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Correlated multiplexity and connectivity of multiplex random networks

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Abstract. Nodes in a complex networked system often engage in more than one type of interactions among them; they form a multiplex network with multiple types of links. In real-world complex systems, a node’s degree for one type of links and that for the other are not randomly distributed but correlated, which we term correlated multiplexity. In this paper, we study a simple model of multiplex random networks and demonstrate that the correlated multiplexity can drastically affect the properties of a giant component in the network. Specifically, when the degrees of a node for different interactions in a duplex Erdős–Rényi network are maximally correlated, the network contains the giant component for any nonzero link density. In contrast, when the degrees of a node are maximally anti-correlated, the emergence of the giant component is significantly delayed, yet the entire network becomes connected into a single component at a finite link density. We also discuss the mixing patterns and the cases with imperfect correlated multiplexity.

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1. Introduction

Over the last decade, the network has proved to be a useful framework to model structural complexity of complex systems [1, 2]. By abstracting a complex system into nodes (constituents) and links (interactions between them), the resulting graph could be efficiently treated analytically and numerically, from which a large body of new physics of complex systems has been acquired [3–5]. Most studies until recently have focused on the properties of isolated, single networks where nodes interact with a single type of links. In most, if not all, real-world complex systems, however, nodes in the system can engage in more than one type of interaction or link: people in a society interact via their friendship, family relationship and/or more formal work-related links, etc. Countries in the global economic system interact via various financial and political channels ranging from commodity trade to political alliance. Even proteins in a cell participate in multiple layers of interactions and regulations, from transcriptional regulations and metabolic synthesis to signaling. Therefore, a more complete description of complex systems would be the multiplex network [6] with more than one type of link connecting nodes in the network. Multiplexity can also have an impact on network dynamics; many dynamic processes occurring in complex network systems such as behavioral cascade in social networks [7] or dynamics of systemic risk in the global economic system [8] should be properly understood from the perspective of multiplex network dynamics. Since its introduction in the social network literature [6], however, only a handful of related earlier studies have been found in the physics literature, notably [9–11], and the understanding of the generic effects of multiplexity remains lacking.

More recently, related concepts such as interacting networks [12] and interdependent networks [13] have been introduced and studied. Leicht and D’Souza [12] studied what they called interacting networks, in which two networks are coupled via inter-network edges, and developed a generating function formalism to study their percolation properties. Buldyrev
et al [13] studied the interdependent networks, in which mutual connectivity in two network layers plays an important role, and found that catastrophic cascades of failure can occur due to the interdependence. Although the specific contexts are different in these studies, if one regards each type of link in a multiplex network to constitute a network layer and the multiplex network as a multilayer network, the multiplex networks and the interacting or interdependent networks may be described by a similar framework at the mathematical level. In this sense, these studies have provided a pioneering insight relevant to multiplex networks that there can be nontrivial effects of having more than one type of link, or channel of interactions, in networks [14, 15].

In [12, 13], network layers were coupled in an uncorrelated way, in the sense that the connections or pairings between nodes in different layers are taken to be random. In real-world complex systems, however, nonrandom structure in network multiplexity can be significant. For example, a person with many links in the friendship layer is likely to also have many links in another social network layer, being a friendly person. Such a nonrandom or correlated multiplexity has recently been observed in the large-scale social network analysis of online game community [16], in the world trade system [17] and in transportation network systems [18, 19], and its impact on network robustness has been studied [18, 20]. In this paper, our main goal is to understand the generic role of correlated multiplexity in the multiplex system’s connectivity structure.

2. Model and formalism

To study how the network connectivity is affected by correlated multiplexity, we consider the following model of multiplex networks with two layers, or *duplex* networks. The network has \( N \) nodes connected by two kinds of links, modeling, for example, individuals participating in two different social interaction channels. The subnetwork formed by each kind of link will be referred to as the network layer. Each network layer \( l \) \((l = 1, 2)\) is specified by the intralayer degree distribution \( \pi^{(l)}(k_l) \), where degree \( k_l \) is the number of links within the specific layer \( l \) of a node. The complete multiplex network can be specified by the joint distribution \( \Pi(k_1, k_2) \) or the conditional degree distribution \( \Pi(k_2 | k_1) \). Generalization to \( l > 2 \) layers is straightforward.

The total degree of a node in the multiplex network is given by \( k = k_1 + k_2 - k_o \), where \( k_o \) denotes the number of overlapping links in the two layers, which can be neglected in the \( N \rightarrow \infty \) limit for random, sparse networks with the largest degree of at most order \( O(\sqrt{N}) \) [21]. One can obtain the total degree distribution \( P(k) \) from the joint degree distribution or the conditional degree distribution as \( P(k) = \sum_{k_1, k_2} \Pi(k_1, k_2) \delta_{k_1, k_1+k_2} = \sum_{k_1} \Pi(k-|k_1|) \pi^{(1)}(k_1) \), where \( \delta \) denotes the Kronecker delta symbol. From \( P(k) \), one can follow the standard generating function technique [22] to study the network structure: the generating function \( g_0(x) \) of the degree distribution \( P(k) \) of the multiplex network can be written as

\[
g_0(x) = \sum_{k=0}^{\infty} P(k) x^k = \sum_{k_1, k_2} \Pi(k_1, k_2) x^{k_1+k_2}.
\]

Emergence of the giant component spanning a finite fraction of the network signals the establishment of connectivity. The size of the giant component \( S \) is obtained via

\[
S = 1 - g_0(u) = 1 - \sum_{k=0}^{\infty} P(k) u^k,
\]
where $u$ is the smallest root of the equation $x = g'_0(x)/g'_0(1) \equiv g_1(x)$, i.e.

$$u = \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k P(k) u^{k-1}. \quad (3)$$

The mean size of the component to which a randomly chosen vertex belongs, $\langle s \rangle$, plays the role of susceptibility and is given by

$$\langle s \rangle = 1 + \frac{g'_0(1)u^2}{g_0(u)[1 - g'_1(u)]}. \quad (4)$$

The condition for existence of the giant component (i.e. $S > 0$) is given by the existence of a nontrivial solution $u < 1$, leading to the so-called Molloy–Reed criterion [22, 24, 25]

$$\sum_k k(k - 2) P(k) = \langle k^2 \rangle - 2 \langle k \rangle > 0. \quad (5)$$

The related recent generalizations of the generating function method for interacting [12] and interdependent networks [23] are worth noting.

For a multiplex network system, the total degree distribution $P(k)$ is determined from the joint degree distribution $\Pi(k_1, k_2)$, which depends on the pattern of correlated multiplexity. Therefore, the presence of correlated multiplexity affects the multiplex system’s connectivity (figure 1). In the following, we specifically consider duplex networks of two ER layers [26] and three limiting cases of correlated multiplexity. Using analytical treatment with mean-field-like approximation as well as extensive numerical simulations, we present how the correlated multiplexity can affect the emergence of the giant component in the multiplex system. Similar procedures can also be applied to multiplex scale-free network models [27].
3. Degree distributions

3.1. Uncorrelated multiplexity

In the absence of correlation between network layers, the joint degree distribution factorizes, \( \Pi_{\text{uncorr}}(k_1, k_2) = \pi^{(1)}(k_1)\pi^{(2)}(k_2) \). The total degree distribution of the multiplex network is given by the convolution of \( \pi^{(1)}(k_1) \) and \( \pi^{(2)}(k_2) \), and its generating function \( g_{\text{uncorr}}(x) = g^{(1)}(x)g^{(2)}(x) \), where \( g^{(l)}(x) \) is the generating function of \( \pi^{(l)}(k_l) \). Using \( g^{(0)}(x) = e^{z(x-1)} \) for the ER network with mean degree \( z \), we have

\[
P(k) = \frac{e^{-z}z^k}{k!}
\]

with \( z = z_1 + z_2 \).

3.2. Maximally positive correlated multiplexity

In the MP correlated multiplex case, a node’s degrees in different layers are maximally correlated in their degree order; the node that is the hub in one layer is also the hub in the other layers, and the node that has the smallest degree in one layer also has the smallest degree in other layers. To obtain the total degree distribution \( P(k) \) of the multiplex network, we use the following mean-field-like scheme ignoring fluctuations in the thermodynamic limit \( (N \to \infty) \), illustrated for the duplex case in figure 2(a). We partition the unit interval into bins of sizes \( \pi^{(l)}(k_\ell) \) sorted in order of \( k_\ell \) for each \( \ell \). Combining the two partitions, we have a new partition that can be used to reconstruct \( P(k) \) as illustrated in figure 2(a).

3.3. Maximally negative correlated multiplexity

In the MN correlated multiplex case, a node’s degrees in different layers are maximally anti-correlated in their degree order; a node that is the hub in one layer has the smallest degree in the other layer, etc. The mean-field-like scheme for the duplex case proceeds in a similar way as the MP case, except that we use two partitions sorted in opposite orders, respectively, as illustrated in figure 2(a).

4. Duplex Erdős–Rényi networks with equal link densities

In this section, we consider ER networks with two layers of equal link densities, for which the degree distributions are most easily calculated.

4.1. Uncorrelated case

The uncorrelated multiplex ER network is simply another ER network with mean degree \( z = 2z_1 \), where \( z_1 \) is the mean degree of the first network layer. The joint degree distribution factorizes, thus the conditional degree distribution \( \Pi(k_2|k_1) \) is independent of \( k_1 \) and \( \Pi(k_2|k_1) = \pi(k_2) \), where \( \pi(k) \) denotes the Poisson distribution with mean degree \( z_1 \). As we increase \( z_1 \), we have a second-order percolation transition at the critical mean intralayer degree \( z_c = 1/2 \).
Figure 2. (a) Schematic illustrations of how the mean-field-like calculation for degree distribution is done. (b, c) Degree distributions of interlaced multiplex ER networks with $z_1 = z_2 = 0.7$ (b) and $z_1 = z_2 = 1.4$ (c). Points are numerical simulation results for the uncorrelated (○), MP (□) and MN (♦) cases, together with lines representing predictions of mean-field-like calculations.

[22, 26], at which the giant component emerges in the duplex network. The giant component size scales in the vicinity of $z_c$ as $S \sim (z_1 - z_c)^\beta$ with $\beta = 1$, and the susceptibility as $\langle s \rangle \sim |z_1 - z_c|^{-\gamma}$ with $\gamma = 1$, following the standard mean-field critical behavior [22].

4.2. Maximally positive case

In this case, in the mean-field-like scheme ($N \to \infty$) each node has exactly the same degrees in both layers, so the conditional degree distribution becomes $\Pi(k_2|k_1) = \delta_{k_2,k_1}$. Thus we have the degree distribution of the duplex network as

$$P(k) = \begin{cases} e^{-z_1}z_1^{k/2}/(k/2)! & (k \text{ even}), \\ 0 & (k \text{ odd}). \end{cases} \quad (7)$$

Therefore, the Molloy–Reed criterion is fulfilled for all nonzero $z_1$, as $\langle k^2 \rangle - 2\langle k \rangle = 4(z_1 + z_1^2) - 2(2z_1) = 4z_1^2 > 0$ for $z_1 \neq 0$. This means that the giant component exists for any nonzero link density, i.e.

$$z_c^{MP} = 0. \quad (8)$$
In fact, one can obtain the solution of $S$ and $\langle s \rangle$ explicitly in this case: as $P(k) = 0$ for odd $k$, (3) has $u = 0$ as a nontrivial solution, from which it follows from (2):

$$S = 1 - P(0) = 1 - e^{-z_1},$$

which grows linearly with the link density near the origin as $S \sim z_1$, i.e. $\beta = 1$. From $u = 0$, the susceptibility $\langle s \rangle = 1$ for all $z_1 > 0$. This means that only isolated nodes are outside the giant component and all the linked nodes form a single giant component. All these predictions are confirmed by numerical simulations (figure 3).

### 4.3. Maximally negative case

In this case one can easily show that distinct regimes appear as $z_1$ increases. Among them, three regimes are of relevance for the giant component properties: (i) for $0 \leq z_1 \leq \ln 2$, more than half of the nodes are of degree zero in each layer so every linked node in one layer is coupled with a degree-0 node in the other layer under MN coupling. In this regime, the conditional degree distribution takes a rather complicated form

$$\Pi(k_2|k_1) = \begin{cases} 
\frac{2\pi(0) - 1}{\pi(0)} & (k_2 = 0, k_1 = 0), \\
\frac{\pi(k_2)}{\pi(0)} & (k_2 \neq 0, k_1 = 0), \\
\delta_{k_2,0} & (k_1 \neq 0),
\end{cases}$$

and thus $P(k)$ is given by

$$P(k) = \begin{cases} 
2\pi(0) - 1 & (k = 0), \\
2\pi(k) & (k \geq 1).
\end{cases}$$
In this regime there is no giant component. (ii) For \( \ln 2 \leq z_1 \leq z^\ast \), a similar consideration leads to \( \Pi(k_2|k_1) \) and \( P(k) \) given by

\[
\Pi(k_2|0) = \begin{cases} 
0 & (k_2 = 0), \\
[2\pi(0) + \pi(1) - 1]/\pi(0) & (k_2 = 1), \\
\pi(k_2)/\pi(0) & (k_2 \geq 2), 
\end{cases} 
\]

\[
\Pi(k_2|1) = \begin{cases} 
[2\pi(0) + \pi(1) - 1]/\pi(1) & (k_2 = 0), \\
[1 - 2\pi(0)]/\pi(1) & (k_2 = 1), \\
0 & (k_2 \geq 2), 
\end{cases} 
\]

\[
\Pi(k_2|k_1) = \delta_{k_2,0} \\ (k_1 \geq 2),
\]

and

\[
P(k) = \begin{cases} 
0 & (k = 0), \\
2[2\pi(0) + \pi(1) - 1] & (k = 1), \\
2\pi(2) - 2\pi(0) + 1 & (k = 2), \\
2\pi(k) & (k \geq 3). 
\end{cases} 
\]

In this regime, \( \langle k^2 \rangle - 2\langle k \rangle = 2(z_1^2 - z_1 - 2e^{-z_1} + 1) \), which becomes positive for \( z_1 > z^{\text{MN}}_c \), where

\[
z^{\text{MN}}_c = 0.838\,587\,497\ldots
\]

Therefore, the giant component emerges at a significantly higher link density than the uncorrelated multiplex case. Being delayed in its birth, the giant component grows more abruptly once formed (see figure 3(a)) than the other two cases. This regime is terminated at \( z = z^\ast \), determined by the condition \( 2\pi(0) + \pi(1) = 1 \), from which we have \( z^\ast = 1.146\,193\,22\ldots \). (iii) For \( z_1 \geq z^\ast \), we have \( P(0) = P(1) = 0 \). In that case, we have \( u = 0 \) from (3) and thereby \( S = 1 \) from (2). This means that the entire network becomes connected into a single component at this finite link density, which can never be achieved for ordinary ER networks. Despite this abnormal behavior and apparent differences in the steepness of the order parameter curve near \( z_c \), the critical behavior is found to be consistent with that of standard mean field, \( \beta = 1 \) and \( \gamma = 1 \) (figure 4).

5. Imperfect correlated multiplexity

In the previous section, we have seen that maximally correlated or anti-correlated multiplexity affects the onset of emergence of the giant component in multiplex ER networks. Despite its mathematical simplicity and tractability, in real-world multiplex systems the correlated multiplexity would hardly be maximal. Therefore it is informative to see how the results obtained for the maximally correlated multiplexity are interpolated when the system possesses partially correlated multiplexity.

To this end, we consider duplex ER networks where a fraction \( q \) of nodes are maximally correlated multiplex while the remaining fraction \( 1-q \) are randomly multiplex. Then, the degree distribution of the interlaced network is given by

\[
P_{\text{partial}}(k) = q P_{\text{maximal}}(k) + (1-q) P(0) + (1-q) P(1)
\]
$P_{\text{uncorr}}(k)$, where maximal is either MP or MN. Following similar steps as in the previous section, we obtain the critical link density as a function of $q$ as

$$z_c = (1 - q)/2$$

for the positively correlated case and

$$z_c = \begin{cases} 
1/(2 - q) & (q < 2 - 1/\ln 2), \\
 z_1(q) & (q > 2 - 1/\ln 2)
\end{cases}$$

for the negatively correlated case, where $z_1(q)$ is the solution of $(2 - q)z_1^2 - z_1 - 2qe^{-z_1} + q = 0$. In figure 5, we show the plot of $z_c$ as a function of $q$ given by (15a) and (15b). This result shows that the effect of correlated multiplexity is not only present for maximally correlated cases but for general $q$.

6. Duplex Erdős–Rényi networks with general link densities

In the previous sections, we focused on the cases with $z_1 = z_2$. In this section, we consider general duplex ER networks with $z_1 \neq z_2$. We performed numerical simulations with $N = 10^4$ for MP, uncorrelated and MN correlated multiplex cases with $z_1$ and $z_2$ in the range 0–3. In figure 6, the giant component sizes $S$ for general $z_1$ and $z_2$ are shown. Similarly to the $z_1 = z_2$ cases, the giant component emerges at lower link densities for the MP case but grows more slowly than the uncorrelated case, whereas it emerges at higher link densities for the MN case but grows more abruptly and connects all the nodes in the network at a finite link density. Therefore the effect of correlated multiplexity is qualitatively the same and generic.

It is worth noting, however, that although the mean-field-like approximation for $P(k)$ introduced in section 3, which was very effective for equal link densities, still yields a qualitatively correct picture, it fails in quantitative agreement with numerical simulation results for general $z_1 \neq z_2$. To understand the origin of this discrepancy, we consider network...
correlations. The assortativity coefficient $r$ [28] defined as the Pearson correlation coefficient between the (total) degrees of nodes connected by a link in the network measures the degree–degree correlations in the network at the two-node level. Nonzero assortativity is known to alter the connectivity properties of networks [28]. To see the role of degree–degree correlations induced by correlated multiplexity, in figures 6(d)–(f) we plot the giant component size obtained from numerical simulations and mean-field calculations, together with the assortativity coefficient as a function $z_1$ with $z_2 = 0.4$ fixed. With uncorrelated multiplexity, the degree–degree correlation is absent, $r = 0$, and the numerical simulation and mean-field calculation agree perfectly (figure 6(e)). For the MP and MN cases, however, the assortativity of the multiplex network is generically nonzero (positive); that is, the multiplex network becomes correlated, which is responsible for the deviations between the numerical simulation and mean-field calculation results. This discrepancy vanishes at $z_1 = z_2$, at which the assortativity also vanishes (figures 6(d) and (f)). Note that there were no degree–degree correlations at the individual network layer level. Therefore this result shows that generically correlated multiplexity not only modulates $P(k)$ but also introduces higher-order correlations in the multiplex network structure. Meanwhile, it is worth noting that such a multiplexity-induced degree correlation has a similar origin as the correlation in colored-edge networks recently studied in a different context of network clustering [29].

7. Conclusion

In conclusion, we have studied the effect of correlated multiplexity on the structural properties of the multiplex network system, a better representation of most real-world complex systems than the single, or simplex, network. We have demonstrated that the correlated multiplexity can dramatically change the giant component properties. With positively correlated multiplexity,
the giant component emerges at a much lower critical link density, which even approaches zero for the MP case, than for the uncorrelated multiplex cases. Once formed, however, the giant component grows much more gradually. With negatively correlated multiplexity, the giant component emerges at a much higher critical density than for uncorrelated multiplex cases, but once formed it grows more abruptly and can establish the full connectivity to connect the entire network into a single component at a finite link density. These results show that a multiplex complex system can exhibit structural properties that cannot be represented by its individual network layer’s properties alone, the impact of which on network dynamics is to be explored in a future study.

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Figure 6. Numerical simulation results of the size of the giant component of duplex ER networks with (a) the MP, (b) uncorrelated and (c) MN cases. In (d–f), the giant component size $S$ (red) is plotted for $z_2 = 0.4$, along with the assortativity coefficient $r$ (blue) for the MP (d), uncorrelated (e) and MN (f) cases. Mean-field-like calculation results (lines) deviate from the numerical simulation results ($\circ$) when the assortativity coefficient ($\square$) becomes nonzero; that is, the network displays degree–degree correlation. Error bars denote standard deviations from $10^4$ independent runs.
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