (the case \( c \) equal 1 is trivial [4]). But it can always be taken to be even, so that the general shape for the circuit of a QW can be expressed as in figure 7. Notice how, in this diagram, each wire carries a \( d \)-dimensional vector \( \psi_{rl}(r, l) \). We will say that the QW has ‘wire dimension’ \( d \). Incoming wires get composed together with a direct sum, to form a \( 2d \)-dimensional vector \( \psi(r, l) \). The state \( \psi(r, l) \) undergoes a \( 2d \times 2d \) unitary gate \( C \) to become some \( \psi'(r, l) = \psi' r + \epsilon, l \) \( \oplus \psi'(r, l + \epsilon), \) etc. The unitary gate \( C \) is called the ‘coin’. Algebraically speaking, this means that a QW can always be assumed to be of the form

\[
\begin{pmatrix}
\ e^{i\theta} \text{Id}_d & 0 \\
0 & e^{i\theta} \text{Id}_d
\end{pmatrix}
\psi = C \psi.
\]
We will not, in this paper, consider QW with additional wires (i.e. more complicated neighborhoods) as the resulting theory would be convoluted, and because they do not fit well with the picture of a lightspeed $c = 1$. Again, they can always be brought back into the above form via space grouping of adjacent cells into supercells [26, 27].

4.1.2. Lorentz transforms for QW. The formalization of a general notion of Lorentz transform for QWs generalizes that presented in section 3. Consider a QW having wire dimension $d$, and whose $2d \times 2d$ unitary coin is $C_m$, where the $m$ are parameters (In the case of the Dirac QW the coin is given explicitly in section 2.1 and there the only parameter is the mass. However, keep in mind that in general $m$ stands for any set of parameters.) A Lorentz transform $L_{\alpha,\beta}$ is specified by:

- a function $m' = f_{\alpha,\beta}(m)$, such that $f_{\alpha',\beta',\beta} = f_{\alpha',\beta} \circ f_{\alpha,\beta}$;
- a family of isometries $E_\alpha$ from $H_d$ to $\bigoplus_\alpha H_d$, such that $(\bigoplus_\alpha E_\alpha) E_\alpha = E_{\alpha'}$. 

Above we used the notation $\bigoplus_\alpha H_d = \bigoplus_{i=1}^d H_d$. Consider $\psi$ a spacetime wavefunction (at this stage it is not necessary to assume that it is a solution of the QW.) Switching to lightlike coordinates, its Lorentz transform $\psi' = L_{\alpha,\beta} \psi$ is obtained by:

- for every $(r, l)$, computing: $\tilde{\psi}_+ = E_\beta \psi_+ = E_\alpha \psi_-$, and $\tilde{\psi} = \tilde{\psi}_+ \oplus \tilde{\psi}_- = \overline{E} \psi$,
- for every $(r, l)$, replacing: the point $(r,l)$ by the lightlike $\alpha \times \beta$ rectangular patch of spacetime

$$ (C_m(i, j) \tilde{\psi}(r, l))_{i=0,\ldots,a-1, j=0,\ldots,b-1} \tag{15} $$

with $C_m(i, j)$ as in remark 1 and figure 3.

Again, figure 5 illustrated an example of such a transformation, for the case of the Dirac QW. The corresponding isometries $E_\alpha$, described in section 3.2, are given by

$$ E_\alpha = \frac{1}{\sqrt{\alpha}} 1_\alpha, $$

where $1_d = (1, ..., 1)^T$ is the $d$-dimensional uniform vector, and the function $f_{\alpha,\beta}$ is given by $f_{\alpha,\beta}(m) = m / \sqrt{\alpha \beta}$. 

Notice that while the focus of this paper is on translation-invariant QW, this is not actually required in order to define the Lorentz transform. The same definition would apply equally well.
if the QW has parameters that depend on the position, i.e., $m = m(r, l)$. We just note that in this case, according to our definition, $f_{\alpha, \beta}$ will not itself depend on the position, and that the new parameters $m'$ would be constant over each lightlike rectangular patch.

4.1.3. Lorentz covariance for QW. The formalization of a general notion of Lorentz covariance for QWs generalizes that presented in section 3.4. Consider a QW having wire dimension $d$ whose $2d \times 2d$ and unitary coin unitary coin $C_m$, where the $m$ are parameters. Consider $\psi$ a spacetime wavefunction which is a solution of this QW. We just gave the formalization of a discrete general notion of Lorentz transform taking a spacetime wavefunction $\psi$ into another spacetime wavefunction $\psi' = L_{\alpha, \beta} \psi$, and parameters $m$ into $m'$. Is it the case, for any $\alpha$ and $\beta$, that the spacetime wavefunction $\psi'$ is a solution of the same QW, but with parameters $m'$? If so, the QW is said to be covariant with respect to the given discrete Lorentz transform. Now, the above-defined discrete Lorentz transform is obtained by replacing each point with a lightlike $\alpha \times \beta$ rectangular patch of spacetime, which, by definition, is internally a piece of spacetime solution of the Dirac QW see equation (15). But again, is it the case that the patches match up to form the entire spacetime wavefunction of a solution? Let us again define

$$\hat{\psi}_+(r, l) = (C_{m'})_+ \hat{\psi} (r, l) \text{ and } \hat{\psi}_-(r, l) = (C_{m'})_- \hat{\psi} (r, l).$$

We need: $\hat{\psi}_+(r, l) = \hat{\psi}_+(r + \epsilon, l)$ and $\hat{\psi}_-(r, l) = \hat{\psi}_-(r, l + \epsilon)$. An equivalent, algebraic way of stating these two requirements is obtained as follows

$$\hat{\psi}_+(r + \epsilon, l) \oplus \hat{\psi}_-(r, l + \epsilon) = \hat{\psi}_+(r, l) \oplus \hat{\psi}_-(r, l).$$

Equivalently,

$$\left( E_\beta \oplus E_{\alpha} \right) \left( \psi_+(r + \epsilon, l) \oplus \psi_-(r, l + \epsilon) \right)
= \left( C_{m'} (\alpha, \cdot ) \oplus C_{m'} (\cdot, \beta) \right) \left( E_\beta \oplus E_{\alpha} \right) \psi (r, l)
\Rightarrow \left( E_\beta \oplus E_{\alpha} \right) C_m \psi (r, l) = C_{m'} \left( E_\beta \oplus E_{\alpha} \right) \psi (r, l)
\Rightarrow \left( E_\beta \oplus E_{\alpha} \right) C_m = C_{m'} \left( E_\beta \oplus E_{\alpha} \right)
\Rightarrow E C_m = C_{m'} E. \tag{16}$$

Notice that in the non-translation invariant case where $m = m(r, l)$, the above is required for every possible values that $m$ can take.

This expresses discrete Lorentz covariance elegantly, as a form of commutation relation between the evolution and the encoding. Diagrammatically this is represented by figure 8(a). The isometry of the $E_{\alpha}$ can also be represented diagrammatically, cf 8(b). Combining both properties straightforwardly leads to

$$C_m = E^\dagger C_{m'} E. \tag{17}$$

This is represented as figure 9(a), which of course can be derived diagrammatically from figure 8. Is this diagrammatic theory powerful enough to be considered an abstract, pictorial theory of Lorentz covariance, in the spirit of [28]?

4.1.4. Diagrammatic Lorentz covariance for QW. Combining the diagrammatic equalities of figure 8, we can almost rewrite the spacetime circuit of a QW with coin $C_m$, into its Lorentz transformed version, for any parameters $\alpha, \beta$… but not quite. A closer inspection shows that this
can only be done over regions such as past cones, by successively: (1) introducing pairs of enclodings via rule figure 8(b) along the border of the past cone; (2) pushing back towards the past the bottom $E$ via rule figure 8(a), thereby unveiling the Lorentz transformed past cone. Whilst this limitation to past-cone-like regions may seem surprising at first, there is a good intuitive reason for that. Indeed, the diagrammatic equalities of figure 8 tell you that you can locally zoom into a spacetime circuit; but you can only locally zoom out if you had zoomed in earlier, otherwise there may be a loss of information. This asymmetry is captured by the fact that figure 8(a) cannot be put upside-down, time-reversed. It follows that you should not be able to equalize an entire spacetime circuit with its complete Lorentz transform, at least not without using further hypotheses. And indeed, when we local Lorentz transform an entire past cone, its border is there to keep track of the fact that this region was locally zoomed into, and that we may later unzoom from it, if we want.

Now, could we add a further diagrammatic rule which would allow us to perform an complete Lorentz transformation, perhaps at the cost of annotating our spacetime circuit diagrams with information on whom has been zoomed into? Those annotations are the dashed lines of figures 8 and 9(a). Clearly, as we use those rules, we know whether some bunch of wires lives in the subspace $S_\alpha$ of the projector $E_\alpha E_\alpha^\dagger$, and we can leave that information behind. Moreover, on this subspace, it is the case that

$$E_\alpha E_\alpha^\dagger = \text{Id}_{S_\alpha}.$$  \hspace{1cm} (18)

Then, representing this last equation in rule figure 9(b), which is conditional on the annotation being there (the other rules are non-conditional, they provide the annotations,) we reach our purpose. Indeed, in order to perform a complete Lorentz-transform we can now apply the rule figure 9(a) everywhere, leading to figure 10, and then remove the encoding gates everywhere via figure 9(b). Thus, it could be said that the rewrite rules of figure 9 provide an abstract, pictorial theory of Lorentz covariance. They allow to equalize, spacetime seen by a certain observer, with spacetime seen by another, inertial observer. Besides their simplicity, the local nature of the rewrite rules is evocative of the local Lorentz covariance of general relativity. This is explored a little further in section 4.3.

---

**Figure 8.** Basic covariance rules. (a) Expresses the fundamental covariance condition of equations (16) and (20). The dashed line is optional, it is an indication which results from using this rule: it tells us that the state of these wires belongs to the subspace $S_\alpha$. The gray and white dots stand for the same unitary interaction, but with different parameters. (b) Expresses the isometry of the enclodings used for the discrete Lorentz transform.
4.1.5. Inverse transformations and equivalence upon rescaling. In analogy with the continuum case, we would like the inverse of a Lorentz transform $L_{\alpha\beta}$ to be $L_{\beta\alpha}$, i.e.,

$$L_{\alpha\beta} L_{\beta\alpha} = \text{Id.} \quad (19)$$

However, according to our definitions of $L_{\ast\ast}$, we know that $L_{\alpha\beta} L_{\beta\alpha}$ is a transformation such that

- each point $(r,l)$ is replaced by the lightlike $\alpha\beta \times \alpha\beta$ square patch of spacetime, with left-incoming wires $F\psi_+(r, l)$, right-incoming wires $F\psi_-(r, l)$, right-outgoing wires $F\psi_+(r + \epsilon, l)$ and left-outgoing wires $F\psi_-(r, l + \epsilon)$, where

$$F = \left( \bigoplus_{\alpha} E_{\beta} \right) E_{\alpha} = E_{\beta\alpha} = \left( \bigoplus_{\beta} E_{\alpha} \right) E_{\beta}$$

- the coin parameter $m$ is mapped to $m' = f_{\alpha\beta,\alpha\beta}(m)$.

Hence, if we are to claim (19) we need to identify any two spacetime diagrams which satisfy these relations. This is achieved as a special case in the completed diagrammatic theory of figure 9.

### 4.2. Over QCA

#### 4.2.1. General form of QCA

Intuitively speaking, a QCA is a multiple walkers QW. The walkers may or may not interact, their numbers may or may not be conserved. Axiomatically
speaking, a QCA is a shift-invariant, causal, unitary evolution over the space $\otimes_c \mathcal{H}$, where $c$ is the dimension of the internal degrees of freedom of each site. Actually, care must be taken when defining such infinite tensor products, but two solutions exist [25–27]. Constructively speaking, it turns out [25–27] that, at the cost of some simple recodings, any QCA can be put in the form of a quantum circuit. This circuit can then be simplified [29] to bear strong resemblance with the circuit of a general QW seen in figure 7. In particular $c$ can always be taken to be $d^2$, so that the general shape for the quantum circuit of a QCA is that of figure 11. Notice how, in this diagram, each wire carries a $d$-dimensional vector $\psi_{\pm r,l}$). We will say that the QCA has ‘wire dimension’ $d$. Incoming wires get composed together with a tensor product, to form a $d^2$-dimensional vector $\psi_{r,l}$. The state $\psi_{r,l}$ undergoes a $d^2 \times d^2$ unitary gate $U$ to become some $\psi_{r+l,e}$, etc. The unitary gate $U$ is called the ‘scattering operator’. Notice how, to some extent, the QCA are alike QW up to replacing $\oplus$ by $\otimes$. Algebraically speaking, the above means that one time-step of a QCA can always be assumed to be of the form

$$\psi \mapsto \left( \bigotimes_{2\mathbb{Z}+1} U \right) \left( \bigotimes_{2\mathbb{Z}} U \right) \psi.$$ 

4.2.2. Lorentz transforms for QCA. The formalization of a general notion of Lorentz transform for QCA is obtained from that over QW essentially by changing occurrences of $\oplus$ into $\otimes$. Indeed, consider a QCA having wire dimension $d$, and whose $d^2 \times d^2$ unitary scattering operator $U$ has parameters $m$. A Lorentz transform $L_{a,\beta}$ is specified by:

- a function $m' = f_{a,\beta}(m)$ such that $f_{a',\beta',\beta} = f_{a',\beta} \circ f_{a,\beta}$,
- a family of isometries $E_a$ from $\mathcal{H}_d$ to $\bigotimes_a \mathcal{H}_d$, such that $(\bigotimes_a E_{a'}) E_a = E_{a'\beta}$.

There is a crucial difference with QWs, however, which is that we cannot easily apply this discrete Lorentz transform to a spacetime wavefunction. Indeed, consider $\psi$ a spacetime
wavefunction. For every time $t$, the state $\psi(t)$ may be a large entangled state across space. What meaning does it have, then, to select another spacelike surface? What meaning does it have to switch to lightlike coordinates? Unfortunately the techniques which were our point of departure for QWs, no longer apply. Fortunately, the algebraic and diagrammatic techniques which were our point of arrival for QWs, apply equally well to QCA, so that we may still speak of Lorentz-covariance.

4.2.3. Lorentz covariance for QCA. Again, the formalization of the notion of Lorentz-covariance for QCA cannot be given in terms of $\psi'$ being a solution if $\psi$ was a solution, because we struggle to speak of $\psi'$. Instead, we define Lorentz-covariance straight from the algebraic view

$$ (E_\beta \otimes E_\alpha)U_m = \overline{U_m} \left( E_\beta \otimes E_\alpha \right) $$

i.e. $E U_m = \overline{U_m} E$. (20)

Diagrammatically this is represented by the same figure as for QWs, namely figure 8(a). The isometry of the $E_\alpha$ is again represented by figure 8(b). Algebraically speaking, combining both properties again leads to

$$ U_m = E^\dagger \overline{U_m} E. $$ (21)

Which diagrammatically this is again represented as figure 9(a). For the same reasons, the conditional rule figure 9(b) again applies: the whole diagrammatic theory carries through unchanged from QWs to QCA.

4.3. Non-homogeneous discrete Lorentz transforms and non-inertial observers

Nothing in the above developed diagrammatic theory forbids us to apply different local discrete Lorentz transforms to different points of spacetime, so long as point $(r, l)$ and point $(r + \epsilon, l)$ (respectively point $(r, l + \epsilon)$) have the same parameter $\beta$ (respectively $\alpha$). This constraint propagates along lightlike lines, so that there can be, at most, one different $\alpha_\epsilon$ (respectively $\beta_\epsilon$) per right-moving (respectively left-moving) lightlike line $r$ (respectively $l$). We call this a non-homogeneous discrete Lorentz transform of parameters $(\alpha_\epsilon), (\beta_\epsilon)$.

The circuit which results from applying such a non-homogeneous discrete Lorentz transform is, in general, a non-homogeneous QWs (respectively QCA), as it may lack shift-invariance in time and space. This is because the coin $C_m$ (respectively scattering unitary $U_m$) of the point $(r,l)$ gets mapped into lightlike $\alpha, \beta$-rectangular patch of spacetime $C'_m$ (respectively
provided that the condition $f_{\alpha\beta} = f$ is met, we can now transform between non-inertial observers by a non-homogeneous discrete Lorentz transform. Figure 12 illustrates this with the simple example of an observer which moves one step right, one step left, until it reaches point $(0, 0)$ where it gets accelerated, and continues moving two steps right, one step left etc. We choose $\beta_i = 1$ for $l < 0$, $\beta_i = 2$ for $l \geq 0$ and $\alpha_r = 1$ for all $r$. This has the effect of slowing down the observer just beyond the point $(0, 0)$. All along his trajectory, he now has to move two steps right for every two steps left that he takes, so that he is now at rest.

In general, suppose that an observer moves $a_k$ steps to the right, $b_k$ steps left, $a_{k+1}$ steps right, etc. He does this starting from position $r_k = r_{k-1} + a_k$ and $l_k = l_{k-1} + b_k$. For every $k$, let $M_k$ be the least common multiple of $a_k$ and $b_k$. We choose $\alpha_r = M_k/a_k$ for $r_{k-1} \leq r < r_k$ and $\beta_l = M_k/b_k$ for $l_{k-1} \leq l < l_k$. Let us perform the non-homogeneous discrete Lorentz transform of parameters $(\alpha_r), (\beta_l)$. Then, the observer now moves $M_k$ steps right for every $M_k$ steps left he takes, and then $M_{k+1}$ steps right for every $M_{k+1}$ steps left etc.

5. The clock QW

Equipped with a formal, general notion of Lorentz transform and Lorentz covariance for QW, we can now seek for an exactly covariant QW.

5.1. Definition

In the classical setting, covariance of random walks has already been explored [8]. The random walk of [8] uses a fair coin, but is nonetheless biased in the following way: after a (fair) coin toss the walker moves during $p$ time steps to the right (respectively during $q$ time steps to the left). There is a reference frame in which the probability distribution is symmetric, namely that with velocity $u = (p - q)/(p + q)$. Changing the parameters $p$ and $q$ corresponds to performing a Lorentz transform of the spacetime diagram.

Now we will make an analogous construction in the quantum setting. The main point is to enlarge the coin space so that the coin operator is idle during $p$, or $q$, time steps. The coin space will be $\mathcal{H}_C = \mathcal{H}_C^p \oplus \mathcal{H}_C^q$, where $\mathcal{H}_C^p \cong \mathcal{H}_C^q = \ell^2(\mathbb{Z})$. The Hilbert space of the QW is then $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathcal{H}_C$, whose basis states will be indicated by $|x, h\rangle$, with $h \in \mathbb{Q}^\geq s = \pm$.

This $\mathcal{H}_C^\pm$ will act as a ‘counter’. When $h > 0$, the walker moves without interaction and the counter is decreased. When the counter reaches 0, the effective coin operator is applied and the counter is reset.

The evolution of the clock QW with parameters $p, q$ is defined on the subspace $\mathcal{H}_C^{p, q}$ of $\mathcal{H}_C$ spanned by the $p + q$ vectors $\left\{ \left| \frac{i}{p} \right\rangle, \left| \frac{j}{q} \right\rangle \right\}$ with $i = 0, ..., p - 1$ and $j = 0, ..., q - 1$, as follows.
This map is unitary provided that the $2 \times 2$ matrix $C$ of coefficients $C_{11} = a$, $C_{12} = b$, $C_{21} = c$, $C_{22} = d$ is unitary. For instance we could choose, as for the Dirac QW, $a = d = \cos(\theta)$, $b = c = -\sin(\theta)$.

The clock QW with parameters $p$ and $q$ will only be used over $\ell^2(\mathbb{Z}) \otimes \mathcal{H}^{p,q}$ where it admits a matrix form which we now provide (over the rest of $\mathcal{H}_C$ it can be assumed to be the identity). From equation (22) we can write $W_{p,q} = T_{p,q}C_{p,q}$ where $T_{p,q}$ is the shift operator.

Figure 12. An inhomogeneous transformation for a non-inertial observer. The region above the red line undergoes a Lorentz transform with parameters $\alpha = 1$ and $\beta = 2$, whilst the region below is left unchanged. After the inhomogeneous transformation, the observer is at rest.
\[ T_{p,q} = \text{diag} \left( e^{-\epsilon \delta_x}, \ldots, e^{-\epsilon \delta_x}, e^{\epsilon \delta_x}, \ldots, e^{\epsilon \delta_x} \right) \]

and \( C_{p,q} \) is the coin operator:

\[
C_{p,q} = \begin{pmatrix}
0 & \text{Id}_{p-1} & 0 & 0 \\
\alpha & 0 & b & 0 \\
0 & 0 & 0 & \text{Id}_{q-1} \\
c & 0 & d & 0
\end{pmatrix}
\]

Hence, the clock QW has an effective coin space of finite dimension \( p + q \). However, we will see that this dimension changes under Lorentz transforms.

### 5.2. Covariance

In order to prove covariance, we need to find isometries satisfying the equation expressed by figure 8(a). Let us consider isometries \( E_{\alpha} : \mathcal{H}_C \to \bigoplus \mathcal{H}_C \) defined by

\[
E_{\alpha} \left| h^i \right\rangle = \left( \left| h^i \right\rangle \bigoplus 0 \bigoplus \cdots \bigoplus a^{-1}\text{times} \right)
\]

(the Hilbert spaces in the direct sum are ordered from the bottom wire to the top one, as in remark 1). In figure 13 it is proved that this choice actually satisfies the covariance relation \( EC_{p,q} = C_{p',q'} E \), where the coin operator parameters have been rescaled as \( p' = \alpha p \) and \( q' = \beta q \). Intuitively, the Lorentz transformation rescales the fractional steps of the clock QW by \( \alpha \) (respectively \( \beta \)), while adding \( \alpha - 1 \) (respectively \( \beta - 1 \)) more points to the lattice. In this way, the counter will reach 0 just at the end of the patch, as it did before the transformation.

### 5.3. Continuum limit of the clock QW

The clock QW does not have a continuum limit because its coin operator is not the identity in the limit \( \epsilon \to 0 \). However, by appropriately sampling the spacetime points, it is possible to take the continuum limit of a solution of the clock QW and show that it converges to a solution of the Dirac equation, subject to a Lorentz transform with parameters \( p, q \). Indeed, the limit can be obtained as follows. First, we divide the spacetime in lightlike rectangular patches of dimension \( p \times q \). Second, we choose as representative value for each patch the point where the interaction is non-trivial, averaged according to the dimensions of the rectangle

\[
\psi'(r, l) = \begin{pmatrix}
\psi_{+}\left(\lfloor r/p \rfloor_e, \lfloor l/q \rfloor_e \right) \\
\sqrt{q} \\
\psi_{-}\left(\lfloor r/p \rfloor_e, \lfloor l/q \rfloor_e \right)
\end{pmatrix}
\]

Finally, by letting \( \epsilon \to 0 \) we obtain

\[
\psi'(r, l) = S \psi\left( r/p, l/q \right),
\]

where now the \( r, l \) coordinates are to be intended as continuous.
Since $\psi'$ is of course a solution of the Dirac equation (with a rescaled mass), this proves that the continuum limit of the clock QW evolution, interpreted as described above, is again the Dirac equation itself.

5.4. Decoupling of the QW and the Klein–Gordon equation

The clock QW does not have a proper continuum limit unless we exclude the intermediate computational steps. Still, as we shall prove in this section, its decoupled form has a proper limit, which turns out to be the Klein–Gordon equation with a rescaled mass. By a decoupled form, we mean the scalar evolution law satisfied by each component of a vector field, individually (see [30]). In the following, we give the decoupled form of the clock QW. The evolution matrix $W$ is sparse and allows for decoupling by simple algebraic manipulations, leading to

$$\tau \psi = \begin{bmatrix} \tau_T \tau_p \tau_C \end{bmatrix} \begin{bmatrix} T^{q+p} - a \tau^{-q} T^p - d \tau^p T^q + \det(C) \tau^{p-q} \end{bmatrix} \psi = 0,$$

where $T = e^{\epsilon \delta}$ and $\tau = e^{\epsilon \alpha}$). This is a discrete evolution law which gives the value of the current step depending on three previous time steps, namely the ones at $t = -p$, $t = -q$ and $t = -p - q$.

By expanding in $\epsilon$ the displacement operators and assuming that the coin operator verifies

$$\det(C) = 1, \quad a = d = 1 + \frac{\epsilon^2 m^2}{2} + O(\epsilon^3)$$

Figure 13. Covariance of the clock QW. This is the transformation given by $\alpha, \beta$ of the clock QW with parameters $p = 1, q = 1$. 
(which is the case if \( a = d = \cos(me) \)) we obtain the continuum limit

\[
\left( \partial_t^2 - \partial_x^2 + \frac{m^2}{pq} \right)\psi = 0. \tag{25}
\]

Up to redefinition of the mass \( m' = m/\sqrt{pq} \), this is the Klein–Gordon equation. This reinforces the interpretation of the clock QW as model for a relativistic particle of mass \( m' \).

6. The clock QCA

One downside of the clock QW is the fact that the dimension of the coin space varies according to the observer. Equipped with a formal, general notion of Lorentz transform and Lorentz covariance for QCA, we can now seek for an exactly covariant QCA of fixed, small, internal degree of freedom.

6.1. From the clock QW to the clock QCA

The idea of the clock QW was to let the walker propagate during a number of steps to the right (respectively to the left), without spreading to the left (respectively to the right). In the absence of any other walker, this had to be performed with the help of an internal clock. In the context of QCA, however, the walker can be made to cross ‘keep going’ signals instead.

The clock QCA has wire dimension \( d = 3 \), with orthonormal basis \(| q \rangle, | 0 \rangle, | 1 \rangle \). Both \(| q \rangle \) and \(| 0 \rangle \) should be understood as vacuum states, but of slightly different natures as we shall see next. \(| 1 \rangle \) should be understood as the presence of a particle. Thus, the clock QCA has scattering unitary a 9×9 matrix \( U \), which we can specify according to its action over the nine basis vectors. First we demand that the vacuum states be stable, i.e.

\[
| q \rangle \otimes | q \rangle \mapsto | q \rangle \otimes | q \rangle, \\
| q \rangle \otimes | 0 \rangle \mapsto | 0 \rangle \otimes | q \rangle, \\
| 0 \rangle \otimes | q \rangle \mapsto | q \rangle \otimes | 0 \rangle, \\
| 0 \rangle \otimes | 0 \rangle \mapsto | 0 \rangle \otimes | 0 \rangle.
\]

Second we demand that multiple walkers do not interact:

\[
| 1 \rangle \otimes | 1 \rangle \mapsto | 1 \rangle \otimes | 1 \rangle.
\]

Third we demand that the interaction between \(| 1 \rangle \) and \(| q \rangle \) be dictated by a massless Dirac QW, or ‘Weyl QW’, i.e. the single walker goes straight ahead:

\[
| 1 \rangle \otimes | q \rangle \mapsto | q \rangle \otimes | 1 \rangle, \\
| q \rangle \otimes | 1 \rangle \mapsto | 1 \rangle \otimes | q \rangle.
\]

Last we demand that the interaction between \(| 1 \rangle \) and \(| 0 \rangle \) be dictated by:

\[
| 1 \rangle \otimes | 0 \rangle \mapsto a (| 0 \rangle \otimes | 1 \rangle) + b (| 1 \rangle \otimes | 0 \rangle), \\
| 0 \rangle \otimes | 1 \rangle \mapsto c (| 0 \rangle \otimes | 1 \rangle) + d (| 1 \rangle \otimes | 0 \rangle).
\]

This map is unitary provided that the 2×2 matrix \( C \) of coefficients \( C_{11} = a, C_{12} = b, C_{21} = c, C_{22} = d \) is unitary. For instance we could choose, as for the Dirac QW, \( a = d = \cos(me), b = c = - \sin(me) \)
The clock QCA is covariant, even though its wire dimension is fixed and small, as we shall see.

6.2. Covariance of the clock QCA

In order to give a precise meaning to the statement according to which the clock QCA is covariant, we must specify our Lorentz transform. According to section 4.2 we must provide a function \( f \), which we take to be the identity, and an encoding \( E_\alpha: \mathcal{H}_d \to \mathcal{H}^{\otimes \alpha}_d \) which we take to be

\[
|a\rangle \mapsto |a\rangle \otimes \otimes_{a-1} |q\rangle,
\]

written from the bottom wire to the top wire as was the convention for QWs. The intuition is that the \((\alpha - 1)\) ancillary wires are just there to stretch out this lightlike direction, but given that \( |q\rangle \) interacts with no one, this stretching will remain innocuous to the physics of the QCA.

Let us prove that things work as planned

\[
UE(|a\rangle \otimes |b\rangle) = \left(\prod_{i=0}^{a-1} |q\rangle^{i,0} \otimes \otimes_{j=1}^{\beta} |q\rangle^{j,0}\right) \left(\prod_{i=0}^{a-1} |q\rangle^{i,0} \otimes \otimes_{j=1}^{\beta} |q\rangle^{j,0}\right) \left(\prod_{i=0}^{a-1} |q\rangle^{i,0} \otimes \otimes_{j=1}^{\beta} |q\rangle^{j,0}\right) = U(|a\rangle \otimes |b\rangle).
\]

Hence, the clock QCA is Lorentz covariant. Notice that things would have worked equally well if \( E_\alpha \) had placed \(|a\rangle\) differently amongst the \(|q\rangle\). It could even have spread out \(|a\rangle\) evenly across the different positions, in a way that is more akin to the Lorentz transform for the Dirac QW.

7. Discussion of the physical interpretation

We formalized discrete Lorentz covariance as a form of commutativity: \( EC_m = C_m E \). In the continuum, Lorentz covariance is not usually expressed as a commutation relation. However, consider a unitary representation of the Poincaré group: \( U(a, \Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}(x - a)) \). Since the representation has to be homomorphic, one has: \( U(0, \Lambda)U(a, \text{id}) = U(\Lambda^{-1}a, \text{id})U(0, \Lambda) \). If \( a \) is along time, then \( U(a, \text{id}) \) is the time evolution (analogous to the discrete, local \( C_m \)), and \( U(\Lambda^{-1}a, \text{id}) \) the time evolution in the new frame (analogous to the discrete, rectangular patch \( C_m \)). Similarly, \( U(0, \Lambda) \) in the continuous encoding (analogous to the discrete \( E \)). Hence, the discrete Lorentz covariance condition is very much akin to the statement of the existence of a unitary representation of the Poincaré group, only with two added twists. First, it is expressed locally within lightlike rectangular patches of spacetime, which is always possible due to the boundedness of the speed of light, but seems more conveniently done in the discrete setting. Second, we do not look for a unitary representation but for an isometric representation, and crucially rely on
covariance up to a mass rescaling under a conformal change of metric (aka scale covariance) of the evolutions of physics (as required by general relativity), in order to fit the encoded $\psi$ back on the grid.

In the continuum setting, the Lagrangian formulation of Lorentz covariance is often preferred to that based on the existence of a unitary representation of the Poincaré group. This is for practical reasons: it suffices to check that the Lagrangian is an invariant scalar in order to prove covariance, whereas exhibiting a unitary representation is not easy. It could be said that the discrete Lorentz covariance suffers precisely the same downside: it is not easy to see whether a QW or a QCA is Lorentz covariant. Yet, the downside is slightly diminished by the fact that $EC_m = \tilde{C}_m \tilde{E}$ is a local expression. Moreover, the fact that scale covariance enters early in the game, puts us on track of which QW or QCA will work. Intuitively, only those which can be rescaled in the sense of the renormalization group, favouring a lightlike direction over the other at will, are suitable. In this paper we saw three mechanisms for implementing these lightlike stretchings: the first-order linear loss of the Dirac QW, the internal states of the clock QW, and the quiescent signals of the clock QCA. It is not clear to us whether there exists a fourth mechanism, which would not be a variant of those three.

The simple theory presented can be criticized on other grounds. First, one may wish for more explicit comparison with the continuum theory. This may be done along the lines of [5, 31, 32]: by letting the lattice spacing $\varepsilon$ go to zero, the convergence of the spacetime wavefunction solution of the Dirac QW can be shown to tend to the solution of the Dirac equation, in a manner which can be quantified. Second, one may argue that the very definition of the Lorentz transform should not depend on the QW under consideration. Similarly, one may argue that the transformed wave function should be a solution of the original QW, without modifications of its parameters. However, recall that $1+1$ dimensional, integral Lorentz transforms are trivial unless we introduce a global rescaling. Thus the discrete Lorentz transform of this paper may be thought of as a biased zooming in. In order to fill in the zoomed in region, one generally has to use the QW in a weakened, reparameterized manner.

8. Summary and perspectives

In the context of QW and QCA, we have formalized a notion of discrete Lorentz transform of parameters $\alpha, \beta$, which consists in replacing each spacetime point with a lightlike $\alpha \times \beta$ rectangular spacetime patch, $\tilde{C}_m \tilde{E}$, where $\tilde{E}$ is an isometric encoding, and $\tilde{C}_m$ is the repeated application of the unitary interaction $C_m$ throughout the patch (see figure 5). We then formalized discrete Lorentz covariance as a form of commutativity: $EC_m = \tilde{C}_m \tilde{E}$. This commutation rule as well as the fact the $\tilde{E}$ is isometric can be expressed diagrammatically in terms of a few local, circuit equivalence rules (see figures 8 and 9), à la [28]. This simple diagrammatic theory allows for non-homogeneous Lorentz transforms (figure 12), which let you switch between non-inertial observers. Actually, it would be interesting to compare the respective powers of covariance under non-homogeneous Lorentz transformations versus general covariance under diffeomorphisms plus local Lorentz covariance.

First we considered the Dirac QW, a natural candidate, given that it has the Dirac equation as continuum limit, which is of course covariant. Unfortunately, we proved the Dirac QW to be covariant only up to first-order in the lattice spacing $\varepsilon$. This is inconvenient if $\varepsilon$ is considered a physically relevant quantity, i.e. if spacetime is really thought of as discrete. But if $\varepsilon$ is thought
of as an infinitesimal, then the second-order failure of Lorentz-covariance is irrelevant. Thus, this result encourages us to take the view that $\epsilon$ is akin to infinitesimals in non-standard analysis. Then, the Dirac QW would be understood as describing an infinitesimal time evolution, but in the same formalism as that of discrete time evolutions. As an alternative language to the Hamiltonian formalism, it has the advantage of sticking to local unitary interactions [33], and that of providing a quantum simulation algorithm.

Exact Lorentz covariance, however, is possible even for finite $\epsilon$. This paper introduces the clock QW, which achieves this property. However the effective dimension of its internal degree of freedom depends on the observer. Furthermore, the clock QW does not admit a continuum limit, unless we appropriately sample the points of the lattice. Yet, its decoupled form does have a continuum limit, which is the Klein–Gordon equation. It is interesting to see that there is a QW evolution which can be interpreted as a relativistic particle (since it satisfies the KG equation), and yet not have a continuum limit for itself.

Eventually, we introduced the clock QCA, which is exactly covariant and has a three dimensional state space for its wires.

Finally, in the physical discussion, we pointed out the difficulty to find other covariant models, not based on the first-order approximation nor on clocked mechanisms. This leaves the following question open: is there a systematic method which given a QW with coin operator $C$, decides whether it exists a Lorentz transform $E_\alpha, f_{\alpha,\beta}$ such that $EC_m = CmE$, i.e. such that the QW is covariant? The same question applies to QCA; answering it would probably confirm the intuition that covariant QWs are scarce amongst QWs.

On the one hand, this paper draws its inspiration from quantum information and a perspective for the future would be to discuss relativistic quantum information theory [34, 35] within this framework. On the other hand, it forms part of a general trend seeking to model quantum field theoretical phenomena via discrete dynamics. For now, little is known on how to build QCA models from first principles, which admit physically relevant Hamiltonians [32, 36–39] as emergent. In this paper we have identified one such first principle, namely the Lorentz covariance symmetry. We plan on studying another fundamental symmetry, namely isotropy, thereby extending this work to higher dimensions.

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Appendix A. First-order-only covariance of the Dirac QW

Uniqueness of encodings. Here we prove that the only encoding compatible with first-order covariance is the flat one, as described in section 3.2. In general, the encoding isometries $E_\alpha, E_\beta$ can be defined in terms of normalized vectors, $v_\pm$ as follows (remember that for the Dirac QW, $\psi_+$ and $\psi_-$ are just scalars)

$$E_\beta\psi_+ = \psi_+ v_+, \quad E_\alpha\psi_- = \psi_- v_-.$$
In order to require covariance, we need to calculate the terms appearing in the commutation relation (16). The rhs of the relation is (see figure A.1 and section 3.4)

$$
\sum \sum \psi_\epsilon \psi_\epsilon \psi_\epsilon = \beta \alpha \prime + \psi_\epsilon \psi_\epsilon \psi_\epsilon
$$

where $d$ is the $d$-dimensional uniform vector, and $\sum = \sum$. On the other hand the lhs is

$$
\psi_\epsilon \psi_\epsilon \psi_\epsilon = \beta \alpha \prime + \psi_\epsilon \psi_\epsilon \psi_\epsilon
$$

Requiring first-order covariance, one obtains

$$
m_\nu_+ = m' \left( \sum \nu_- \right) 1_\beta, \quad m_\nu_- = m' \left( \sum \nu_+ \right) 1_\alpha
$$

which, together with the normalization of $\nu_\pm$, gives

$$
m' = \frac{m}{\sqrt{\alpha \beta}}, \quad \nu_+ = \frac{e^{i2\alpha}}{\sqrt{\beta}} 1_\beta, \quad \nu_- = \frac{e^{i2\beta}}{\sqrt{\alpha}} 1_\alpha
$$
thereby proving that the only possible encoding compatible with first-order covariance is the flat one (up to irrelevant phases).

**Failure at second order.** The Dirac QW can similarly be expanded to the second order. This time, however, the patches that make up $\psi'$ do not match up. A simple counter-example supporting this fact arises with $\alpha = 2$ and $\beta = 1$ already, as illustrated in figure A.2. Notice that we ought to have $C(0, 1)\psi_\omega(0, 0) = \overline{C(1, 1)\psi_\omega(0, 0)}$, if we want those outcoming wires to match up with the corresponding incoming wires of the next patch $\psi_\omega(0, 1) = \overline{\psi_\omega(0, 1)} = \psi_\omega(0, 1)/\sqrt{2}$. But it turns out that those outcoming wires verify $\overline{C(0, 1)\psi_\omega(0, 0)} \neq \overline{C(1, 1)\psi_\omega(0, 0)}$ due to a term in $\varepsilon^2$.

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