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Fully covariant radiation force on a polarizable particle

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Abstract. The electromagnetic force on a polarizable particle is calculated in a covariant framework. Local equilibrium temperatures for the electromagnetic field and the particle's dipole moment are assumed, using a relativistic formulation of the fluctuation–dissipation theorem. Two examples illustrate radiative friction forces: a particle moving through a homogeneous radiation background and above a planar interface. Previous results for arbitrary relative velocities are recovered in a compact way.

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1. Introduction

Friction is ubiquitous in everyday life and omnipresent in almost every mechanical system, but still very difficult to grasp at the level of elementary forces. For example, internal friction in fluids (viscosity) or contact friction between two solids can be attributed to electromagnetic interactions who couple individual atoms. Building a bridge from the microscopic realm to the macroscopic world is a challenge because of the huge diversity at the atomistic level: surface reconstruction, adsorbates, roughness, etc. A common feature of friction forces, however, is the conversion of directed motion into thermal motion or material excitations. This feature can be studied with the help of simple models at the fundamental level. In recent years, studies of moving objects have been developed from first principles: consider, for example, a flat surface separated by vacuum from another body in constant parallel motion. On a length scale of a few nanometres, the electromagnetic interactions between the two can be characterized by a few macroscopic parameters (refractive index, conductivity, surface impedance, etc). In this framework, consistent quantum field theories have been formulated [1–3] that in principle can take full advantage of Lorentz invariance. An example is the reflection of light from a moving plate that is evaluated by transforming the incident field into the plate’s rest (or co-moving) frame and back. In the same spirit, this co-moving frame is a natural candidate for local thermodynamic equilibrium, at least for macroscopic objects. (For a discussion on the transformation law of temperature and its dependence on the definition of relative motion, see [4].) From the viewpoint of relativistic thermodynamics [5, 6], the two situations of bodies in relative motion or fixed at different temperatures indeed represent very similar non-equilibrium settings.

In this paper, we construct a fully covariant formulation of radiation-induced forces on a small neutral particle. We consider stationary, non-equilibrium motion at arbitrary speed parallel to a planar surface. Local temperatures are assigned to the particle and the surface, in their respective rest frames. Covariance is maintained from the beginning and expresses in a compact form the transformation properties of the electromagnetic field, the material polarization and

the particle's dipole moments. The fluctuation–dissipation theorem that determines the spectra of thermal fluctuations is formulated in local (co-moving) frames of particle or surface, respectively, which provides a natural link to relativistic thermodynamics. We check our general expression for the radiation force by specializing to motion through the blackbody radiation field and to a particle above a dielectric surface. In the two cases, we use different gauges, but come to results fully consistent with previous work [7, 8]. We believe that the present formulation is useful because it is compact and flexible, and illustrates the assumptions behind the macroscopic quantum field theory in a physically transparent way. This may pave the way to interpret electromagnetic friction phenomena that have attracted some interest in recent years [9–13].

The outline is as follows: the covariant framework is constructed in section 2, resulting in the fluctuation–dissipation theorem for the electromagnetic field and the particle's dipole moment in section 2.3. The force is split in two contributions that can be attributed to radiation reaction and vacuum fluctuations [14, 15], both are given in general form in section 3. We specialize to blackbody friction in section 3.3 and to radiation forces above a surface in section 3.4.

2. Covariant framework

2.1. Polarization and force density

In this part, we introduce a covariant expression for the force acting on a polarizable body. We start with some basic identities from electrodynamics: polarization and magnetization fields $\mathbf{P}$ and $\mathbf{M}$ are defined from

$$\rho = -\nabla \cdot \mathbf{P}, \quad \mathbf{j} = \partial_t \mathbf{P} + \nabla \times \mathbf{M}, \quad (1)$$

where ρ is the charge density and $\mathbf{j}$ the spatial current density. In terms of the four vector $(j^\mu) = (\rho, \mathbf{j}) = (\rho, j^1, j^2, j^3)$:

$$j^\mu = \partial_\nu M^{\nu\mu}, \quad (2)$$

where $M^{\nu\mu}$ is the polarization tensor with the matrix representation

$$(M^{\nu\mu}) = \begin{pmatrix} 0 & P^1 & P^2 & P^3 \\ -P^1 & 0 & M^3 & -M^2 \\ -P^2 & -M^3 & 0 & M^1 \\ -P^3 & M^2 & -M^1 & 0 \end{pmatrix}. \quad (3)$$

Its antisymmetry ensures charge conservation. The electromagnetic force density

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} \quad (4)$$

is part of the four-vector

$$f_\mu = F_{\mu\nu} j^\nu, \quad (5)$$

where the field strength (Faraday) tensor has components

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}. \quad (6)$$

Pulling these relations together, we can write the covariant force density in terms of the field and polarization tensors

$$f_\mu = F_{\mu\nu} \partial_\kappa M^{\kappa\nu}. \quad (7)$$

The total force on a body is found by integrating over a volume containing the body. Up to surface terms in this integral, equation (7) is equivalent to the Einstein–Laub formula for the force on polarizable matter [16, 17]. There are some experiments where this formula is not appropriate, as effects like electrostriction must be taken into account; for a discussion, see [18]. A recent suggestion that the Lorentz force equation (4) is not compatible with special relativity [19] has been met with criticism².

In the following, we focus on the situation that the force f_μ (equation (7)) arises from fluctuations of the field and of the body’s polarization. In the spirit of perturbation theory, we split the polarization, for example, into

$$M^{\mu\nu}(x) = M_{\text{fl}}^{\mu\nu}(x) + M_{\text{in}}^{\mu\nu}(x), \quad (8)$$

where the first term ‘fl’ describes the free fluctuations of the electric and magnetic dipole moments, while the second term ‘in’ (induced) gives their response to a field. This split has a long tradition in quantum optics, related to the distinction between ‘vacuum fluctuations’ and ‘radiation reaction’, see for example Milonni’s book [20]. In [14], the two terms of equation (8) arise from the homogeneous plus particular solution to the equations of motion for the polarization field. An alternative justification can be found in the [appendix](#). A similar split for the fields yields an average force density

$$f_\mu(x) = \langle F_{\mu\nu}^{\text{fl}}(x) \partial_\sigma M_{\text{in}}^{\sigma\nu}(x) \rangle + \langle F_{\mu\nu}^{\text{in}}(x) \partial_\sigma M_{\text{fl}}^{\sigma\nu}(x) \rangle. \quad (9)$$

In the lowest non-vanishing order of perturbation theory (see the [appendix](#)), the operators appearing here evolve freely. In particular, the field and body observables can be averaged with respect to local thermal equilibrium (temperatures T_F and T_A). To make the two terms real-valued, the operator products must be symmetrized (see after equation (21)), as discussed in [14]. In the following sections, we spell out the linear response functions and the fluctuation spectra, respectively.

2.2. Response functions

2.2.1. Polarizability. For simplicity, we focus on a pointlike particle (‘atom’) at position $\mathbf{x}_A$ with an electric dipole polarizability which is often the dominant response. The particle carries an electric dipole moment that responds to the electric field vector

$$\mathbf{d}(t) = \alpha \mathbf{E}(t, \mathbf{x}_A), \quad (10)$$

where α is called the polarizability. It is actually frequency dependent (dispersion) and can be calculated from a dipole correlation function, according to linear response theory:

$$\frac{i}{\hbar} \langle [d_i(t), d_j(t')] \rangle \Theta(t - t') = \int d\omega e^{-i\omega(t-t')} \alpha_{ij}(\omega). \quad (11)$$

This formula holds in a frame where the atom is stationary. We assume in the following an isotropic response, $\alpha_{ij}(\omega) = \delta_{ij} \alpha(\omega)$.

² See ‘The Net Advance in Physics’ at <http://web.mit.edu/redingtn/www/netadv/srLzMa.html>.

Introducing the four-velocity u^μ tangent to the particle's worldline x_A , we define the covariant polarization density

$$M^{\mu\nu}(x) = [u^\mu(t)d^\nu(t) - d^\mu(t)u^\nu(t)]\delta(\mathbf{x} - \mathbf{x}_A(t)). \quad (12)$$

The component of d^μ parallel to u^μ can be chosen arbitrarily, it drops out from this construction. The atom responds to the electric field in the co-moving frame so that the covariant form of equation (10) becomes

$$d^\mu(x_A) = \alpha g^{\mu\kappa} F_{\kappa\lambda}(x_A) u^\lambda, \quad (13)$$

where $g^{\mu\kappa}$ is the metric tensor. This four-vector d^μ is indeed perpendicular to u^μ . A similar construction can be given for the magnetic polarizability. Making the approximation that the atom's worldline is inertial (constant u^μ), we get in Fourier space ($k \cdot x = k_\mu x^\mu$)

$$M^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} M^{\mu\nu}(k) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4h}{(2\pi)^4} e^{-ik \cdot x} \alpha^{\mu\nu\kappa\lambda}(k, h) F_{\kappa\lambda}(h), \quad (14)$$

$$\alpha^{\mu\nu\kappa\lambda}(k, h) = 2\pi \delta(u \cdot (k - h)) \alpha(u \cdot h) \{u^\mu g^{\nu\kappa} u^\lambda + u^\nu g^{\mu\lambda} u^\kappa\} e^{i(k-h) \cdot x_A}. \quad (15)$$

Here we have restored dispersion via the argument of the polarizability: the quantity $\omega'_A = u \cdot h$ plays the role of the frequency of a wavevector component h_μ of the applied field, as seen in the atom's rest frame. This describes in particular the first- and second-order Doppler shifts. The dipole radiates at the same frequency as the applied field (linear response), hence the $\delta(u \cdot (k - h))$. We work in this paper with retarded response functions: the dipole responds to the electric field in its past and therefore, $\alpha(\omega'_A)$ is analytic in the upper half-plane of complexified frequencies ω'_A . For the radiation force (9), equation (14) gives the polarization $M_{\text{in}}^{\mu\nu}$ induced by the fluctuating field $F_{\kappa\lambda}^{\text{fl}}$.

2.2.2. Green function. It is well known that for the electromagnetic field, the vector potential created by a source current can be found with the help of a Green function

$$A_\mu(x) = \int d^4y \mathcal{G}_{\mu\nu}(x, x') j^\nu(x'). \quad (16)$$

In free space, for example, we have, adopting the Feynman gauge ($k^2 = k \cdot k = \omega^2 - \mathbf{k}^2$, we set $c = \varepsilon_0 = 1$)

$$\mathcal{G}_{\mu\nu}(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-x')} \frac{-g_{\mu\nu}}{k^2 + i0 \text{sgn } \omega}. \quad (17)$$

In the general case, translation invariance in time or space can only hold under special circumstances, so we adopt the Fourier expansion

$$\mathcal{G}_{\mu\nu}(x, x') = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4h}{(2\pi)^4} e^{-i(k \cdot x - h \cdot x')} \mathcal{G}_{\mu\nu}(k, -h). \quad (18)$$

If we represent the current density in terms of the polarization $M^{\kappa\lambda}$ (equation (2)), we get the field amplitude from

$$F_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4h}{(2\pi)^4} e^{-ik \cdot x} \mathcal{G}_{\mu\nu\kappa\lambda}(k, -h) M^{\kappa\lambda}(h), \quad (19)$$

$$\mathcal{G}_{\mu\nu\kappa\lambda}(k, -h) = k_\mu \mathcal{G}_{\nu\kappa}(k, -h) h_\lambda + k_\nu \mathcal{G}_{\mu\lambda}(k, -h) h_\kappa, \quad (20)$$

where the antisymmetry of $M^{\kappa\lambda}$ was used. Note the formal analogy to the fourth rank polarization tensor $\alpha^{\mu\nu\kappa\lambda}$ (equation (15)). For the radiation force (9), we shall use equation (19) to express the field $F_{\mu\nu}^{\text{in}}$ radiated by the fluctuating polarization $M_{\text{fl}}^{\kappa\lambda}$.

2.3. Fluctuation spectra

To evaluate the radiation force (9), we need correlation functions of the fluctuating polarization and fields. These are provided by the fluctuation–dissipation theorem, assuming thermal states for atom and field.

2.3.1. Dipole and polarization. The non-relativistic form of the dipole correlation function is [21, 22] (i, j are spatial components)

$$\langle d^i(\omega), d^j(\omega') \rangle_A = 2\pi \hbar \delta(\omega + \omega') \delta^{ij} \coth \left(\frac{\hbar \omega}{2k_B T_A} \right) \text{Im} \alpha(\omega), \quad (21)$$

where the operator product is symmetrized: $\langle B, C \rangle_A = \frac{1}{2} \langle BC + CB \rangle_A = \frac{1}{2} \text{tr} \{ \rho_A (BC + CB) \}$ with the equilibrium density operator ρ_A . $\text{Im} \alpha(\omega)$ describes the spectral distribution of the atomic oscillator strength that may also depend on the atomic temperature T_A . In the relativistic formulation, the inverse temperature becomes a time-like four-vector $\beta_A^\mu = (\hbar/k_B T_A) u^\mu$ tangent to the atom's worldline [5, 6]. We recover equation (21) in the spacelike hypersurface perpendicular to u^μ when the following correlation function for the polarization field $M^{\mu\nu}(x)$ localized on the atom is assumed (written in four-dimensional Fourier space)

$$\langle M_{\text{fl}}^{\mu\nu}(k), M_{\text{fl}}^{\kappa\lambda}(h) \rangle = 2\pi \hbar \delta(u \cdot (k+h)) \Gamma^{[\mu\nu][\kappa\lambda]} \coth \left(\frac{\beta_A \cdot k}{2} \right) \text{Im} \alpha(u \cdot k) e^{i(k+h) \cdot x_A}, \quad (22)$$

$$\Gamma^{[\mu\nu][\kappa\lambda]} = u^{[\mu} g^{\nu][\kappa} u^{\lambda]}. \quad (23)$$

Here the square brackets are denoting odd combinations of paired indices

$$u^{[\mu} g^{\nu][\kappa} u^{\lambda]} = u^\mu g^{\nu\kappa} u^\lambda - u^\nu g^{\mu\kappa} u^\lambda - u^\mu g^{\nu\lambda} u^\kappa + u^\nu g^{\mu\lambda} u^\kappa \quad (24)$$

and ensure the antisymmetry of the fluctuating polarization tensor. For a non-inverted atom, the absorption spectrum $\sim \omega \text{Im} \alpha(\omega)$ is positive for all frequencies; this property is inherited by the correlation spectrum (22).

2.3.2. Field correlations. The correlations of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are well known at thermal equilibrium in the rest frame [22, 23]:

$$\langle E_i(\omega, \mathbf{x}), E_j(\omega', \mathbf{x}') \rangle_F = i\hbar \pi \delta(\omega + \omega') \coth \frac{\beta_F \omega}{2} \omega^2 [\mathcal{G}_{ij}(\omega; \mathbf{x}, \mathbf{x}') - \mathcal{G}_{ji}^*(\omega; \mathbf{x}', \mathbf{x})], \quad (25)$$

$$\langle E_i(\omega, \mathbf{x}), B_j(\omega', \mathbf{x}') \rangle_F = -\hbar \pi \delta(\omega + \omega') \coth \frac{\beta_F \omega}{2} \omega \epsilon_{klj} \partial'_k [\mathcal{G}_{il}(\omega; \mathbf{x}, \mathbf{x}') - \mathcal{G}_{li}^*(\omega; \mathbf{x}', \mathbf{x})], \quad (26)$$

$$\langle B_i(\omega, \mathbf{x}), B_j(\omega', \mathbf{x}') \rangle_F = i\hbar \pi \delta(\omega + \omega') \coth \frac{\beta_F \omega}{2} \epsilon_{kli} \epsilon_{mnj} \partial_k \partial'_m [\mathcal{G}_{ln}(\omega; \mathbf{x}, \mathbf{x}') - \mathcal{G}_{nl}^*(\omega; \mathbf{x}', \mathbf{x})], \quad (27)$$

where now β_F is the inverse field temperature. In the presence of macroscopic bodies, we assume that the field relaxes to their temperature. The Green tensor in equations (25)–(27) is calculated from the Kubo formula

$$\mathcal{G}_{ik}(\omega; \mathbf{x}, \mathbf{x}') = -\frac{i}{\hbar} \int_{t'}^{\infty} dt e^{i\omega(t-t')} \langle A_i(t, \mathbf{x}) A_k(t', \mathbf{x}') - A_k(t', \mathbf{x}') A_i(t, \mathbf{x}) \rangle_F \quad (28)$$

in the Dzyaloshinskii gauge (zero scalar potential). Note that this definition assumes that the field is stationary in the chosen frame, i.e. correlations $\langle A_i(x) A_k(x') \rangle_F$ of the vector potential depend only on the time difference $t - t'$. The integration is over retarded times $t > t'$ in this frame. It is a fundamental result of linear response theory that the correlation function of equation (28) also provides the Green function for the vector potential, i.e. the kernel in equation (16).

We proceed to combine the field spectra into a covariant formulation. In a first step, we complete equations (25)–(27) by performing a spatial Fourier transformation, yielding ($h = (\omega', \mathbf{h})$)

$$\langle E_i(k), E_j(h) \rangle_F = i\hbar\pi\delta(\omega + \omega') \coth \frac{\beta_F\omega}{2} \omega^2 [\mathcal{G}_{ij}(\omega, \mathbf{k}, \mathbf{h}) - \mathcal{G}_{ji}^*(\omega, -\mathbf{h}, -\mathbf{k})], \quad (29)$$

$$\langle E_i(k), B_j(h) \rangle_F = -i\hbar\pi\delta(\omega + \omega') \coth \frac{\beta_F\omega}{2} \epsilon_{jkl}\omega h_k [\mathcal{G}_{il}(\omega, \mathbf{k}, \mathbf{h}) - \mathcal{G}_{li}^*(\omega, -\mathbf{h}, -\mathbf{k})], \quad (30)$$

$$\langle B_i(k), B_j(h) \rangle_F = -i\hbar\pi\delta(\omega + \omega') \coth \frac{\beta_F\omega}{2} \epsilon_{ikl}\epsilon_{jmn} k_k h_m [\mathcal{G}_{ln}(\omega, \mathbf{k}, \mathbf{h}) - \mathcal{G}_{nl}^*(\omega, -\mathbf{h}, -\mathbf{k})] \quad (31)$$

using for the Green tensor the expansions

$$\mathcal{G}_{ij}(\omega, \mathbf{x}, \mathbf{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{d^3h}{(2\pi)^3} \mathcal{G}_{ij}(\omega, \mathbf{k}, \mathbf{h}) e^{i(\mathbf{k}\cdot\mathbf{x} + \mathbf{h}\cdot\mathbf{x}')}, \quad (32)$$

$$\mathcal{G}_{ji}^*(\omega, \mathbf{x}', \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{d^3h}{(2\pi)^3} \mathcal{G}_{ji}^*(\omega, -\mathbf{h}, -\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} + \mathbf{h}\cdot\mathbf{x}')}. \quad (33)$$

In the frame where the field is in equilibrium and in the Dzyaloshinskii gauge, we construct the covariant Green tensor $\mathcal{G}_{\mu\nu}(k, h)$ from

$$\mathcal{G}_{ij}(k, h) = 2\pi\delta(\omega + \omega') \mathcal{G}_{ij}(\omega, \mathbf{k}, \mathbf{h}), \quad \mathcal{G}_{0\nu} = \mathcal{G}_{\mu 0} = 0. \quad (34)$$

One can check straightforwardly that the field correlations (29–31) are equivalent to the following spectrum of the Faraday tensor:

$$\langle F_{\mu\nu}^{\text{fl}}(k), F_{\kappa\lambda}^{\text{fl}}(h) \rangle_F = \frac{i\hbar}{2} \coth \frac{\beta_F \cdot k}{2} [k_{[\mu} \mathcal{G}_{\nu]\kappa}(k, h) h_{\lambda]} - h_{[\lambda} \mathcal{G}_{\kappa] \nu}^*(-h, -k) k_{\mu}], \quad (35)$$

where the brackets denote again antisymmetrized pairs of indices (see equation (24)). As in equation (22), we have expressed the inverse temperature in terms of the four-vector $\beta_F^\mu = (\hbar/k_B T_F) u_F^\mu$, where u_F^μ is the velocity of the field's equilibrium frame.

With equation (35), the fluctuation–dissipation theorem for the fields is now in manifestly covariant form. We emphasize that the only input parameters required are the photon propagator and the four-velocity of the field's equilibrium frame. A general observer notices the Doppler shift via the contraction $u_F^\mu k_\mu$ and a net energy flow (Poynting vector) parallel to u_F^μ . A general proof that this expression is also gauge-invariant will be given elsewhere. We show below that with a Green tensor $\mathcal{G}_{\mu\nu}$ in the generalized Coulomb gauge [2], we recover the spectrum of the radiation force in the presence of a macroscopic medium discussed in [8, 24]. We have also checked that in free space, the Green tensor in the Feynman gauge (equation (17)) reproduces

from the fluctuation–dissipation theorem (35) the well-known expressions for the electric and magnetic field spectra.

As a side remark, we note that the fluctuation–dissipation theorems (22), (35) display tensor structures that are manifestly antisymmetric in the double indices carried by the polarization and electromagnetic fields. The response functions written in equations (15) and (20) are only apparently of lower symmetry. They do preserve the parity of the index pairs, however, so that an antisymmetric field induces an antisymmetric polarization, for example. But since the response functions are read off from the linear relations between antisymmetric quantities (see, e.g., equation (19)), their symmetric part actually remains undetermined. In other words, we can also use in the fourth rank polarizability (15) a tensor with the structure

$$\frac{1}{4} \{ u^{[\mu} g^{\nu][\kappa} u^{\lambda]} + u^{[\nu} g^{\mu][\lambda} u^{\kappa]} \} = \frac{\Gamma^{[\mu\nu][\kappa\lambda]}}{2}, \quad (36)$$

so that the structural simplicity in the fluctuation–dissipation relation is preserved even in the covariant formulation. (For a simplification of the field spectra, see the planar geometry discussed below.)

3. Calculation of the force

The two terms in the radiation force density (9) yield a force acting on the atom, combining the response functions with the fluctuation–dissipation theorems for field and polarization,

$$\begin{aligned} F &= F^{(1)} + F^{(2)} \\ &= \int d^3x \{ f^{(1)}(x) + f^{(2)}(x) \}, \end{aligned} \quad (37)$$

$$f_{\mu}^{(1)}(x) = -i \int \frac{d^4k}{(2\pi)^4} \frac{d^4w}{(2\pi)^4} \frac{d^4h}{(2\pi)^4} w_{\eta} \alpha^{\eta\nu\kappa\lambda}(w, h) \langle F_{\mu\nu}^{\text{fl}}(k) F_{\kappa\lambda}^{\text{fl}}(h) \rangle e^{-i(k+w)\cdot x}, \quad (38)$$

$$f_{\mu}^{(2)}(x) = -i \int \frac{d^4k}{(2\pi)^4} \frac{d^4w}{(2\pi)^4} \frac{d^4h}{(2\pi)^4} w_{\eta} \mathcal{G}_{\mu\nu\kappa\lambda}(k, -h) \langle M_{\text{fl}}^{\kappa\lambda}(h) M_{\text{fl}}^{\eta\nu}(w) \rangle e^{-i(k+w)\cdot x}. \quad (39)$$

The two contributions are worked out separately in sections 3.1 and 3.2. We specialize first the geometry of the field to a stationary situation.

We focus on a planar geometry for the equilibrium field, with the atom moving along a translation-invariant direction in the xy -plane. We choose a frame where the field is in equilibrium, and denote spatial projections onto the xy -plane by the index $\parallel$. In this situation, translational invariance entails that the Green tensor has the property

$$\begin{aligned} \mathcal{G}_{\mu\nu}(k, h) &= (2\pi)^3 \delta(k_0 + h_0) \delta(\mathbf{k}_{\parallel} + \mathbf{h}_{\parallel}) \mathcal{G}_{\mu\nu}(k, h_z), \\ \mathcal{G}_{\nu\mu}^*(-h, -k) &= (2\pi)^3 \delta(h_0 + k_0) \delta(\mathbf{h}_{\parallel} + \mathbf{k}_{\parallel}) \mathcal{G}_{\mu\nu}^*(-h, -k_z). \end{aligned} \quad (40)$$

Note that this property holds in any inertial frame moving parallel to the xy -plane. This leaves only the integration over h_z in equations (38) and (39). In addition, the spatial integration over $\mathbf{x}$ in equation (37) yields a $\delta(\mathbf{k} + \mathbf{w})$.

3.1. Field fluctuations

In the equilibrium frame of the field, we have $\beta_F \cdot k = \beta k_0$. Inserting the correlation spectrum of $F_{\mu\nu}$ in equation (38) and performing the simplifications mentioned above, the contribution of field fluctuations to the force can be written in the form ($w^\mu = (w_0, -\mathbf{k})$, $h^\mu = (-k_0, -\mathbf{k}_\parallel, h_z)$)

$$F_\mu^{(1)} = \frac{\hbar}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{dw_0}{2\pi} \frac{dh_z}{2\pi} \coth\left(\frac{\beta k_0}{2}\right) w_\eta \alpha^{\eta\nu\kappa\lambda}(w, h) \times [k_{[\mu} \mathcal{G}_{\nu][\kappa}(k, h_z) h_\lambda] - h_{[\lambda} \mathcal{G}_{\kappa][\nu]}^*(-h, -k_z) k_\mu] e^{-i(w_0+k_0)t}. \quad (41)$$

The polarizability $\alpha^{\eta\nu\kappa\lambda}(w, h)$ (equation (15)) contains a δ -function whose argument becomes for an atom with velocity $u^\mu = \gamma(1, \mathbf{v})$ in the xy -plane:

$$u \cdot (w - h) = \gamma(w_0 + k_0) + \gamma \mathbf{v} \cdot (\mathbf{k}_\parallel - \mathbf{k}_\parallel) = \gamma(w_0 + k_0). \quad (42)$$

This allows to perform the w_0 integral and completes the identification $w = -k$. We are therefore justified to work with the following expression for the polarizability tensor:

$$\alpha^{\eta\nu\kappa\lambda}(w, h) = 2\pi \frac{\delta(w_0 + k_0)}{\gamma} \alpha(-u \cdot k) \{u^\eta g^{\nu\kappa} u^\lambda + u^\nu g^{\eta\lambda} u^\kappa\} e^{i(k_z+h_z)z_A}. \quad (43)$$

As expected, the force resulting from equation (41) is constant in time and depends only on the atom-surface distance z_A .

The contraction of the two terms in brackets in equation (41) with the fourth rank tensor in the polarizability (43) gives

$$-k_\eta \{u^\eta g^{\nu\kappa} u^\lambda + u^\nu g^{\eta\lambda} u^\kappa\} k_{[\mu} \mathcal{G}_{\nu][\kappa}(k, h) h_\lambda] = k_\mu \phi(k, h), \quad (44)$$

$$-k_\eta \{u^\eta g^{\nu\kappa} u^\lambda + u^\nu g^{\eta\lambda} u^\kappa\} h_{[\lambda} \mathcal{G}_{\kappa][\nu]}^*(-h, -k) k_\mu] = k_\mu \bar{\phi}(-h, -k), \quad (45)$$

where we defined the scalar function

$$\phi(k, h) = -(u \cdot k)(u \cdot h) \mathcal{G}_\kappa(k, h) - (k \cdot h) u^\nu u^\kappa \mathcal{G}_{\nu\kappa}(k, h) + (u \cdot h) u^\nu k^\kappa \mathcal{G}_{\nu\kappa}(k, h) + (u \cdot k) h^\nu u^\kappa \mathcal{G}_{\nu\kappa}(k, h). \quad (46)$$

The function $\bar{\phi}(-h, -k)$ in equation (45) is obtained by the replacement $\mathcal{G}_{\nu\delta}(k, h) \mapsto \mathcal{G}_{\delta\nu}^*(-h, -k)$ in the definition (46) of $\phi(k, h)$. We note that this formula is valid for arbitrary k and h .

We end up with the following integral for this piece of the radiation force ($k^\mu = (\omega, \mathbf{k}_\parallel, k_z)$ and $h^\mu = (-\omega, -\mathbf{k}_\parallel, h_z)$):

$$F_\mu^{(1)} = \frac{\hbar}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{dh_z}{2\pi} \coth\left(\frac{\beta_F \omega}{2}\right) \frac{k_\mu}{\gamma} \alpha(-u \cdot k) [\phi(k, h) - \bar{\phi}(-h, -k)] e^{i(k_z+h_z)z_A}. \quad (47)$$

This formula and equation (49) below, giving the two pieces in F_μ , are our main result.

3.2. Dipole fluctuations

The contribution of dipole fluctuations (equation (39)) is worked out in a similar way. We find again that only $w = -k$ is relevant. The contraction of the fourth rank tensors involves the remarkable identity

$$-k_\eta \{k_\mu \mathcal{G}_{\nu\kappa}(k, -h) h_\lambda + k_\nu \mathcal{G}_{\mu\lambda}(k, -h) h_\kappa\} u^{[\kappa} g^{\lambda][\eta} u^{\nu]} = k_\mu \Phi(k, -h), \quad (48)$$

where the function $\Phi(k, -h)$ has the same definition as $\phi(k, h)$ in equation (46), except for the replacement $\mathcal{G}_{\nu\delta}(k, h) \mapsto \mathcal{G}_{\nu\delta}(k, -h)$. The force due to dipole fluctuations finally takes the form (here, $h = (k_0, \mathbf{k}_\parallel, h_z)$)

$$F_\mu^{(2)} = -i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{dh_z}{2\pi} \coth\left(\frac{\beta_A \cdot k}{2}\right) \frac{k_\mu}{\gamma} \text{Im} \alpha(u \cdot k) \Phi(k, -h) e^{i(k_z - h_z)z_A}, \quad (49)$$

which has a structure quite similar to $F_\mu^{(1)}$ (equation (47)).

3.3. Blackbody friction

As an illustration and check of this approach, we consider two simple situations: motion in blackbody radiation (this section) and above a planar dielectric (section 3.4). In the first case, we use the free-space photon Green's function (17) in the Feynman gauge, in the second case, the Green function obtained in [2] in a generalized Coulomb gauge. Recall that friction forces in the blackbody radiation field have been studied in the early days of quantum theory by Einstein and Hopf [25] and Einstein [26].

The full translational symmetry of the photon propagator (17) entails the additional δ -function $\delta(k_z + h_z)$ in $\mathcal{G}_{\mu\nu}(k, h)$ (equation (40)). Using the resulting values for the four-vector h , the scalar functions in the two pieces $F^{(1)}$ and $F^{(2)}$ are worked out to be

$$\phi(k, h), \bar{\phi}(-h, -k) \mapsto -\frac{2(u \cdot k)^2 + k^2}{k^2 \pm i0 \text{sgn}(\omega)}, \quad (50)$$

$$\Phi(k, -h) \mapsto \frac{2(u \cdot k)^2 + k^2}{k^2 + i0 \text{sgn}(\omega)}. \quad (51)$$

The combination $\phi(k, h) - \bar{\phi}(-h, -k)$ becomes proportional to a δ -function localized on the light cone $k^2 = 0$, thus removing the term k^2 in the numerator. The integration over the direction of $\mathbf{k}$ remains: we observe that only the components F_0 and F_x are non-zero (due to parity) in a frame where $\mathbf{v}$ is along the x -axis. Under the reflection $(\omega, k_x) \mapsto (-\omega, -k_x)$ that flips the sign of $u \cdot k$, only the imaginary part $\text{Im} \alpha(-u \cdot k) = -\text{Im} \alpha(u \cdot k)$ gives an even integrand in equation (47). Pulling these facts together, we find (solid angle $d\Omega$ for unit vector $\mathbf{k}/|\omega|$)

$$F_\mu^{(1)} = \frac{\hbar}{2\pi\gamma} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d\Omega}{4\pi} \coth\left(\frac{\beta_F \omega}{2}\right) \omega k_\mu (u \cdot k)^2 \text{Im} \alpha(u \cdot k). \quad (52)$$

For the second contribution

$$F_\mu^{(2)} = -i\hbar \int \frac{d^4k}{(2\pi)^4} \coth\left(\frac{\beta_A \cdot k}{2}\right) \frac{k_\mu}{\gamma} \text{Im} \alpha(u \cdot k) \frac{2(u \cdot k)^2 + k^2}{k^2 + i0 \text{sgn}(\omega)}, \quad (53)$$

we make the integrand even under the reflection mentioned above equation (52) and get again an $-2\pi i \text{sgn}(\omega) \delta(k^2)$ from the last term. The integral then reduces to

$$F_\mu^{(2)} = -\frac{\hbar}{2\pi\gamma} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d\Omega}{4\pi} \coth\left(\frac{\beta_A \cdot k}{2}\right) \omega k_\mu (u \cdot k)^2 \text{Im} \alpha(u \cdot k), \quad (54)$$

which has nearly the same structure as $F^{(1)}$ (equation (52)) except that $\coth(\beta_A \cdot k/2)$ involves the atomic temperature and the Doppler-shifted frequency $\omega'_A = u \cdot k$ in the atom's rest frame.

The total force thus features the difference

$$\coth \frac{\beta_A \cdot k}{2} - \coth \frac{\beta_F \omega}{2} = 2N(\omega'_A, T_A) - 2N(\omega, T_F), \quad (55)$$

where $N(\omega, T)$ is the Bose–Einstein distribution. (The co-moving frequency ω'_A has necessary the same sign as ω for field modes on the light cone.) This makes the frequency integration converge exponentially fast at $|\omega| \rightarrow \infty$. And it is easy to check that the sum of equations (52) and (54) is equal to the radiative friction force calculated in [7, 24] for the case of a small polarizable particle (for a review and discussion, see [27, 28]).

3.4. Friction above a dielectric surface

We finally address the situation that a neutral particle is moving parallel to a dielectric surface where the radiation force acts as friction. This issue has recently received some regain of interest in particular after the claim in [12] that for two macroscopic bodies in relative motion, the frictional stress should vanish for $T \rightarrow 0$. (See [29–31] for further discussion.)

3.4.1. Reflected photon Green function. The starting point is the photon Green function $\mathcal{G}_{\mu\nu}(x, x')$ for which we take the expression derived in [2] (both $z, z' > 0$, outside the surface)

$$\mathcal{G}_{\mu\nu}(x, x') = - \int \frac{dk_0}{2\pi} \frac{d^2 k_{\parallel}}{(2\pi)^2} e^{-i[k_0(t-t') - \mathbf{k}_{\parallel} \cdot (\mathbf{x} - \mathbf{x}')]}$$

$$\times \left\{ \int \frac{dk_z}{2\pi} \frac{g_{\mu\nu} e^{ik_z(z-z')}}{k^2 + i0 \operatorname{sgn} \omega} + \int_{\mathcal{C}} \frac{dk_z}{2\pi} \sum_{\sigma} \frac{r_{\sigma} P_{\mu\nu}^{(\sigma)} e^{ik_z(z+z')}}{k^2 + i0 \operatorname{sgn} \omega} \right\}, \quad (56)$$

where σ is a polarization index and the quantities r_{σ} , $P_{\mu\nu}^{(\sigma)}$ are detailed in equation (57) below. This is written in the rest frame of the dielectric medium. The first term in curly brackets is the same as in free space (equation (17)), and we can focus here on the second term. It is built from waves that are reflected from the planar surface, as illustrated by the sign flip in front of the second coordinate z' . The integral is over a contour $\mathcal{C}$ in the complex k_z plane, including the real axis and two segments running on opposite sides of the imaginary axis. In our notation where $z, z' > 0$, these segments are between $k_z = 0$ and $k_z = i|\mathbf{k}_{\parallel}|\sqrt{1 - 1/n^2}$ where $n > 1$ is the refractive index of the dielectric medium below the surface [2]. The reflection matrices involve two transverse polarizations $\sigma = s, p$ which are gauge-independent, and two gauge-dependent ones, scalar $\sigma = l$ and longitudinal $\sigma = k$:

$$r_s = \frac{k_z - k'_z}{k_z + k'_z}, \quad P^{(s)}_{ij} = \begin{pmatrix} -k_y^2 & k_x k_y & 0 \\ k_x k_y & -k_x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\mathbf{k}_{\parallel}^2},$$

$$r_p = \frac{n^2 k_z - k'_z}{n^2 k_z + k'_z}, \quad P^{(p)}_{ij} = \begin{pmatrix} k_z^2 k_x^2 & k_z^2 k_x k_y & k_z k_x \mathbf{k}_{\parallel}^2 \\ k_z^2 k_x k_y & k_z^2 k_y^2 & k_z k_y \mathbf{k}_{\parallel}^2 \\ -k_z k_x \mathbf{k}_{\parallel}^2 & -k_z k_y \mathbf{k}_{\parallel}^2 & -\mathbf{k}_{\parallel}^4 \end{pmatrix} \frac{1}{\mathbf{k}_{\parallel}^2 \mathbf{k}^2}, \quad (57)$$

$$r_l = \frac{k_z - n^2 k'_z}{k_z + n^2 k'_z}, \quad P^{(l)}_{00} = 1,$$

$$r_k = \frac{k_z - n^2 k'_z}{k_z + n^2 k'_z}, \quad P^{(k)}_{ij} = \begin{pmatrix} -k_x^2 & -k_x k_y & k_x k_z \\ -k_x k_y & -k_y^2 & k_y k_z \\ -k_x k_z & -k_y k_z & k_z^2 \end{pmatrix} \frac{1}{\mathbf{k}^2}.$$

All other components vanish. (The gauge chosen in [2] is a generalized Coulomb one.) Note that the scalar and longitudinal polarizations have the same reflection amplitude $r_l = r_k$. Finally, the reflection amplitudes involve the medium wavevector k'_z given by

$$k'_z = \sqrt{n^2 k_z^2 + (n^2 - 1) \mathbf{k}_\parallel^2}, \quad (58)$$

where the square root must be evaluated with a branch cut joining the points $k_z = -i|\mathbf{k}_\parallel|\sqrt{1 - 1/n^2}$ and $k_z = +i|\mathbf{k}_\parallel|\sqrt{1 - 1/n^2}$. The integration contour $\mathcal{C}$ avoids this branch cut from above. In the following, we denote by $\mathcal{R}$ the reflected part of the photon Green function and keep only this contribution. From the expression (56) of $\mathcal{R}(x, x')$, we read off that the double Fourier representation used before in equation (32) has the property

$$\mathcal{R}_{\mu\nu}(k, h) = (2\pi)^4 \delta(k_0 + h_0) \delta(\mathbf{k}_\parallel + \mathbf{h}_\parallel) \delta(k_z - h_z) \mathcal{R}_{\mu\nu}(k), \quad (59)$$

so that $h = -k_r = -(k_0, \mathbf{k}_\parallel, -k_z)$ is fixed to the reflected wavevector.

We have checked that this formulation (with the contour integral over k_z and the relation (59) fixing h_z) carries through the previous calculation in place of the ordinary Fourier integrals over k_z and h_z . A similar expansion also holds for the conjugate tensor $\mathcal{R}^*$. Indeed, using the fact that the contour is mapped according to $\mathcal{C}^* = -\mathcal{C}^{-1}$, and the properties of the scattering amplitudes under complex conjugation compiled in [2], one can convince oneself that the conjugate tensor $[\mathcal{R}_{\nu\mu}(x', x)]^*$ can be written exactly as the second term in equation (56), except that the retarded denominator must be replaced by the advanced one, $k^2 - i0 \operatorname{sgn}(\omega)$.

3.4.2. Reduction to the light cone. We perform the integration over k_z by closing the contour $\mathcal{C}$ with a half-circle at infinity in the upper half-plane (observe $e^{ik_z(z+z')}$ in equation (56)). There are two poles, one at $k_z = i|\mathbf{k}_\parallel|$ from the normalization factor $\mathbf{k}^2$ in the projectors, and another one at

$$k_z = i\sqrt{\mathbf{k}_\parallel^2 - (\omega + i0)^2} \equiv i\kappa \quad (60)$$

from the photon propagator (on the light cone). It is easy to check that the residues at the former pole compensate between the p - and longitudinal polarizations, the reflection coefficients taking the values $r_p = (n^2 - 1)/(n^2 + 1) = -r_k$. For more technical details, see [32].

Another cancellation happens between the longitudinal and scalar polarizations on the light cone when the scalar function $\phi(k, h)$ (equation (46)) is evaluated. Let us write $\phi_\sigma(k)$ ($\sigma = l, k$) for the corresponding expressions when the projectors $P_{\mu\nu}^{(\sigma)}$ are replaced for $\mathcal{R}_{\mu\nu}(k, h)$ (we use $h = -k_r$):

$$\phi_\sigma(k) = (u \cdot k)(u \cdot k_r) P^{(\sigma)\lambda}_{\lambda} + (k \cdot k_r) u^\nu u^\lambda P^{(\sigma)}_{\nu\lambda} - (u \cdot k_r) u^\nu k^\lambda P^{(\sigma)}_{\nu\lambda} - (u \cdot k) k_r^\nu u^\lambda P^{(\sigma)}_{\nu\lambda}. \quad (61)$$

The scalar polarization picks the time-like components, while the longitudinal polarization projects onto $\mathbf{k}$ and $\mathbf{k}_r$. Indeed, the latter projector is re-written (on the light cone) as

$$P_{ij}^{(k)} = -\frac{k_i k_{rj}}{\mathbf{k}^2} = -\frac{k_i k_{rj}}{\omega^2}, \quad (62)$$

where $\mathbf{k}_r = (\mathbf{k}_\parallel, -k_z)$ is the reflected wavevector. Straightforward algebra shows that

$$k^2 = 0 \quad \Rightarrow \quad r_l \phi_l(k) + r_k \phi_k(k) = 0. \quad (63)$$

This is again an indication that our covariant expression for the radiation force is gauge invariant.

The conjugate Green tensor $\mathcal{R}^*$ is handled in a similar way, taking care of the positions of the poles in k_z . The sum of the two terms in $F^{(1)}$ (equation (47)) results in:

$$\int_C \frac{dk_z dh_z}{2\pi} k_\mu [\phi(k, h) - \bar{\phi}(-h, -k)] e^{i(k_z + h_z)z_A} = \frac{1}{2} \sum_{\sigma=s,p} \phi_\sigma(k) \left[k_\mu \frac{r_\sigma e^{-2\kappa z_A}}{\kappa} - \bar{k}_\mu \frac{r_\sigma^* e^{-2\kappa^* z_A}}{\kappa^*} \right], \quad (64)$$

where the light-like vectors k^μ and $\bar{k}^\mu$ have z -components given by $i\kappa$ (equation (60)) and $i\kappa^*$, respectively. The polarizations come with real-valued weight functions

$$\phi_s(k) = \gamma^2(\omega - \mathbf{v} \cdot \mathbf{k}_\parallel)^2 + 2\gamma^2(\mathbf{v} \times \mathbf{k}_\parallel)^2 \left(1 - \frac{\omega^2}{\mathbf{k}_\parallel^2} \right), \quad (65)$$

$$\phi_p(k) = \gamma^2(\omega - \mathbf{v} \cdot \mathbf{k}_\parallel)^2 + 2\gamma^2(\mathbf{k}_\parallel^2 - (\mathbf{v} \cdot \mathbf{k}_\parallel)^2) \left(1 - \frac{\omega^2}{\mathbf{k}_\parallel^2} \right). \quad (66)$$

3.4.3. Field fluctuations. For comparison with [8], we work out the friction force F_x parallel to $\mathbf{v}$. The terms in brackets in equation (64) are then complex conjugates one of the other. From equation (60) for κ and the properties of the reflection coefficients in [2], we observe that this function is odd under a sign flip of both ω and k_x . Keeping only even terms in the integrand, we end up from equations (47) and (64) with the manifestly real expression for that part of the force that depends on the field temperature

$$F_x^{(1)} = \frac{\hbar}{2\gamma} \int \frac{d\omega}{2\pi} \frac{d^2 k_\parallel}{(2\pi)^2} \coth\left(\frac{\beta_F \omega}{2}\right) k_x \operatorname{Im} \alpha(u \cdot k) \sum_{\sigma=s,p} \phi_\sigma(k) \operatorname{Im} \left(\frac{r_\sigma e^{-2\kappa z_A}}{\kappa} \right). \quad (67)$$

3.4.4. Dipole fluctuations. The integral over dipole fluctuations (49) is similar. The Green function $\mathcal{R}_{\mu\nu}(k, -h)$ (equation (59)) involves $\delta(k_r - h)$ and fixes h . Performing the k_z -integration, we get

$$h = k_r : \int_C \frac{dk_z dh_z}{2\pi} k_\mu \Phi(k, -h) e^{i(k_z - h_z)z_A} = \sum_{\sigma=s,p} (-\phi_\sigma(k)) \frac{r_\sigma e^{-2\kappa z_A}}{2\kappa}, \quad (68)$$

where the weight functions defined in equation (61) appear again. Putting this into equation (49) and picking the even part of the integrand, we get

$$F_x^{(2)} = -\frac{\hbar}{2\gamma} \int \frac{d\omega}{2\pi} \frac{d^2 k_\parallel}{(2\pi)^2} \coth\left(\frac{\beta_A \cdot k}{2}\right) k_x \operatorname{Im} \alpha(u \cdot k) \sum_{\sigma=s,p} \phi_\sigma(k) \operatorname{Im} \left(\frac{r_\sigma e^{-2\kappa z_A}}{\kappa} \right). \quad (69)$$

The net force thus involves the same difference of thermal occupations as in free space (equation (55)).

3.4.5. Comparison to previous results. In [8, 24], the same problem was treated in a not manifestly covariant way. We compare here to equation (13) in the review [33] which is the sum of the friction force in free space (section 3.3) and above a magneto-dielectric surface. The free-space piece is equal to the sum of equations (52) and (54), as can be shown with the

Table 1. Dictionary of symbols. In [33], Gauss units are used.

	Particle temperature	Field temperature	Particle velocity	(Electric) polarizability	Co-moving frequency	Parallel wave vector	Propagating waves	Evanescent waves	Reflection amplitude	Polarization weight
Reference [33]	T_1	T_2	$\beta c, V$	$4\pi\alpha_e$	$\gamma\omega^-$	$\mathbf{k}$	$\tilde{q}_0$	iq_0	$\tilde{\Delta}_{m,e}, \Delta_{m,e}$	$\gamma^2\chi_{m,e}^-$
This paper	T_A	T_F	v_x	α	$u \cdot k$	$\mathbf{k}_{\parallel}$	$k_z \in \mathbb{R}$	$k_z = i\kappa$	$r_{s,p}$	$\phi_{s,p}$

translation table 1. We have checked that also the surface friction is the same as our result, in both propagating and evanescent sectors ($k_{\parallel} < \omega/c$ and $k_{\parallel} > \omega/c$, respectively), taking care of the symmetry of the integrand with respect to the signs of ω and k_x . In [13], the scenario for calculating the force is slightly different because the parameters in the atomic polarizability (frequency shift, linewidth) are modified themselves due to the interaction with the surface. We assume here that $\alpha(\omega)$ is an independent input parameter. Alternatively, one may work with a ‘dressed polarizability’ that depends, in general, on the atom-surface distance.

In [11], the near-surface change in the polarizability is emphasized as well, and contributions up to order $\mathcal{O}[\alpha^2(\omega)]$ are calculated. The starting point for radiative friction is not equation (4) for the (average) force. A friction coefficient to linear order in the velocity v is calculated from a force autocorrelation function, essentially similar to the Kubo formula in equation (28). The terms to lowest order in the polarizability α and for a common temperature coincide with the result of [8] and therefore with ours.

In [34], perturbation theory to the second order is used for the power dissipated at $T = 0$ by a neutral atom coupled in the electrostatic limit to the surface plasmon modes of a conducting half-space. The surface plasmons are modelled as harmonic oscillators using the Huttner–Barnett model. This microscopic description entails the atomic polarizability and the dielectric function of the half-space. The final results are calculated for an atom moving with a non-relativistic velocity and agree with [13], on the one hand, and with [33], on the other [35]. In this limit (low velocities, short distances and zero temperature), the preceding approaches including ours are therefore consistent.

3.4.6. Normal force. For completeness, we also give here the two contributions to the normal component of the radiation force. This provides the generalization of the well-known Casimir–Polder interaction to the situation of a moving particle at a different temperature than the surface. The calculations are the same and lead to

$$F_z^{(1)} = -\frac{\hbar}{2\gamma} \int \frac{d\omega}{2\pi} \int \frac{d^2k_{\parallel}}{(2\pi)^2} \coth\left(\frac{\beta_F \omega}{2}\right) \operatorname{Re} \alpha(u \cdot k) \sum_{\sigma=s,p} \phi_{\sigma}(k) \operatorname{Im} [r_{\sigma} e^{-2\kappa z_A}], \quad (70)$$

$$F_z^{(2)} = -\frac{\hbar}{2\gamma} \int \frac{d\omega}{2\pi} \int \frac{d^2k_{\parallel}}{(2\pi)^2} \coth\left(\frac{\beta_A \cdot k}{2}\right) \operatorname{Im} \alpha(u \cdot k) \sum_{\sigma=s,p} \phi_{\sigma}(k) \operatorname{Re} [r_{\sigma} e^{-2\kappa z_A}]. \quad (71)$$

Both contributions are manifestly real. We have checked that these results coincide with equation (12) of [33] for both propagating and evanescent modes.

4. Conclusions

The problem of radiative friction on neutral particles near macroscopic bodies or between two such objects has been addressed by several authors in recent years, using different approaches. We have constructed here a framework that embodies several of these results, and has the advantage of being manifestly compatible with the requirements of special relativity. The formulation highlights the different geometric objects that are involved in the electromagnetic coupling, the material polarization is for example an antisymmetric rank-two tensor, conjugate to the Faraday tensor. We believe that one advantage of the formulation is to expound clearly the concept of local thermodynamic equilibrium which is a prerequisite to apply the fluctuation–dissipation theorem in the relativistic context. From this viewpoint, the two-temperature situations that have been studied quite intensively over the previous years, appear on the same footing as two objects in relative motion.

We found hints that the radiative force on the particle is a gauge-independent quantity, by retrieving previous results from different choices for the relativistic photon propagator (Green tensor). It is also interesting that the covariant formulation displays the force (equations (47) and (49)) as being proportional to the four-wavevector k_μ . This may help to interpret the associated potential energy in a covariant way and to compare with other results.

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Appendix

We give here an elementary justification of equation (8) where the polarization operator is split into a fluctuating and an induced part. For the sake of simplicity, it is not necessary to evoke a relativistic formalism. Consider the following Hamiltonian:

$$H = H_F + H_A + H_{\text{int}} \quad (\text{A.1})$$

in an obvious notation where the atom–field interaction is

$$H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}(\mathbf{x}_A) \quad (\text{A.2})$$

for a point-like atom at position $\mathbf{x}_A$. We want to calculate the average force on the atom [33]

$$\mathbf{f}(t) = \langle d_j(t) \nabla E_j(t, \mathbf{x}_A) \rangle \quad (\text{A.3})$$

in the limit of large times $t \rightarrow \infty$ (implicit summation over cartesian index j). We work in the interaction picture and approximate the time evolution operator up to first order in the dipole moment

$$U(t) = 1 - \frac{i}{\hbar} \int_0^t dt' d_k(t') E_k(t', \mathbf{x}_A). \quad (\text{A.4})$$

We thus get

$$\mathbf{f}(t) = \langle U^\dagger(t) d_j(t) \nabla E_j(t, \mathbf{x}_A) U(t) \rangle \quad (\text{A.5})$$

$$= \frac{i}{\hbar} \int_0^t dt' (\langle d_k(t') E_k(t', \mathbf{x}_A) d_j(t) \nabla E_j(t, \mathbf{x}_A) \rangle - \langle d_j(t) \nabla E_j(t, \mathbf{x}_A) d_k(t') E_k(t', \mathbf{x}_A) \rangle). \quad (\text{A.6})$$

In this order of perturbation theory, the operators evolve according to the respective non-perturbed Hamiltonians of atom and field. Assuming that the two are in a factorized state initially, we get

$$\langle d_j(t) \nabla E_j(t, \mathbf{x}_A) \rangle = 0. \quad (\text{A.7})$$

With the operator identity

$$ab = \frac{1}{2} (\{a, b\} + [a, b]) \quad (\text{A.8})$$

and neglecting commutators that lead to higher order terms, we find

$$\begin{aligned} \mathbf{f}(t) = \frac{i}{2\hbar} \int_0^t dt' (\langle [d_j(t), d_k(t')] \rangle \langle \nabla E_j(t, \mathbf{x}_A), E_k(t', \mathbf{x}_A) \rangle \\ + \langle \nabla E_j(t, \mathbf{x}_A), E_k(t', \mathbf{x}_A) \rangle \langle \{d_k(t'), d_j(t)\} \rangle). \end{aligned} \quad (\text{A.9})$$

We can now recognize the commutators as response functions (polarizability and Green tensor). Because the system is in a stationary state, both the response functions and the anti-symmetrized correlations only depend on the time difference $t - t'$. The force becomes stationary in the limit that the initial time $t = 0 \rightarrow -\infty$:

$$\begin{aligned} \mathbf{f}(t) = \frac{i}{\hbar} \int_{-\infty}^{\infty} dt' \left[\alpha_{ij}(t - t') \left\langle \frac{1}{2} \{ \nabla E_j(t, \mathbf{x}_A), E_k(t', \mathbf{x}_A) \} \right\rangle \right. \\ \left. + \nabla_i \mathcal{G}_{ij}(t - t'; \mathbf{x}_A, \mathbf{x}_A) \left\langle \frac{1}{2} \{ d_k(t'), d_j(t) \} \right\rangle \right], \end{aligned} \quad (\text{A.10})$$

where the gradient ∇_1 in the last line indicates differentiation with respect to the first argument. In the main text, these integrals are worked out in Fourier space.

Equation (A.10) can now be given a physical interpretation: in the first line, the symmetrized correlations characterize the quantum fluctuations of the electric fields which polarize the atom that responds linearly (polarizability α). The reciprocal is true for the second line: the atomic dipole fluctuations generate radiation reaction fields (Green tensor $\mathcal{G}$). Equation (A.10) is consistent with the heuristic splitting of the field and dipole into induced and fluctuating parts (equation (8)) and formalizes the way to take the corresponding averages in this interpretation (equation (9)).

One can apply the same reasoning to the following interaction in covariant form:

$$H = F_{\mu\nu} M^{\mu\nu} \quad (\text{A.11})$$

to get the average four-force density

$$f_\mu = \langle F^{\mu\nu} \partial_\nu M^{\kappa\nu} \rangle. \quad (\text{A.12})$$

This leads to equations (38) and (39) of the main text.

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