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Elimination of perturbative crossings in adiabatic quantum optimization

Neil G Dickson

D-Wave Systems Inc., 100-4401 Still Creek Drive, Burnaby, BC
V5C 6G9, Canada

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Abstract. It was recently shown that, for solving NP-complete problems, adiabatic paths always exist without finite-order perturbative crossings between local and global minima, which could lead to anticrossings with exponentially small energy gaps if present. However, it was not shown whether such a path could be found easily. Here, we give a simple construction that deterministically eliminates all such anticrossings in polynomial time, space and energy, for any Ising models with polynomial final gap. Thus, in order for adiabatic quantum optimization to require exponential time to solve any NP-complete problem, some quality other than this type of anticrossing must be unavoidable and necessitate exponentially long runtimes.

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1. Introduction

The usefulness of adiabatic quantum optimization (AQO) for solving NP-complete problems, as originally proposed in [1], has been the subject of much debate in recent years. In AQO, the Hamiltonian of the quantum system begins as H_B , whose ground (eigen)state is easy to prepare as the initial state of the system, and is evolved to a final Hamiltonian H_P , whose ground state is the optimal solution to an optimization problem. The running time, t_f , required to obtain a large amplitude of the ground state at the end of the evolution is proportional to g_{\min}^{-2} , where g_{\min} is the minimum energy gap between the ground and first excited states, excluding excited states simply heading toward final degeneracy with the ground state, during the evolution. The complexity of AQO is therefore determined by the scaling of g_{\min} with the problem size.

An example of an NP-hard¹ optimization problem well suited to being represented by H_P is that of finding the ground state of an Ising model with arbitrary couplings, J_{ij} , and arbitrary local fields, h_i :

$$H_P = \sum_i h_i \sigma_z^{(i)} + \sum_{i < j} J_{ij} \sigma_z^{(i)} \sigma_z^{(j)}, \quad (1)$$

where i and j are qubit indices, and $\sigma_z^{(i)}$ and $\sigma_z^{(j)}$ are the corresponding Pauli matrices that assign energy values to the computation (Z) basis states. It is easily seen that this H_P is identical to the classical Ising model Hamiltonian, so its ground state is naturally identical to the ground state being sought (referred to here as the ‘global minimum’ to avoid confusion where applicable). Of course, determining such a ground state is not to be confused with bringing an actual classical Ising system, such as a bulk solid state system, to its ground state, which is not discussed here. H_B is often chosen to be of the form

$$H_B = - \sum_i \Delta_i \sigma_x^{(i)}, \quad \Delta_i > 0, \quad (2)$$

whose ground state is the uniform superposition (i.e. equal amplitudes in phase and magnitude) of all Z-basis states. If representations of H_P and H_B can be found for a problem such that g_{\min} decreases only polynomially with the number of qubits, n , that problem can be solved in polynomial time using AQO. This, of course, assumes that the quantum system used to implement AQO is sufficiently isolated from a thermal environment, but effects of such an environment are not examined here.

The complexity of AQO for solving NP-complete problems is a contentious issue, and has recently sparked intense debate. It was originally conjectured that the gap size may scale polynomially with problem size [1], but counterexamples were found [2, 3], which were subsequently defeated by modifying the Hamiltonian [4, 5]. Using perturbation expansion, Amin and Choi [6] showed that an eigenstate corresponding with a local minimum of H_P (anti)crossing with that of the global minimum near the end of the evolution, sometimes called a first-order quantum phase transition, can result in an exponentially small g_{\min} . In particular, when a local minimum has more low-energy states, possibly degenerate with it, in its vicinity than the global minimum, the eigenstate corresponding to the local minimum may cross that of the global minimum. Using the same perturbation argument, Altshuler *et al* [7] showed that for one representation of random exact cover instances, the probability of having problematic

¹ A problem is NP-hard if solving it in polynomial time allows all NP-complete problems to be solved in polynomial time.

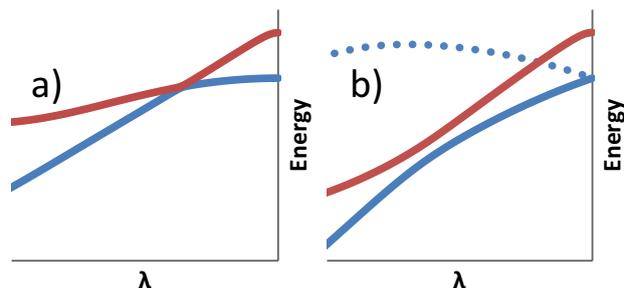


Figure 1. (a) The two lowest energy eigenstates in an AQO spectrum, which to low-order perturbation appear to cross, but actually anticross, causing a small g_{\min} , and (b) three lowest eigenstates after changing the final ground state to be twofold degenerate. If the final ground state is degenerate, the corresponding eigenstates (solid and dotted blue) will repel each other away from the end. If this degeneracy can be introduced without significantly affecting the excited state, the ground state energy (solid blue) will consequently be pushed away from that of the excited state (solid red).

crossings increases with the system size, and claimed that these crossings are unavoidable. Others made similar observations using different techniques [8, 9].

Others argued that the analysis in [7] had not rigorously proven that the crossings are unavoidable, since problem structure that can impact the presence of such crossings was neglected [10–13]. Dickson and Amin [13] showed that there is always *some* selection of H_B and H_P that guarantees no crossings of local and global minima, but it was left open whether or not an *efficient* method of selecting the Hamiltonian exists.

For ease of description below, an eigenstate is referred to as degenerate when there exist other eigenstates of the same energy, i.e. when there exists a degenerate manifold including that state. The degeneracy of an eigenstate refers to the total number of eigenstates at that energy. Only degeneracy in H_P is discussed here.

Here, we present the first general method for eliminating perturbative crossings, based on the effects of degeneracy of the eigenstates of H_P , illustrated in figure 1.

2. The concept

The main concept of the construction we present here is to force the ground state to diverge from all other states away from the end of the evolution by strategically introducing degeneracy, as depicted in figures 1 and 2, eliminating any crossings that may have existed before applying the construction. This can be done by adding extra qubits in such a way that the ground state becomes the most degenerate, the first excited state less degenerate, continuing this up to the highest excited state, which will be the least degenerate (possibly nondegenerate). This degeneracy makes the eigenstates corresponding with the global minimum repel each other from the end more than all other eigenstates, so the ground state diverges from all other eigenstates as desired. This must be done without any knowledge of the global minimum, apart from that it is the lowest energy state of H_P .

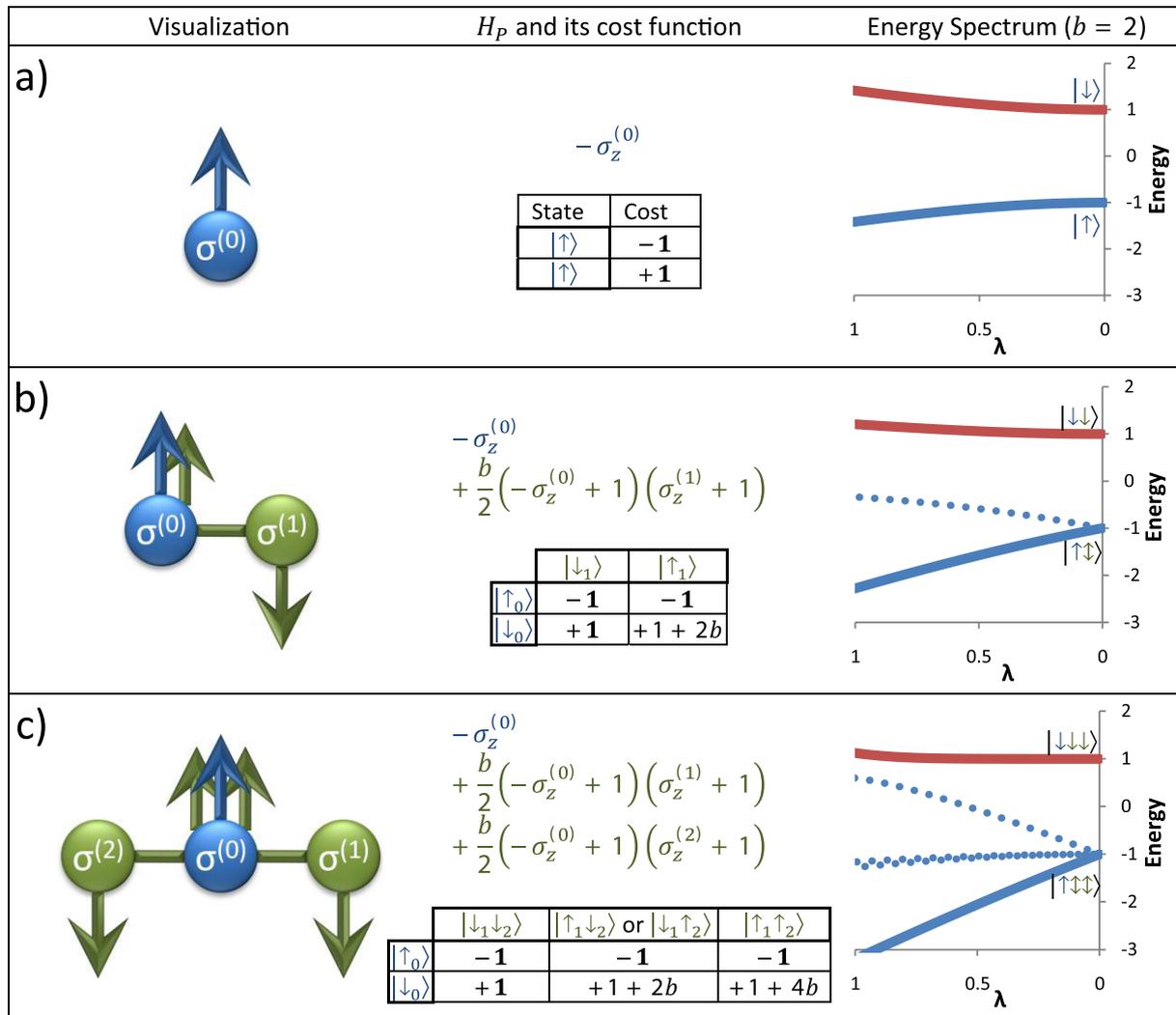


Figure 2. Adding extra qubits in such a way that the ground state becomes degenerate, but the excited state remains nondegenerate, causes the final ground states to repel each other. The ground state energy (solid blue) then diverges from the excited state energy (solid red). Here, this is depicted for one original qubit (blue), and (a) zero, (b) one and (c) two extra qubits (green). λ is the perturbation parameter used in the analysis below. For brevity, $|\uparrow\rangle \otimes (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ is written as $|\uparrow\downarrow\rangle$, and similar for $|\uparrow\downarrow\downarrow\rangle$. Note that the same method can be applied to a term $+\sigma_z^{(0)}$ instead of $-\sigma_z^{(0)}$ simply by substituting one for the other in H_P . Likewise, the same method can be applied to a pair of original qubits coupled with a J term $-\sigma_z^{(0)}\sigma_z^{(1)}$ or $+\sigma_z^{(0)}\sigma_z^{(1)}$, again by substituting over $-\sigma_z^{(0)}$.

For simplicity, we will begin with the case of minimizing arbitrarily-connected Ising models where all $h, J \in \{-1, 0, +1\}$, which is still NP-hard. This can be easily generalized to arbitrary h or J values (of polynomial precision) without resorting to expensive reduction mappings onto $\{-1, 0, +1\}$, as described later. Even though the only structure in a general, dense Ising model is that at most 2 spins participate in each term of its Hamiltonian, this small

amount of structure is enough to perform the required transformation. This does not apply to random energy models, studied in [14, 15], since they have no applicable structure.

The construction introduces extra qubits for each nonzero h or J term of H_P such that when the h or J term is satisfied (i.e. at its negative value), the extra qubits corresponding to that term are at degeneracy (i.e. flipping them does not change the energy of the system), and when the term is unsatisfied, the extra qubits are *not* at degeneracy. This is shown for an h term in figure 2, and can be applied equivalently to a J term. Naturally, states with more terms satisfied will then be more degenerate than states with fewer terms satisfied, since states with more satisfied terms have more qubits at degeneracy. In the case where $h, J \in \{-1, 0, +1\}$, the global minimum has the most terms satisfied, and will thus have the most qubits at degeneracy. As we will see, the slope (first-order perturbation energy correction) of a state will be proportional to the number of qubits at degeneracy in that state, and so the ground state will have the largest negative slope, diverging from all other states.

To find the ground state of an Ising model, M , given h and J as input, construct H_P as described in the following section, using n qubits that directly correspond with the spins of M and a polynomial number of extra qubits. Initialize the system in the ground state of the initial Hamiltonian, H_B , evolve the Hamiltonian into H_P , then measure (in the Z basis) the state of the n qubits corresponding with M . If the evolution was adiabatic, the measured state will be the ground state of M , because by design, those qubits represent the ground state of M in all ground states of H_P .

3. The construction

As mentioned above, we will begin by examining the case of Ising models with $h, J \in \{-1, 0, +1\}$. The Hamiltonian of the original Ising model, M , is given as

$$H_M = \sum_{i \in M} h_i \sigma_z^{(i)} + \sum_{\{i, j\} \in M} J_{ij} \sigma_z^{(i)} \sigma_z^{(j)}. \quad (3)$$

To introduce and amplify degeneracy of the ground state as desired, for each of the m nonzero terms in M , we need to be able to add an extra qubit such that when the term is satisfied, the energy is unchanged regardless of whether the extra qubit is -1 or $+1$. However, when the term is unsatisfied, the energy must be unchanged when the extra qubit is -1 , but higher by energy $2b > 0$ when the extra qubit is $+1$. This can be summarized as

Term	Added energy cost	
	extra = -1	extra = $+1$
Satisfied	0	0
Unsatisfied	0	$2b$

Note that if the energy didn't remain unchanged for some value of the extra qubit for both the satisfied and unsatisfied case, we would effectively be changing the energy of the original states, which is not what we intend, so all three zeros are important. The added cost $2b$ serves only to break the degeneracy when the corresponding term is unsatisfied.

To construct the new terms to appear in H_P , one may first observe that the above table is simply a multiple of the truth table for a Boolean AND operator, which is equivalent to

binary variable multiplication, so we need only to convert our two conditions ('unsatisfied' and 'extra = +1') to binary variables and multiply by $2b$. Since nonzero h and J are limited to ± 1 , this gives us $(\mathbf{term} + 1)/2$ and $(\mathbf{extra} + 1)/2$, so the term added to the Hamiltonian for the extra qubit should be

$$b(\mathbf{term} + 1)(\mathbf{extra} + 1)/2. \quad (4)$$

This is illustrated in figure 2. Now that we have a mechanism for adding these extra qubits, for each $h_i \neq 0$, add $a \geq 1$ extra qubits $\{i_1, \dots, i_a\} \in M_h$ with the following terms to the Hamiltonian. These are in exactly the same form as (4)

$$\sum_{k=1}^a b(h_i \sigma_z^{(i)} + 1)(\sigma_z^{(i_k)} + 1)/2. \quad (5)$$

Similarly, for each $J_{ij} \neq 0$, add a extra qubits $\{ij_1, \dots, ij_a\} \in M_J$ with the following terms to the Hamiltonian

$$\sum_{k=1}^a b(J_{ij} \sigma_z^{(i)} \sigma_z^{(j)} + 1)(\sigma_z^{(ij_k)} + 1)/2. \quad (6)$$

It is important to note that while this introduces 3-local (ZZZ) terms, equivalent, albeit much less intuitive, 2-local terms with the same property can be introduced by adding another extra qubit for each term. This is described in the [appendix](#) below.

Since there are m nonzero terms in the original Ising model, M , we have added a total of am extra qubits, each with an associated term coupling it to M . This gives us

$$H_P = \sum_{i \in M} \left(h_i \sigma_z^{(i)} + \sum_{k=1}^a b(h_i \sigma_z^{(i)} + 1)(\sigma_z^{(i_k)} + 1)/2 \right) + \sum_{\{i,j\} \in M} \left(J_{ij} \sigma_z^{(i)} \sigma_z^{(j)} + \sum_{k=1}^a b(J_{ij} \sigma_z^{(i)} \sigma_z^{(j)} + 1)(\sigma_z^{(ij_k)} + 1)/2 \right). \quad (7)$$

Special consideration must be made when M may already have single bit flip degeneracies. The above construction still works for sufficiently (polynomially) large a and b , but the analysis is made more complicated, so for simplicity of analysis in these cases, all qubits that may have a 1 bit flip degeneracy can be replaced by a strongly ferromagnetically coupled pair of qubits. Since the pair is strongly coupled to align with each other, they will act logically as the original single qubit, replacing any 1 bit flip degeneracy with a 2 bit flip degeneracy (i.e. both qubits in the pair must flip to remain at the same energy). We are currently limiting the focus to J values of ± 1 , but this strong ferromagnetism can be expressed with multiple (identical) $J = -1$ terms between the two qubits in the pair. Thus, the analysis below safely assumes that there are no 1 bit flip degeneracies in the original Ising model.

Unfortunately, performing a full numerical simulation of this construction for relevant cases is infeasible, primarily due to the large number of added qubits necessary to apply it to Ising models sufficiently complex to have perturbative crossings. Thus, it is examined only analytically below. However, it is numerically feasible to demonstrate that changing the degeneracy, by adding extra qubits in a *similar* manner, can be used to create and then eliminate these crossings. An example is illustrated in figure 3.

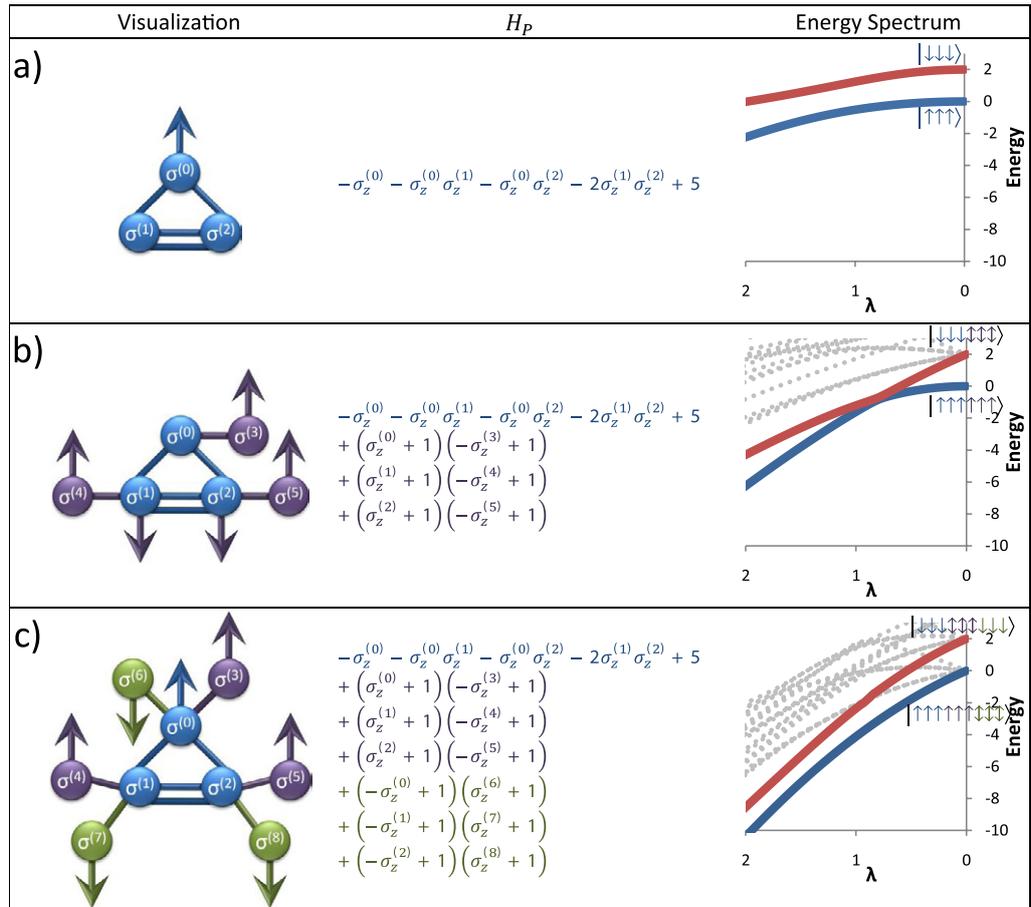


Figure 3. (a) Starting from a simple 3-qubit Ising model without a perturbative (anti)crossing, (b) the known local minimum, $|\downarrow\downarrow\downarrow\rangle$, can be made more degenerate (8-fold in this case) until a harsh crossing is created between the eigenstates corresponding with it and the global minimum, $|\uparrow\uparrow\uparrow\rangle$. (c) Now that there is a crossing, it can be eliminated again by increasing the degeneracy of the known global minimum to match or preferably exceed that of the local minimum. Although this does not apply the full construction (exact diagonalization is infeasible for the 42 qubits needed to eliminate the crossing), which does not require knowledge of the local or global minima, this example demonstrates the principle upon which the construction is based. (Note: For easier viewing, biases that exactly cancel are not depicted in the visualization.)

4. Analysis

We will be performing a perturbation analysis on the above construction by expressing the time-dependent Hamiltonian, H , as

$$H = H_P + \lambda H_B,$$

$$H_B = - \sum_{i \in M \cup M_h \cup M_J} \Delta_i \sigma_x^{(i)}, \quad (8)$$

where λ is initially ∞ (though proportional to abm suffices), and is monotonically decreased toward zero over time. For now, we select all $\Delta_i = 1$.

This analysis focuses on the end of the evolution, where λ is small enough that the perturbation expansion, in λ , of the energy of an eigenstate of H holds. At infinite-order perturbation, the true radius of convergence for any eigenstate falls at the anticrossing it encounters having the smallest λ . However, up to orders of perturbation less than the number of bit flips (Hamming distance) between two anticrossing eigenstates, the associated asymptotic expansion may have a larger radius of convergence. Anticrossings within this radius are referred to here as perturbative crossings, since the perturbed states cross in energy near where the exact eigenstates anticross [6]. This type of crossing guarantees a g_{\min} that is exponentially small in the Hamming distance of the crossing states [6], which for many cases is naturally $\Theta(n)$ [7], hence the focus on this case.

It must be emphasized that this analysis does not apply to nonperturbative anticrossings, occurring at or beyond the radius of convergence, such as those examined in [9]. These may remain after elimination of perturbative crossings, and it is unclear how the construction presented here affects them. As explained in [7], extending the radius of convergence by simple means only shifts the locations of anticrossings, not eliminating them or significantly changing their properties. Having a radius of convergence diverge with n may in effect hide relevant anticrossings from the analysis, so below, we ensure that this does not occur.

As mentioned above, the energies of the original states of M (with the additional qubits in their -1 states) are unchanged by the addition of the new qubits, and all added states with higher energy are 1 bit flip away from states with lower energy, and thus are not local minima.

Supposing first that M has a nondegenerate state α , after applying the construction, α is now degenerate (unless of course all terms were unsatisfied, which means that α is the highest energy state). In particular, upon introducing small $\lambda > 0$, the lowest eigenstate of the degeneracy, $|\alpha\rangle$, is always a uniform superposition of α , since this uniquely minimizes $\langle\alpha|H_B|\alpha\rangle$, the coefficient of the first-order term below. We wish to examine the perturbation expansion of this eigenstate $|\alpha\rangle$,

$$E_{|\alpha\rangle}(\lambda) = E_{|\alpha\rangle}^{(0)} + E_{|\alpha\rangle}^{(1)}\lambda + E_{|\alpha\rangle}^{(2)}\lambda^2 + \dots \quad (9)$$

Because nonzero h, J are limited to ± 1 , we have that

$$\begin{aligned} E_{|\alpha\rangle}^{(0)} &= -(\# \text{ of terms satisfied in } \alpha) + (\# \text{ unsat. in } \alpha) \\ &= -2(\# \text{ of terms satisfied in } \alpha) + m. \end{aligned} \quad (10)$$

Then, because $|\alpha\rangle$ is a uniform superposition over the states corresponding with α , the first-order perturbative energy correction of $|\alpha\rangle$ is

$$\begin{aligned} E_{|\alpha\rangle}^{(1)} &= \langle\alpha|H_B|\alpha\rangle \\ &= -\sum_{i \text{ degen. in } \alpha} \Delta_i \\ &= -a(\# \text{ of terms satisfied in } \alpha) \\ &= \frac{a}{2}(E_{|\alpha\rangle}^{(0)} - m) \\ &= \frac{a}{2}E_{|\alpha\rangle}^{(0)} - \text{constant}. \end{aligned} \quad (11)$$

This means that a final state α of lower energy than a final state β , will have a lowest eigenstate $|\alpha\rangle$ with larger negative slope than $|\beta\rangle$. Thus, first-order perturbation predicts that $|\alpha\rangle$ and $|\beta\rangle$ will diverge and not cross.

In the case where α is degenerate in the original Ising model (but with no 1 bit flip degeneracy), it can be seen that $|\alpha\rangle$ is a combination of uniform superpositions over each state's new degeneracy. For example, if α originally contained states k and l , $|\alpha\rangle$ contains some combination of a uniform superposition over the states corresponding with k and a uniform superposition over the states corresponding with l . Since there are assumed to be no first-order degeneracies in the original α , this means that $E_{|\alpha\rangle}^{(1)}$ is the same as in the nondegenerate case.

However, in order to prove that this can eliminate perturbative crossings, one must examine the impact of this construction on higher orders of perturbation. Consider a single state α_* in the eigenstate $|\alpha\rangle$. If α_* is a local minimum (a state with which a crossing could occur with the global minimum), all states 1 bit flip from α_* have higher energy, and therefore have at least one h or J unsatisfied that is satisfied in α_* . As described above, for any h or J satisfied in α_* , each of the corresponding a added qubits contributes half of the states in the new superposition $|\alpha_*\rangle$. This means that flipping any 1 bit, i , from α_* , which had incurred energy cost $B_{\alpha_*,i}$, now incurs at least cost $(1+2kb)B_{\alpha_*,i}$ for the portion of $|\alpha_*\rangle$ that has k added qubits in their $+1$ states. Since the choice of α_* was arbitrary, this applies to all of $|\alpha\rangle$. For simplicity of analysis, we choose $a = b = n^2 \rightarrow \infty$, though this is extremely excessive in practice. We then find that the second-order energy correction is bounded by

$$\begin{aligned}
 \left| E_{|\alpha\rangle}^{(2)} \right| &\leq \left| E_{\alpha \text{ orig.}}^{(2)} \right| \frac{1}{2a} \sum_{k=0}^a \binom{a}{k} (1+2kb)^{-1} + \sum_{\substack{\text{added } i \\ \text{nondegen. in } \alpha}} \frac{\Delta_i^2}{2b} \\
 &\rightarrow \left| E_{\alpha \text{ orig.}}^{(2)} \right| \frac{1}{ab} + \frac{a}{2b} (\# \text{ of terms unsatisfied in } \alpha) \\
 &= \left| E_{\alpha \text{ orig.}}^{(2)} \right| \frac{1}{n^4} + \frac{1}{4} (E_{|\alpha\rangle}^{(0)} + m) \\
 &= O(n^{-2}) + \frac{1}{4} (E_{|\alpha\rangle}^{(0)} + m) \\
 &\rightarrow \frac{1}{4} E_{|\alpha\rangle}^{(0)} + \text{constant.} \tag{12}
 \end{aligned}$$

One may wonder why $a = b$ was chosen instead of $b \gg a$, since the latter would make the second-order term approach zero, instead of favouring higher-energy final states as it currently does. That, however, could also make higher-order terms approach zero in a similar manner, dramatically increasing the radius of convergence of the series. By setting $a = b$ and expanding a component of the perturbation, one finds that for $q \geq 2$, the q th-order term will have a component of magnitude $(E_{|\alpha\rangle}^{(0)}/4)^{q-1}$ (of alternating sign) and no components proportional to a . For example, it is easily checked that the third-order term is dominated by $(1/16)(E_{|\alpha\rangle}^{(0)})^2$. This means that the radius of convergence scales proportional to $1/E_{|\alpha\rangle}^{(0)}$, getting smaller as the problem size increases. Since the magnitude of the first-order term is proportional to a , and the value of a was chosen to be $\Theta(n^2)$, within the radius of convergence, first-order should dominate $\Omega(n^2 E_{|\alpha\rangle}^{(0)})$ orders of perturbation.

Choosing such a large a also guarantees elimination of crossings if M had first-order degeneracy of its local minima, since for any M with at least one nonzero term, $E_{\alpha \text{ orig.}}^{(1)} > -n$.

After adding the extra qubits, the difference between the first-order terms will still be $\Omega(n^2)$, in favour of the global minimum. As n increases, global minima will become an increasing portion of the states within energy b of the minimum. Thus, for sufficiently large a and b , the actual sizes of the degenerate manifolds do not need to be known.

One may argue that adding qubits with such a large factor b could simply add a new crossing at a point where $\lambda \sim b$ and the original diagonal part of the Hamiltonian can be neglected, effectively giving a diagonal part of just

$$\sum_{i \in M} \sum_{k=1}^a b(h_i \sigma_z^{(i)} + 1)(\sigma_z^{(ik)} + 1)/2 + \sum_{\{i,j\} \in M} \sum_{k=1}^a b(J_{ij} \sigma_z^{(i)} \sigma_z^{(j)} + 1)(\sigma_z^{(ijk)} + 1)/2. \quad (13)$$

However, the ground state of this Hamiltonian is exponentially degenerate and one of these states is always trivially found. This is because there are no local minima: any nonzero term can be made zero by flipping the corresponding added qubit (i_k or ij_k) from $+1$ to -1 . Nonetheless, the effect of the exponentially many excited states of this Hamiltonian on nonperturbative anticrossings is unclear, as the states play a significant role only at or beyond the radius of convergence.

It is important to emphasize that all of the energy scales introduced are polynomial in n , and the number of added qubits is also polynomial in n , since without these constraints, g_{\min} could be trivially made constant.

Given such a dramatic change in degeneracy, it is clear that the construction presented here would also have some type of effect on a classical annealing system. For example, global minima can be made to have an arbitrarily high total probability in the Boltzmann distribution. However, it is not clear whether this is associated with finding these global minima faster, since it is associated with, once a global minimum has been reached, being effectively trapped in the global minima for an exponentially long time. This is because at all but one of the exponentially many global minima, a polynomially high barrier in energy must be surpassed to escape. The same is true of low local minima, meaning that if such a construction were to have any benefit classically, it would have to direct the system away from low local minima. For AQO, the construction does exactly that, causing the ground state, within the radius of convergence, to avoid local minima of H_P .

5. Extending to arbitrary coefficients

The above construction for Ising model terms with coefficients in $\{-1, 0, +1\}$ is most easily extended to integer coefficients. For a term with integer coefficient $+k$, simply rewrite it as k identical terms with coefficient $+1$ and apply the construction as before on all k terms. The same splitting works with coefficients of $-k$ becoming k terms with coefficient -1 .

To extend this beyond integers to polynomial precision real numbers, one could simply rescale M to be polynomially approximated by integers, but a more reasonable construction is available from that

$$E_{|\alpha\rangle}^{(1)} = - \sum_{i \text{ degen. in } \alpha} \Delta_i. \quad (14)$$

One can simply round a coefficient down to the nearest integer below, apply the construction as before, then add a extra qubits as above whose Δ_i values are selected to be the remainder that was rounded off, instead of 1. This has negligible impact on higher order terms in that the radius

of convergence is still proportional to $1/E_{|\alpha}^{(0)}$ (unless the coefficients were chosen to be scaled such that the bulk of weight lies in the rounded off components, which is easily fixed by scaling up the coefficients into a more reasonable range).

6. Conclusion

Above, we have presented a simple method of posing the NP-hard problem of finding the ground state of an arbitrarily-connected Ising model with local fields as an AQO with no perturbative crossings between local and global minima. Thus, all NP-complete problems can be solved using AQO without encountering these crossings.

It is critical to note that this does *not* prove that AQO can solve NP-complete problems in polynomial time. However, it does mean that proving otherwise requires identifying some effect other than perturbative crossings that unavoidably results in exponentially long adiabatic runtimes.

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Appendix. Reduction to 2-local

As mentioned above, the prescription above introduces a 3-local term when adding an extra qubit corresponding with a J term in the original Ising model. However, a 2-local function with the necessary properties can be constructed by adding another extra qubit. The properties to maintain are:

1. All values of the function are ≥ 0 .
2. All configurations of qubits i and j that satisfy the associated $J_{ij} \neq 0$ term must have 2 configurations of the additional qubits, 1 bit flip apart, where the function is 0.
3. All configurations of qubits i and j that do not satisfy the associated $J_{ij} \neq 0$ term must have 1 configuration of the additional qubits where the function is 0.
4. All other configurations must give values $\geq 2b$.

The following 4-qubit, 2-local function satisfies these properties for a positive J . $(\sigma_z^{(i)} + 1)/2$ is abbreviated as x_i , etc

$$f_{(+)}(\sigma_z^{(i)}, \sigma_z^{(j)}, \sigma_z^{(ijk)}, \sigma_z^{(ijk*)}) = \frac{b}{2} \left(4(x_i x_j + x_j x_{ijk} + x_{ijk} x_i) + 6(x_i + x_j + x_{ijk})(1 - 2x_{ijk*}) + 8x_{ijk*} - 1 + \sigma_z^{(i)} \sigma_z^{(j)} + \sigma_z^{(ijk)} + 1 \right). \quad (\text{A.1})$$

The terms inside the outer parentheses up to and including the -1 encode a $+\sigma_z^{(i)} \sigma_z^{(j)} \sigma_z^{(ijk)}$ term. Note that alternate encodings of this 3-local term where exactly 2 of x_i , x_j , and x_{ijk} are flipped

(e.g. replaced with $1 - x_i$, etc) do not maintain the needed properties of the overall function. The cost table of $f_{(+)}$ is as follows:

			$f_{(+)} / 2b$	
$\sigma_z^{(i)}$	$\sigma_z^{(j)}$	$\sigma_z^{(ijk)}$	$\sigma_z^{(ijk*)} = -1$	$\sigma_z^{(ijk*)} = +1$
-1	-1	-1	0	2
-1	-1	+1	2	1
-1	+1	-1	1	0
-1	+1	+1	4	0
+1	-1	-1	1	0
+1	-1	+1	4	0
+1	+1	-1	4	0
+1	+1	+1	8	1

It's clearly visible that for some value of new qubit ijk^* , qubit ijk is at degeneracy when the original J term is satisfied, but when the J term is unsatisfied, neither qubit is at degeneracy.

The following 4-qubit, 2-local function satisfies these properties for a negative J :

$$f_{(-)}(\sigma_z^{(i)}, \sigma_z^{(j)}, \sigma_z^{(ijk)}, \sigma_z^{(ijk*)}) = \frac{b}{2} \left(4 \left((1 - x_i)x_j + x_jx_{ijk} + x_{ijk}(1 - x_i) \right) + 6 \left((1 - x_i) + x_j + x_{ijk} \right) (1 - 2x_{ijk*}) + 8x_{ijk*} - 1 - \sigma_z^{(i)}\sigma_z^{(j)} + \sigma_z^{(ijk)} + 1 \right). \quad (\text{A.2})$$

The terms inside the outer parentheses up to and including the -1 encode a $-\sigma_z^{(i)}\sigma_z^{(j)}\sigma_z^{(ijk)}$ term. Instead of flipping x_i (i.e. replacing with $1 - x_i$), it also works to instead flip either x_j or x_{ijk} , so long as it is only 1 of the 3. Note that an alternate encoding of this 3-local term where all 3 are flipped does not maintain the needed properties of the overall function. The cost table of $f_{(-)}$ as defined above is as follows:

			$f_{(-)} / 2b$	
$\sigma_z^{(i)}$	$\sigma_z^{(j)}$	$\sigma_z^{(ijk)}$	$\sigma_z^{(ijk*)} = -1$	$\sigma_z^{(ijk*)} = +1$
-1	-1	-1	1	0
-1	-1	+1	4	0
-1	+1	-1	4	0
-1	+1	+1	8	1
+1	-1	-1	0	2
+1	-1	+1	2	1
+1	+1	-1	1	0
+1	+1	+1	4	0

It's again visible that qubit ijk is at degeneracy when and only when the original J term is satisfied.

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