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Allometric exponent and randomness

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Abstract. An allometric height–mass exponent \(\gamma\) gives an approximative power-law relation \(\langle M \rangle \propto H^\gamma\) between the average mass \(\langle M \rangle\) and the height \(H\) for a sample of individuals. The individuals in the present study are humans but could be any biological organism. The sampling can be for a specific age of the individuals or for an age interval. The body mass index is often used for practical purposes when characterizing humans and it is based on the allometric exponent \(\gamma = 2\). It is shown here that the actual value of \(\gamma\) is to a large extent determined by the degree of correlation between mass and height within the sample studied: no correlation between mass and height means \(\gamma = 0\), whereas if there was a precise relation between mass and height such that all individuals had the same shape and density then \(\gamma = 3\). The connection is demonstrated by showing that the value of \(\gamma\) can be obtained directly from three numbers characterizing the spreads of the relevant random Gaussian statistical distributions: the spread of the height and mass distributions together with the spread of the mass distribution for the average height. Possible implications for allometric relations, in general, are discussed.

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1. Introduction

Allometric relations in biology describe how a quantity $Y$ scales with the body mass $M$, i.e. $Y = AM^\gamma$, where $\gamma$ is an allometric exponent. Allometric relations have a long history with pioneering work by D’Arcy Thomson [1] and Huxley [2]. Among others, the allometric relation for the metabolic rate $B$ has drawn much interest: Kleiber’s law [3] states that $B \sim M^p$, with $p = 1/\gamma \approx 3/4$, and has been tested in e.g. [4–6]. For a review of allometric relations see [7].

In this paper, we focus on the allometric relation between height and mass for humans. This mass ($M$)–height ($H$) relation has an even longer history, dating back to the pioneering work by Quetelet from 1842 [8], where the allometric relation was introduced to define a normal man so that $M/H^\gamma$ becomes a Gaussian variable. The precise definition of the allometric exponent used in the present study is $\langle M \rangle_H \propto H^\gamma$, where $\langle M \rangle_H$ is the average mass of individuals of height $H$ in the sample. Note that the allometric exponents $\gamma = 1, 2$ and $3$ correspond to that mass is proportional to height, body surface and body volume, respectively. For practical purposes $\gamma = 2$ is often a good approximation for humans, as shown in [9]. This approximation is the basis for the body mass index (BMI) $A$ given by $\langle M \rangle_H = AH^2$, provided mass is in kilograms and height in meters. More recently, it has been suggested that a larger allometric index $2 < \gamma \leq 3$ should be more appropriate [10–12]. In particular, Burton in [11] suggests that $\gamma = 2$ is an underestimate caused by randomness. This is in accordance with the conclusions reached in the present investigation.

The objective of the present investigation is to understand the relation between the exponent $\gamma$ and the randomness for a given sample of individuals. The issue is best illustrated by a specific example. Figures 1(a) and (b) show the height and mass distributions, $P(H)$ and $P(M)$, respectively, for 25 000 children 18 years old from Hong Kong [13]. Figure 1(c) in addition shows the distribution $P(M|H = \langle H \rangle)$ for children of average height $\langle H \rangle$. All these three statistical distributions are, to very good approximation, Gaussians. This means that the variables in all three cases are randomly distributed around their respective average values. The random spreads are in all three cases characterized by the normalized standard deviations, which we denote as $\bar{\sigma}_H$, $\bar{\sigma}_M$ and $\bar{\sigma}$ for random spreads of height, mass and mass-for-average-height, respectively. The relation derived in the present paper states that $\gamma$ to good approximation should be given by $\gamma = \sqrt{\frac{\bar{\sigma}_M^2}{\bar{\sigma}_H^2}} - 1$. From the random spreads in figure 1, one then finds $\gamma = 1.63$. Figure 2(a) shows that this is a very accurate prediction. This means that the allometric exponent is entirely determined by the randomness of the three distributions. Why is this so and what does it imply? These are questions which come to mind.
Figure 1. (a) The height distribution $P(H)$ and (b) the mass distribution $P(M)$ for 25,000 18 year old Hong Kong children [13]. (c) The conditional probability distribution $P(M|H=\langle H\rangle)$ obtained for 3670 children whose heights are in the interval $(171.82, 173.62)$ around $\langle H\rangle = 172.72$. In (a)–(c), the crosses are the data and the full-drawn curves are the corresponding Gaussian approximations. The numbers of bins are 81 for (a) and (b), and 31 for (c). $H$ in (a) and $M$ in (b) and (c) are in units of cm and kg. (d) The standard deviation $\sigma (H)$ of the distribution $P(M|H)$ in equation (3) as a function of height $H$. The horizontal line shows that the standard deviation $\sigma (H)$ is independent of $H$ in a range around the average height $\langle H \rangle \approx 172.72$ cm.

In section 2, the relation between $\gamma$ and the random spreads is derived. Comparisons with data are made in section 3, and in section 4 we sum up and discuss the results.

2. Allometric exponent expressed in normalized standard deviations

The point made in this paper is that the exponent $\gamma$ can be estimated from the sole knowledge of the first and second moments of the mass and height distributions. In order to derive such a relation, we assume that the mass and height distributions are approximately Gaussians. This is, as is illustrated by the data sets in figure 1, often a fair approximation around the maxima of the distributions. It means that the probability distributions for the mass and height are
Figure 2. Allometric relations for the Hong Kong data in (a) and (b) and for the Swedish data in (c) and (d). (a) Log–log plot of the average mass $\langle M \rangle_H$ as a function of height $H$. Symbols correspond to the average for a length interval of 1.26 cm. The data fall on a straight line in accordance with the allometric relation $\langle M \rangle_H \propto H^\gamma$. The value of $\gamma$ determined by the least-squares fit to the data is $\gamma_{ex} = 1.63$. The straight line is the prediction in equation (11) in terms of the random spreads given by equation (10). The prediction is very accurate for this data set. (b) Log–log plot of the ordered data $M_{\text{ordered}}(H)$ as a function of $H$. The data are well represented by $M_{\text{ordered}} \propto H^\gamma$ with $\gamma_{\text{ordered}} = 3.29$. Panels (c) and (d) are the same as (a) and (b) for the Swedish data. The straight lines are least-squares fits to the data giving respectively $\gamma_{ex} = 2.35$ and $\gamma_{\text{ordered}} = 3.30$. These predictions are again in good agreement with the predictions given in table 1. Note that the allometric exponents between the Hong Kong data set in (a) and the Swedish data set in (c) are significantly different, whereas they are almost identical for within the ordered data-representation given by (b) and (d). These features are explained in this paper.

approximately given by

$$P_M(M) = \frac{1}{\sqrt{2\pi\sigma_M^2}} \exp\left(-\frac{(M - \langle M \rangle)^2}{2\sigma_M^2}\right),$$

(1)

$$P_H(H) = \frac{1}{\sqrt{2\pi\sigma_H^2}} \exp\left(-\frac{(H - \langle H \rangle)^2}{2\sigma_H^2}\right),$$

(2)
respectively. Note that these two distributions are characterized by the four explicit numbers \( \langle M \rangle, \langle H \rangle, \sigma_M \) and \( \sigma_H \). The degree of correlation between the mass and height is then given by the mass distribution for a given height, which we likewise assume to be approximately Gaussian and is given by the conditional probability

\[
P(M|H) = \frac{1}{\sqrt{2\pi\sigma(H)^2}} \exp\left( -\frac{(M - \langle M \rangle_H)^2}{2\sigma(H)^2} \right),
\]

where \( \langle M \rangle_H \) and \( \sigma(H) \) are the average and the standard deviation of mass obtained for all individuals of height \( H \). Note that the standard deviation \( \sigma_X \) for a stochastic variable \( X \) is related to the first and second moments by \( \sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2 \). A particular feature in the present context is that the distribution of mass for a given height can approximately be characterized by a constant standard deviation \( \sigma \), since in practice it turns out that \( \sigma(H) \) is only weakly dependent on \( H \) in the vicinity of \( \langle H \rangle \) (see figure 1(d)). Thus equation (3) can approximately be reduced to

\[
P(M|H) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( -\frac{(M - \langle M \rangle_H)^2}{2\sigma^2} \right).
\]

Another particular feature of the mass relation is that the average mass for a given height, \( \langle M \rangle_H \), monotonously increases with height. We can use this one-to-one correspondence by changing the variable in equation (2) from \( H \) to \( \langle M \rangle_H \). The height distribution in terms of \( \langle M \rangle_H \) is then just \( P_H(H(\langle M \rangle_H)) \). This means that there exists a precise relation between the three distributions (equations (1), (2) and (4)) given by

\[
P_M(M) = \int \, d\langle M \rangle_H \, P(M|\langle M \rangle_H) \, P_H(H(\langle M \rangle_H)) \, \frac{dH(\langle M \rangle_H)}{d\langle M \rangle_H}.
\]

Another crucial feature of the data is that \( \langle M \rangle_H \) to some approximation is described by the power-law relation

\[
\langle M \rangle_H = AH^{\gamma}.
\]

This is the allometric relation in focus and discussed in this paper. Here \( A \) and \( \gamma \) are two constants. The constant \( A \) can be expressed as \( A = \langle M \rangle_H H^{-\gamma} = \langle M \rangle_H \langle H \rangle^{-\gamma} \). However, for Gaussian distributions \( \langle H \rangle \) corresponds to the peak position of the height distribution and, since the individuals in this peak also to good approximation have average mass, it follows that \( \langle M \rangle_H \approx \langle M \rangle \). In the following, we will consequently use the simplified estimate \( A = \langle M \rangle \langle H \rangle^{-\gamma} \). The argument for \( P_H \) in equation (2) is

\[
H(\langle M \rangle_H) = \left( \frac{\langle M \rangle_H}{A} \right)^{\frac{1}{\gamma}} = \langle H \rangle \left( \frac{\langle M \rangle_H}{\langle M \rangle} \right)^{\frac{1}{\gamma}}.
\]

Close to the peak of this distribution at \( \langle M \rangle_H \) we can use the linear approximation

\[
H(\langle M \rangle_H) = \langle H \rangle \left( \frac{\langle M \rangle_H}{\langle M \rangle} \right)^{\frac{1}{\gamma}} \approx \langle H \rangle \left( 1 + \frac{1}{\gamma} \left( \langle M \rangle_H - \langle M \rangle \right) \right).
\]

Inserting this approximation into the relation

\[
P_H(\langle M \rangle) = P_H(H(\langle M \rangle_H)) \frac{dH(\langle M \rangle_H)}{d\langle M \rangle_H}
\]
leads to the Gaussian distribution

\[ P_H((M)) = \frac{1}{\sqrt{2\pi \sigma_H^2}} \exp\left( -\frac{((M)_H - (M_H))^2}{2\sigma_H^2} \right). \]  

(9)

Using equation (9) together with equation (4) means that the right-hand side of equation (5) becomes a convolution of two Gaussians. Since the convolution of two Gaussians with standard deviations \( \sigma_1 \) and \( \sigma_2 \) becomes a Gaussian with standard deviation \( \sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2} \), it follows that \( \sigma_M^2 = \sigma^2 + \sigma_H^2 \gamma^2 / (\langle M \rangle)^2 \) or equivalently

\[ \gamma = \frac{\langle H \rangle \sqrt{\sigma_M^2 - \sigma^2}}{\sigma_H} = \frac{\sqrt{\sigma_M^2 - \sigma^2}}{\sigma_H}, \]  

(10)

where we have introduced the normalized standard deviations \( \tilde{\sigma}_M = \sigma_M / \langle M \rangle, \tilde{\sigma} = \sigma / \langle M \rangle \) and \( \tilde{\sigma}_H = \sigma_H / \langle H \rangle \). Equation (10) is the central relation in the present investigation and shows that \( \gamma \) can be approximately obtained from the three dimensionless numbers \( \tilde{\sigma}_M, \tilde{\sigma} \) and \( \tilde{\sigma}_H \), which measures the random spread of the data in units of, respectively, the average mass and height of the individuals.

Also note that \( \gamma \) given by equation (10) is what you get when an allometric relation is used as an ansatz. It does not a priori say anything about whether or not an allometric relation is a good approximation of the data.

3. Comparison with data

In the light of the above theoretical underpinning we return to the data for 25000 18 year old children [13]. One notes in figure 1 that both the height and the mass distributions to good approximations are Gaussians for this data set. The average mass for a child is \( \langle M \rangle \approx 57.7 \) kg and height \( \langle H \rangle \approx 172.7 \) cm. The standard deviations are \( \sigma_M \approx 5.3 \) kg and \( \sigma_H \approx 4.8 \) cm. Figure 1(c) shows that also the distributions of mass for given heights are Gaussians and figure 1(d) shows that the standard deviation \( \sigma (H) \) in equation (3) is constant in a broad range of \( H \) around \( \langle H \rangle \), so that equation (4) gives a very good description. As shown in section 2, under these conditions the relation given by equation (10) applies. This relation states that if there is a power-law relation between average mass and height, \( \langle M \rangle_H = A H^\gamma \), then the best prediction for the given information is

\[ \langle M \rangle_H = \langle M \rangle \left( \frac{H}{\langle H \rangle} \right)^{\frac{\sqrt{\tilde{\sigma}_M^2 - \tilde{\sigma}^2}}{\tilde{\sigma}_H}}. \]  

(11)

Table 1 gives the average height and mass (in cm and kg, respectively) together with the three normalized standard deviations for the statistical distributions: \( \tilde{\sigma}_H, \tilde{\sigma}_M \) and \( \tilde{\sigma} \). The resulting power-law exponent predicted by \( \gamma_h = \frac{\sqrt{\tilde{\sigma}_M^2 - \tilde{\sigma}^2}}{\tilde{\sigma}_H} \), as well as \( \gamma_ex \) obtained by direct fitting to the data (see figure 2(a)), is also listed. The agreement between \( \gamma_h \) and \( \gamma_ex \) is very precise and confirms that there really exists a relation between the spreads and the power-law exponent. The question is what it implies.

In order to get an idea of what this means, we note that if \( \sigma = 0 \) then there exists a one-to-one function between \( M \) and \( H \) and according to equation (10) we obtain

\[ \gamma = \frac{\tilde{\sigma}_M}{\tilde{\sigma}_H}. \]  

(12)
Table 1. Summary of data for 25 000 18 year old children from Hong Kong [13] (first row) and for 11 300 13.5–19 year old Swedish children [14] (second row). The average height \( \langle H \rangle \) and the average mass \( \langle M \rangle \) are in units of cm and kg. The normalized dimensionless standard deviations for the height \( \bar{\sigma}_H \), the mass \( \bar{\sigma}_M \), and the mass distribution at average height \( \bar{\sigma} \) are listed. The theoretical prediction \( \gamma_{\text{th}} \) from equation (10) and \( \gamma_{\text{ex}} \), obtained from the least-squares fit to the data presented in figures 2(a) and (c), are in good agreement. \( \gamma_{\sigma=0} \) from equation (10) with \( \sigma = 0 \) and \( \gamma_{\text{ordered}} \), obtained from the least-squares fit to the data in figures 2(b) and (d), also agree with each other. Note the close agreements between fitted and predicted values of the allometric exponents \( \gamma \) in all cases.

<table>
<thead>
<tr>
<th></th>
<th>( \langle H \rangle )</th>
<th>( \langle M \rangle )</th>
<th>( \bar{\sigma}_H )</th>
<th>( \bar{\sigma}_M )</th>
<th>( \bar{\sigma} )</th>
<th>( \gamma_{\text{th}} )</th>
<th>( \gamma_{\text{ex}} )</th>
<th>( \gamma_{\sigma=0} )</th>
<th>( \gamma_{\text{ordered}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hong Kong</td>
<td>172.72</td>
<td>57.68</td>
<td>0.0280</td>
<td>0.0912</td>
<td>0.079</td>
<td>1.63</td>
<td>1.63</td>
<td>3.26</td>
<td>3.29</td>
</tr>
<tr>
<td>Sweden</td>
<td>170.48</td>
<td>60.52</td>
<td>0.0573</td>
<td>0.1862</td>
<td>0.130</td>
<td>2.33</td>
<td>2.35</td>
<td>3.25</td>
<td>3.30</td>
</tr>
</tbody>
</table>

Figure 3. Plot of the \( \gamma_{\text{th}} \) obtained from equation (10). The standard deviations \( \bar{\sigma}_M \) and \( \bar{\sigma}_H \) from the children data are used to plot \( \gamma_{\text{th}} \) as a function of the standard deviation ratio \( \sigma / \bar{\sigma}_H \). Note that, since \( \gamma_{\sigma=0} = \bar{\sigma}_M / \bar{\sigma}_H \) is almost identical for the Hong Kong and Swedish data, both predictions are obtained from the same curve. The black dot represents the predicted maximum value \( \gamma_{\sigma=0} \) for \( \sigma = 0 \). The cross and triangle represent the prediction of \( \gamma_{\text{th}} \) for, respectively, the Hong Kong and Swedish children data.

Changing \( \bar{\sigma} \) in the Hong Kong children data to \( \bar{\sigma} = 0 \) changes the prediction for \( \gamma_{\text{th}} \) from 1.63 to 3.26 (compare table 1 and figure 3). We can test this prediction against the children data by re-ordering so that the children are assigned masses which strictly follow the heights of the
and figures gives various Pearson’s correlation coefficient $r$ between the height $H$ and various quantities ($M, M/H^2, M/H^3, M/H^{7/6}$ and $M/H^{γ_{ex}}$) are computed for Hong Kong children [13] (first row) and Swedish children [14] (second row). The height–mass correlation ($r$ for $H$ versus $M$) is of course positively significant in both cases. For the Hong Kong data in the first row, the correlations between $H$ and $M/H^3$, and between $H$ and $M/H^2$ are negative, implying that the exponents 2 and 3 are overestimations. In contrast, $M/H^{γ_{ex}}$ and $M/H^{7/6}$ exhibit neutral correlation with $H$, which shows that our estimation $γ_{th} ≈ 1.63$ describes data much better than the conventional BMI value $γ = 2$. Likewise in the second row for the Swedish data, the correlations between $H$ and $M/H^3$, and between $H$ and $M/H^2$ are, respectively, negative and positive, implying an exponent between 2 and 3. This is again in agreement with our analysis and theory.

<table>
<thead>
<tr>
<th>Application</th>
<th>$H$ versus $M$</th>
<th>$H$ versus $M/H^3$</th>
<th>$H$ versus $M/H^2$</th>
<th>$H$ versus $M/H^{7/6}$</th>
<th>$H$ versus $M/H^{γ_{ex}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Hong Kong)</td>
<td>0.50</td>
<td>−0.43</td>
<td>−0.12</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>(Sweden)</td>
<td>0.64</td>
<td>−0.21</td>
<td>0.14</td>
<td>0.03</td>
<td>0.02</td>
</tr>
</tbody>
</table>

children. For this re-ordered data, $σ$ is indeed zero and as shown in figure 2(b) the slope for this re-ordered data is indeed again close to the prediction. This gives a direct demonstration of the connection between spread and power-law exponent.

Table 2 gives various Pearson’s $r$-coefficients computed for various pairs of quantities. For the Hong Kong children data (first row of the table), the height–mass correlation ($H$ versus $M$) is significantly positive, implying that in general the taller child has the heavier mass. It is to be noted that the correlations for $M/H^3$ and $M/H^2$ deviate much from zero, while $M/H^{γ_{ex}}$ and $M/H^{7/6}$ exhibit neutral correlation with the height, suggesting that our estimation $γ_{th} ≈ 1.63$ describes data much better than the conventionally used BMI value $γ = 2$.

In order to rule out that there is anything accidental or fortuitous about the results presented, we have investigated a second data set in the same way. This second data set gives the height and mass for Swedish children between 13.5 and 19 years old (more precisely between 5000 and 7000 days old containing in total 11 327 data points) [14]. The results are presented in figures 2(c) and (d) with parameters given in the second rows of tables 1 and 2. From table 1, one can see that the average height and mass for these two data sets are roughly the same. However, since the Swedish children data span over a longer age period than the Hong Kong data, the standard deviations for height, $σ_H$, and mass, $σ_M$, are larger by about a factor of 2. This is of course because children during a longer period grow more. Yet the ratio between the standard deviations, $σ_M/σ_H$, is closely equal for the two data sets (3.26 and 3.25, respectively.) This ratio is in fact $γ_{σ=0}$ and, as seen from table 1 and figures 2(b) and (d), $γ_{σ=0}$ gives very precise estimates of the allometric exponents $γ_{ordered}$. The fact that the exponents $γ_{ordered}$ are very nearly the same for the two data sets suggests that, in this particular aspect, children from Hong Kong and Sweden are very similar. Also for the Swedish data set, there is good agreement between the experimental allometric exponent $γ_{ex}$ and the prediction $γ_{th}$ from equation (10) (compare figures 2(c) and table 1). However, there is a significant difference between the allometric

exponents $\gamma_{\text{ex}}$ for the two data sets: $\gamma_{\text{ex}} = 1.63$ and 2.35 for, respectively, Hong Kong and Swedish children. The close agreement between $\gamma_{\text{ex}}$ and the prediction $\gamma_{\text{th}}$ for both data sets suggests that the difference in the value of the allometric exponent $\gamma_{\text{ex}}$ can be attributed to a relatively larger spread in weight for children of average height for Hong Kong compared to Swedish data. From this point of view, it is a sampling difference rather than some difference in trait of a Hong Kong or Swedish individual. A possible explanation could be that compared to Sweden, Hong Kong has for a long time been a human hotspot with an influx of people of great variety from both genetical and cultural backgrounds, and this has resulted in a relatively larger spread of weight for a given height in a particular age interval.

4. Discussion

The implication of these results becomes clearer when comparing figures 2(a) and (b). Both represent data with the same two Gaussian distributions for mass and height given in figure 1. The difference is that the data in figure 2(a) also have a spread of mass for individuals with a given height, as shown in figure 1. For the artificial data in figure 2(b), there is no such spread. The data in figure 2(b) represent a true allometric relation between mass and height of the form $M \propto H^{\gamma}$: as soon as you pick a person with a certain height, you also to very good approximation know his mass. Since $\gamma$ for this artificial data set is 3.29, it means either that these artificial people all have the same shape but get a bit denser with increasing height, or that they just become somewhat disproportionally fatter with increasing height. The point is that you in this case can relate the allometric exponent to some property of the individual. However, for the real data in figure 2(a) this becomes more problematic. This is because for a given height the individuals have a random mass distributed about the mean, as shown in figure 1(c). This random spread can have a multitude of different causes, such as availability of food, climate, diseases, genetics, etc. Different individuals are affected by this multitude of causes in different ways.

An alternative way of describing this is as follows. Suppose you have a data set like the one discussed in this paper and suppose that each individual can be characterized by the allometric relation $M \propto \langle H \rangle_M^{\gamma_0}$, where $\langle H \rangle_M$ is the average height for individuals of mass $M$. This means that if you pick an individual with mass $M$, then you know that his most likely height is given by the allometric relation with $\gamma_0$. If in addition there is no randomness in the height, then you for certain know his height and the result is given by figure 2(b). But if there is a randomness in the height caused by many causes, so that the probability of the height for an individual is described by a Gaussian probability distribution, then the allometric exponent becomes smaller and you can end up with something like figure 2(a) instead. The difference is that figure 2(a) describes the collective data set, whereas figure 2(b) corresponds to an allometric relation on an individual basis.

In figures 2(a) and (b), it is precisely the random spread which causes the decrease of the allometric exponent from 3.29 to 1.63. So you cannot any longer associate the allometric exponent $\gamma$ with some unique growth property of the individuals. It is rather like that the spread in figure 1(c) just tells you how much the masses and heights for individuals are random and uncorrelated. This is also reflected by the prediction given by equation (10) which decreases from its maximum value $\gamma = \frac{\bar{\sigma}_M}{\bar{\sigma}_H}$ to zero with increasing random spread $\bar{\sigma}$, as illustrated in figure 3.
Some further insight into this is given by the comparison between the Hong Kong children and the Swedish children. The allometric exponent for the Hong Kong children is significantly smaller than for the Swedish children. According to the present analysis, this difference can be traced to the difference in the relatively larger spread in mass for a given height in the case of the Hong Kong data set. One possible explanation for this difference in spread could be a sampling difference. Compared to Sweden, Hong Kong has been a historical hotspot leading to a greater variety of people from both genetical and cultural backgrounds. Such greater variety is also likely to cause a larger variety in weight for a given height.

The present analysis is quite general and its implication is likely to have a wider range of applicability within allometric relations than just to the illustrative example of the mass–height relation for humans discussed here: it cautions against attributing too much specific cause to the precise value of an allometric exponent. The crucial point is that the allometric exponent for an individual is, because of randomness, not the same as the allometric exponent of the collective data set.

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