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Correlations between spectra with different symmetries: any chance to be observed?

P Braun[1,2], **F Leyvraz**[3,4,5] **and T H Seligman**[3,4]
[1] Fachbereich Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany
[2] Institute of Physics, Saint Petersburg University, 198504 Saint Petersburg, Russia
[3] Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Cuernavaca, Morelos, México
[4] Centro Internacional de Ciencias, Cuernavaca, Morelos, México
E-mail: leyvraz@fis.unam.mx

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Abstract. A standard assumption in quantum chaology is the absence of correlation between spectra pertaining to different symmetries. Doubts were raised about this statement for several reasons, in particular because in semiclassics the spectra of different symmetries are expressed in terms of the same set of periodic orbits. We re-examine this question and notice the absence of correlations in the universal regime. In the case of continuous symmetry, the problem is reduced to parametric correlation, and we expect correlations to be present up to a certain time which is essentially classical but larger than the ballistic time.

[5] Author to whom any correspondence should be addressed.

Contents

1. Introduction

The topic we are to consider is the cross-correlation between spectra of different symmetries for the so-called quantum chaotic systems, i.e. quantum systems whose classical analogue is chaotic. The instinctive reaction would be to say that states of different symmetries live in different spaces, do not talk to each other and cannot conceivably display any correlation [1]. Doubts appear when one recalls that in the semiclassical approach the level correlations of different symmetries can be expressed in terms of sums over the same set of periodic orbits, and such 'brotherhood in the parent orbits' may introduce a correlation. Such 'brotherhood in parent classical trajectories' is known to lead to interesting physical effects, e.g. phenomena such as 'weak localization' or 'elastic enhancement', in the transport through symmetric cavities [2]; it may thus be suspected to introduce cross-symmetry spectral correlations. In nuclear physics, some evidence has been assembled that correlations in two-body random ensembles with finite sets of states exist, which is not too surprising considering the small number of independent two-body matrix elements [3], and certain nuclear scattering data support the existence of correlations to some extent [4], but a study of correlations between spectra of opposite parity in the limit of many particles revealed no correlations [5].

These findings make it all the more desirable to analyze the usual assumption of independence in the semiclassical limit in some detail, as we shall do here. We obtain basically a negative result in the case of both discrete and continuous symmetries, i.e. the absence of correlations is confirmed in the universal regime.

These conclusions are to be taken with a grain of salt. Correlations may still exist in certain pathological systems (mostly those with disconnected phase space, often due to unusual behavior under time-reversal symmetry [6–8]. Our reasoning also does not apply to arithmetic billiards which have chaotic dynamics, but an exponentially large number of geodesics with identical actions [9]. Nevertheless, it has been shown, by explicit evaluation of the Selberg trace sum formula for the correlation function, that eigenvalues for even and odd parity in the modular domain on the surface with constant negative curvature are indeed uncorrelated [10]. More importantly, for continuous symmetries correlations may exist up to a certain time which is essentially classical in nature, if the difference of quantum numbers characterizing the irreducible representations (IRs) is of the order of one as the quantum numbers themselves get large. Note that this effect can be easily overlooked, as times are often given in terms of the Heisenberg time, which, in the semiclassical limit, goes to infinity.

The paper is structured as follows. After laying foundations we discuss the discrete symmetries in what we consider to be the simplest case. This is the reflection symmetry in a two-dimensional system when we can pass to a half-space with the Dirichlet or Neumann boundary conditions. We obtain the relevant results for these two particular cases using periodic orbit expansions and find the expected absence of correlations. Starting with the diagonal approximation, we later discuss the off-diagonal contributions and times beyond the Heisenberg time. After that, we study general discrete symmetries. Trivial generalization of the described technique is possible only for groups that induce a decomposition of space into fundamental domains. This is not always true and we shall present the general argument using the symmetry decomposition of periodic orbits, as proposed in [11].

In the next section, we pass to continuous symmetry and consider the effect of a change in the index of IR as parametric correlation [12] within the symmetry-reduced Hilbert space. Again, we predict the absence of correlations in the universal regime, i.e. on the scale of the Heisenberg time; however, correlations on a classical, i.e. $\hbar$-independent, time scale are expected. Finally, we present some conclusions and an outlook about possible observations.

2. Definitions: the spectral cross-correlation function and the form factor

We start with some definitions that will be essential to the later arguments, and at the same time we fix the notations we shall use.

Let $\rho^{(1)}(E)=\sum_k \delta(E-E_k^{(1)})$ and $\rho^{(2)}(E)=\sum_k \delta(E-E_k^{(2)})$ be two spectral densities describing either two subspectra of different symmetries of the same Hamiltonian or the spectra of two different Hamiltonians. We shall assume that both spectra are unfolded to the same constant mean level density Δ (see e.g. [1]). The precise manner of the unfolding procedure is a somewhat thorny issue and will be discussed later.

The two-level cross-correlation function is defined as

$$R^{(1,2)}(E,\varepsilon)=\frac{\langle \rho^{(1)}(E+\frac{\varepsilon\Delta}{2\pi})\rho^{(2)}(E-\frac{\varepsilon\Delta}{2\pi})\rangle}{\langle\rho^{(1)}(E)\rangle\langle\rho^{(2)}(E)\rangle}-1, \tag{1}$$

where $\langle\cdots\rangle$ denotes averaging over an interval of E and smoothing over a window of the dimensionless energy offset ε; since the spectra are unfolded we have $\langle\rho^{(1)}(E)\rangle=\langle\rho^{(2)}(E)\rangle=\Delta^{-1}$.

It is often more convenient to work with the Fourier transform of $R^{(1,2)}(E,\varepsilon)$ with respect to ε. The result is the cross form factor, which is a double sum,

$$K^{(1,2)}(\tau)=\sum_{k,l}\mathrm{e}^{\mathrm{i}(E_k^{(1)}-E_l^{(2)})\tau T_{\mathrm{H}}/\hbar}; \tag{2}$$

the dimensionless $\tau=T/T_{\mathrm{H}}$ is time in units of the Heisenberg time $T_{\mathrm{H}}=2\pi\hbar/\Delta$.

A semiclassical representation for the spectral correlator and the form factor follows from the Gutzwiller expansion of the fluctuating part of the spectral density in terms of the classical periodic orbits γ,

$$\rho_{\mathrm{osc}}(E)\sim\sum_{\gamma^{(1)}}A_\gamma\,\mathrm{e}^{\mathrm{i}S_\gamma/\hbar-\mathrm{i}\mu_\gamma\pi/2}. \tag{3}$$

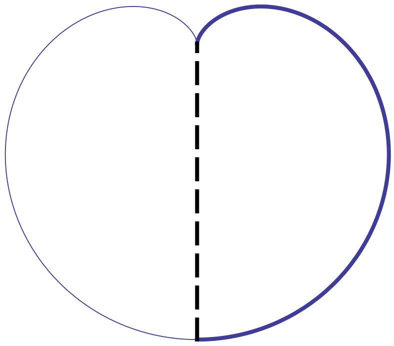

Figure 1. The cardioid billiard. The fundamental domain is bordered by the symmetry line (dashed) and the right (bold) half of the billiard wall.

Here S_γ, A_γ and μ_γ are the action, the stability coefficient and the Maslov index of the orbit γ. Strictly speaking this expression should also include the periodic orbit repetitions; these, however, become irrelevant in the semiclassical limit. Replacing, in (2), both spectral densities by the respective expansions (3), we obtain

$$K^{(1,2)}(\tau) \sim \frac{1}{\delta}\left\langle \sum_{T_{\mathrm{H}}\tau<T_{\gamma_1},\, T_{\gamma_2}<T_{\mathrm{H}}\tau+\delta} A_{\gamma_1}A_{\gamma_2}\mathrm{e}^{\mathrm{i}(S^{(1)}_{\gamma_1}-S^{(2)}_{\gamma_2})/\hbar-\mathrm{i}(\mu^{(1)}_{\gamma_1}-\mu^{(2)}_{\gamma_2})\pi/2}\right\rangle; \tag{4}$$

the brackets $\langle\cdots\rangle$ denote averaging over E and smoothing over a small window of τ. The sum is taken over periodic orbits γ_1 of system 1 and γ_2 of system 2. Their periods T_{γ_1} and T_{γ_2} must lie in the interval $[T, T+\delta]$ whose width δ is small compared with $T = \tau T_{\mathrm{H}}$.

3. Discrete symmetry

To illustrate the central point of our argument we shall start with the simplest possible example of a system with a single reflection symmetry.

3.1. A simple example: reflection symmetry

Consider the spectral correlation between the even and odd spectra in a chaotic system with two degrees of freedom such as the cardioid billiard [13–15]; see figure 1. On the line of symmetry, the even and odd wave functions either have zero normal derivative or themselves turn into zero. The subspectra of definite parity can be found separately by solving the Schrödinger equation in the desymmetrized billiard (the fundamental domain) obtained by cutting the billiard along the symmetry line and dropping one of the halves. Imposing the Neumann (resp. Dirichlet) boundary condition on the cut, we shall obtain the even (resp. odd) part of the spectrum.

The semiclassical level density consists of the smooth part (the Weyl term) and the fluctuating part which is the Gutzwiller sum over periodic orbits of the desymmetrized billiard with the mirror-reflecting wall at the symmetry line. The difference in the Weyl term between the even and odd cases is responsible for the fact that the Nth eigenvalue in the symmetric sector E_N^S is lower than the Nth eigenvalue in the antisymmetric sector E_N^A by an amount of the order of $\sqrt{E_N^{A,S}}$; see [16] and references therein. This issue has some consequences but can be taken care by an appropriate unfolding procedure. In the Gutzwiller sum, the even and odd cases differ by the effective Maslov indices of the periodic orbits since each reflection in the Dirichlet and Neumann boundary contributes π or zero, respectively, to the phase of the orbit contribution.

We shall denote by n_γ the number of strikes of the periodic orbit γ of the desymmetrized billiard against the line of symmetry. It is easy to see that orbits with even n_γ correspond to the periodic orbits of the full billiard with the same period; the latter are non-symmetric unless γ is self-retracing. On the other hand, orbits with odd n_γ are associated with symmetric orbits of the full billiard whose period is doubled compared with γ.

We assume that the number of orbits γ with periods in a certain interval $[T, T+\delta]$ with T large may not depend on parity of n_γ, as well as on parity of the number of visits to any side of the desymmetrized billiard. This assumption is physically plausible and must hold at least for generic systems; we shall see that it guarantees the absence of spectral cross-correlation. Note that we consider here cases in which all orbits unconnected by time reversal have *different* actions. In non-generic cases, for which classical orbits have exactly degenerate actions due to additional symmetries, we may find counterexamples: such anomalous behavior is found, for example, in the so-called arithmetic billiards on the surface of constant negative curvature [17].

The semiclassical cross form factor now becomes a double sum over the same set of periodic orbits of the desymmetrized system. In the following expression, summation over γ_1 and γ_2 is connected with the Gutzwiller expansion of the even and odd spectral densities, respectively,

$$K_{\mathrm{gu}}(\tau) \sim \frac{1}{\delta}\left\langle \sum_{T_{\mathrm{H}}\tau<T_{\gamma_1},\, T_{\gamma_2}<T_{\mathrm{H}}\tau+\delta} A_{\gamma_2}\,(-1)^{n_{\gamma_2}}\, \mathrm{e}^{\mathrm{i}\left(S_{\gamma_1}-S_{\gamma_2}\right)/\hbar} \right\rangle; \tag{5}$$

the factor $(-1)^{n_{\gamma_2}}$ emerges because each visit to the symmetry line with the Dirichlet boundary condition yields the Maslov phase π.

3.2. Diagonal approximation

Dropping, in (5), all summands with γ_1 not coinciding with γ_2 (up to time reversal since we have time-reversal invariance), we come to the diagonal approximation

$$K_{\mathrm{gu,diag}}(\tau) \sim \frac{1}{\delta}\left\langle \sum_{T_{\mathrm{H}}\tau<T_{\gamma}<T_{\mathrm{H}}\tau+\delta} A_{\gamma}^{2}\,(-1)^{n_\gamma} \right\rangle.$$

It differs from Berry's diagonal approximation for the autocorrelation form factor [18] by the presence of the sign factor $(-1)^{\gamma}$ leading to destructive interference among the contributions. Assuming that for the long orbits with very large n_γ the number of orbits with even and odd n_γ is the same and not correlated with the stability coefficients, we obtain $K_{\mathrm{gu,diag}}(\tau) = 0$.

A warning has to be made here. Our basic argument (even and odd spectral densities are created by the same set of orbits which contribute with the same action but a different Maslov index) implicitly assumes ergodicity of the classical motion and loses its validity if ergodicity is violated. Consider, for example, a particle moving in a symmetric double-well potential in two dimensions. For energies below the dividing barrier the classical motion is clearly non-ergodic since the available phase space disconnects into two symmetric domains; even and odd levels in the quantum problem are then almost degenerate, up to the tunnelling splitting. Consequently, the cross $g \longleftrightarrow u$ form factor will practically coincide with the autocorrelation form factor of the spectrum in a single well.

Another trivial issue should also be pointed out here: for very short times (ballistic times) corresponding to the period of the shortest classical orbits, there are no more cancellations between different orbits and correlations certainly arise.

3.3. Off-diagonal contributions

The off-diagonal contribution for the auto form factor differing from (5) by the absence of the sign factor $(-1)^{n_\gamma}$ is known to stem from pairs of the so-called orbit-partners. A partner γ' of an orbit γ is the orbit that practically coincides with γ everywhere but in the 'encounters', i.e. a set of two or more orbits stretches almost parallel and abnormally close to each other. The lengths of γ and γ' can be so close that their action difference can be comparable with or smaller than $\hbar$. Consequently, the corresponding contributions to the auto form factor will avoid the destructive interference, which, in the case of randomly composed pairs, would lead to annihilation of the contribution. The leading off-diagonal correction is due to the so-called Sieber–Richter pairs [19]; higher order terms giving the auto form factor as a series in powers of τ are calculated in [20]. Analytically continuing the result, one obtains the auto form factor for times smaller than the Heisenberg time, i.e. for $\tau < 1$; for larger times a different approach is needed (see subsection 3.4).

Let us now address the cross form factor. Again, contributions of randomly composed pairs are expected to average to zero. Limiting summation only to pairs of the orbit-partners, we may reduce (5) to

$$K_{\rm gu}(\tau) \sim \left\langle \frac{1}{\delta} \sum_{\tau T_{\rm H} < \bar{T}_\gamma < \tau T_{\rm H} + \delta} (-1)^{n_\gamma} A_\gamma^2 \left(\sum_{\gamma'(\gamma)} {\rm e}^{\frac{{\rm i}\Delta \bar{S}_{\gamma\gamma'}}{\hbar}} \right) \right\rangle .$$

The inner sum is taken over all partners γ' of the orbit γ including the diagonal ones. Since the partner orbit coincides with the original orbit everywhere except the relatively short encounter stretches, the Maslov indices of γ and γ', coincide, apart from the contribution of reflections from the Dirichlet boundary. Again, we assume that there is no correlation between parity of the number of strikes of the orbit γ against the cut and the value of the sum over partners of γ multiplied by its stability factor. Hence the expected value of $K_{\rm gu}(\tau)$, with the off-diagonal contributions taken into account, is zero, at least for $\tau < 1$.

3.4. Times larger than the Heisenberg time

The fact that maxima (and, in the case of GSE, even singularities) of form factors can occur at the Heisenberg time makes it necessary to investigate the long-time behavior in the discrete symmetry case. Semiclassical investigation of spectral auto- and cross-correlation beyond the Heisenberg time was started by Bogomolny and Keating [21]. Complete expansion of the auto-correlation functions using the formalism of the four-determinant generating functions is given in [22, 23]; the results of the latter can be generalized to cross-correlations.

The appropriate generating function of the two-point cross-correlation functions for the spectra of Hamiltonians $H^{(1)}$ and $H^{(2)}$ is a ratio of four spectral determinants,

$$Z^{(12)}(\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D) = \left\langle \frac{\det\left(E + \varepsilon_C \Delta/2\pi - H^{(1)}\right) \det\left(E + \varepsilon_D \Delta/2\pi - H^{(2)}\right)}{\det\left(E + \varepsilon_A \Delta/2\pi - H^{(1)}\right) \det\left(E + \varepsilon_B \Delta/2\pi - H^{(2)}\right)} \right\rangle_E , \qquad (6)$$

where $\langle\cdots\rangle_E$ as usual indicates averaging over the center energy E and smoothing over small windows of the energy offsets from E; unlike the generating function in the problem of autocorrelation it is not symmetric with respect to the exchange of ε_C and ε_D. Using the identity

$$\frac{\partial \ln\det(E-H)}{\partial E} = \mathrm{Tr}\,(E-H)^{-1} \tag{7}$$

and proportionality between $\mathrm{Im}\,\mathrm{Tr}(E-H)^{-1}$ and the level density $\rho(E)$, we obtain the cross-correlation function as the second derivative,

$$\begin{aligned} R^{(12)}(\varepsilon) &= \mathrm{Re}\lim_{\eta\to+0}\left\langle \mathrm{Tr}\left(E+\frac{\varepsilon^+\Delta}{2\pi}-H^{(1)}\right)^{-1}\mathrm{Tr}\left(E-\frac{\varepsilon^+\Delta}{2\pi}-H^{(2)}\right)^{-1}\right\rangle_E - 1 \\ &= 2\mathrm{Re}\lim_{\eta\to+0}\frac{\partial^2}{\partial\varepsilon_A\partial\varepsilon_B}Z^{(12)}(\varepsilon_A,\varepsilon_B,\varepsilon_C,\varepsilon_D)\bigg|_{\parallel}, \end{aligned} \tag{8}$$

$$\varepsilon^+ = \varepsilon + \mathrm{i}\eta. \tag{9}$$

Here η is a small positive number and the symbol '$\parallel$' denotes the substitution of the arguments,

$$\varepsilon_A = \varepsilon^+,\quad \varepsilon_B = -\varepsilon^+,$$

$$\varepsilon_C = \varepsilon,\quad \varepsilon_D = -\varepsilon.$$

The semiclassical approximation for the generating function follows from that for the spectral determinants. Using the so-called 'Riemann–Siegel look-alike' asymptotics of $\det(E-H)$ introduced by Berry and Keating [24–26], one can show that the semiclassical estimate of $Z^{(12)}$ consists of two terms. One of these substituted into (11) leads to a correlation function whose Fourier transformation reproduces the form factor for small times. The second term in $Z^{(12)}$ is responsible for the behavior of the form factor for times larger than T_{H}. Each term in the semiclassical generating function is obtained as an infinite product over the periodic orbits, averaged over the central energy E. The leading order contribution is obtained by the diagonal approximation, which, in the formalism of the generating functions, means that the factors associated with different periodic orbits are assumed independent (with the obvious exception of the mutually time-reversed orbits); consequently, averages of products over the periodic orbits are replaced by the products of averages.

The diagonal approximation for the generating function describing the gu correlation is deduced in the same way, step by step, as in the case of autocorrelation; for details, see [22, 23]. At a certain stage, one obtains

$$Z^{(\mathrm{gu})}(\varepsilon_A,\varepsilon_B,\varepsilon_C,\varepsilon_D) = Z_{\mathrm{I}} + Z_{\mathrm{II}},$$

$$Z_{\mathrm{I}} = \mathrm{e}^{\mathrm{i}/2(\varepsilon_A-\varepsilon_B-\varepsilon_C+\varepsilon_D)}\prod_\gamma \exp g\vartheta_{\mathrm{I},\gamma},$$

$$Z_{\mathrm{II}} = \mathrm{e}^{\mathrm{i}/2(\varepsilon_A-\varepsilon_B-\varepsilon_D+\varepsilon_C)}\prod_\gamma \exp g\vartheta_{\mathrm{II},\gamma}.$$

The factor g in the exponents is equal to 2 for systems with time reversal allowed (such as the cardioid billiard) where it reflects the existence of pairs of mutually time-reversed orbits; in the absence of time reversal $g = 1$. The exponents are given by

$$\vartheta_{\mathrm{I},\gamma} = |F_\gamma|^2(\mathrm{e}^{\mathrm{i}\tau_\gamma\varepsilon_A} - \mathrm{e}^{\mathrm{i}\tau_\gamma\varepsilon_C})(\mathrm{e}^{-\mathrm{i}\tau_\gamma\varepsilon_B} - \mathrm{e}^{-\mathrm{i}\tau_\gamma\varepsilon_D})(-1)^{n_\gamma},$$
$$\vartheta_{\mathrm{II},\gamma} = -|F_\gamma|^2[\mathrm{e}^{\mathrm{i}\tau_\gamma\varepsilon_A} - (-1)^{n_\gamma}\mathrm{e}^{\mathrm{i}\tau_\gamma\varepsilon_D}][\mathrm{e}^{-\mathrm{i}\tau_\gamma\varepsilon_C} - (-1)^{n_\gamma}\mathrm{e}^{-\mathrm{i}\tau_\gamma\varepsilon_B}].$$

We denote here by $\tau_\gamma = T_\gamma / T_{\mathrm{H}}$ the period of a periodic orbit γ in terms of the Heisenberg time; F_γ is its stability coefficient. As before, n_γ is the number of visits of the orbit γ to the cut in the desymmetrized billiard; if the factors $(-1)^{n_\gamma}$ were dropped, we would return to the case of autocorrelation.

What remains is to perform summation over the periodic orbits in the exponents of Z_{I}, Z_{II}. Again, we assume absence of correlation between parity of n_γ, on the one hand, and the orbit period and the stability coefficient, on the other. Then the summands containing the factor $(-1)^{n_\gamma}$ disappear after summation over γ such that

$$\sum_\gamma \vartheta_{\mathrm{I},\gamma} \to 0, \tag{10}$$

$$\sum_\gamma \vartheta_{\mathrm{II},\gamma} \to -\sum_\gamma |F_\gamma|^2[\mathrm{e}^{\mathrm{i}\tau_\gamma(\varepsilon_A-\varepsilon_C)} + \mathrm{e}^{\mathrm{i}\tau_\gamma(\varepsilon_D-\epsilon_B)}]; \tag{11}$$

consequently the term Z_{I} in the generating function is an irrelevant constant.

The sum in (11) can be evaluated using the sum rule of Hannay and Ozorio de Almeida [27] for the stability coefficients F_γ,

$$\sum_{T<T_\gamma<T+\Delta T} |F_\gamma|^2 = \frac{\Delta T}{T},$$

which gives

$$\sum_\gamma \vartheta_{\mathrm{II},\gamma} \approx -\int_{\tau_0}^{\infty} \frac{\mathrm{d}\tau}{\tau}[\mathrm{e}^{\mathrm{i}\tau(\varepsilon_A-\varepsilon_C)} + \mathrm{e}^{\mathrm{i}\tau(\varepsilon_D-\epsilon_B)}]$$
$$\approx \ln[(\varepsilon_A-\varepsilon_C)(\varepsilon_D-\epsilon_B)] + 2\ln\tau_0 + 2\gamma - \mathrm{i}\pi,$$

with γ denoting the Euler constant. The lower integration limit is $\tau_0 = T_0/T_{\mathrm{H}}$ with T_0 standing for some minimal period above which periodic orbits behave approximately ergodically; in the semiclassical limit τ_0 tends to zero. The second term in the generating function is thus

$$Z_{\mathrm{II}}(\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D) \propto \mathrm{e}^{\mathrm{i}/2(\varepsilon_A-\varepsilon_B-\varepsilon_D+\varepsilon_C)}[\tau_0^2(\varepsilon_A-\varepsilon_C)(\varepsilon_D-\epsilon_B)]^g.$$

The respective two-point correlation function is zero in the presence of time reversal when $g = 2$. Without time reversal ($g = 1$) the term Z_{II} contributes like const $\tau_0^2\mathrm{e}^{\mathrm{i}2\varepsilon}$ to the cross-correlation function or, equivalently, like a delta-peak to the time-dependent cross form factor $K_{\mathrm{gu}}(\tau)$ at $\tau = 1$, i.e. at the Heisenberg time. However, since the term is proportional to τ_0^2, this contribution vanishes in the semiclassical limit. Therefore, as could be expected, correlations between the even and odd parts of the spectrum do not exist also for times larger than the Heisenberg time.

3.5. General discrete symmetries

In this section, we give a formal treatment of the problem in the presence of more complicated discrete symmetry groups G. The level density $\rho(E)$ falls into a sum of subdensities $\rho_m(E)$ connected with different IRs m of G. The semiclassical approximation for the fluctuating part of $\rho_m(E)$ is provided by the symmetry-adapted Gutzwiller trace formula [11]. In the following, we shall follow the ideas expounded in [11], so that our treatment includes the case in which the reduction of the configuration space M of the system to a fundamental domain $\bar{M}$, which tesselates the full M when the symmetry operations are applied to it, is difficult or impossible.

Let us look at the cross form factor defined as in (2)

$$K^{(m_1,m_2)}(\tau) = \sum_{k,l} \mathrm{e}^{\mathrm{i}\left(E_{k,m_1}-E_{l,m_2}\right)t/\hbar}; \tag{12}$$

throughout this section the argument of the form factor $\tau = t/T_{\mathrm{H}}$, i.e. time measured in units of the Heisenberg time. What we want to show is that this quantity vanishes. At the abstract level this can be rewritten as

$$K^{(m_1,m_2)}(\tau) = \left[\mathrm{Tr}\left(P_{m_1}\mathrm{e}^{-\mathrm{i}Ht/\hbar}P_{m_1}\right)\right]^{\star} \cdot \left[\mathrm{Tr}\left(P_{m_2}\mathrm{e}^{-\mathrm{i}Ht/\hbar}P_{m_2}\right)\right], \tag{13}$$

where P_m is the projector of the Hilbert space on the IR m. It is given by the expression

$$P_m = \frac{1}{|G|}\sum_{g\in G}\chi_m(g)g, \tag{14}$$

where $\chi_m(g)$ is the character corresponding to the IR m and g denotes the group element, which is assumed to act directly on the elements of the Hilbert space.

As is well known, traces can be estimated semiclassically as integrals over coherent states $|p,q\rangle$. Assuming that the symmetry operations g have a well-defined semiclassical limit, we know how they act on the phase space as well as on the Hilbert space and thus are able to define such quantities as $g(p,q)$. One therefore obtains

$$K^{(m_1,m_2)}(\tau) = \frac{1}{|G|^2}\sum_{g,g'\in G}\int \frac{\mathrm{d}p_1\,\mathrm{d}q_1\,\mathrm{d}p_2\,\mathrm{d}q_2}{(2\pi)^{2d}}[\chi_{m_1}(g)\langle p_1,q_1|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|g(p_1,q_1)\rangle]^* \\ \times\chi_{m_2}(g')\langle p_2,q_2|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|g'(p_2,q_2)\rangle, \tag{15}$$

where we have used the identity $\mathrm{Tr}(PAP) = \mathrm{Tr}(AP)$, where P is an arbitrary projector. Since the $|p,q\rangle$ are well localized in phase space, the kind of expressions occurring in (15) only differs from zero when the phase space point (p,q) is (at least approximately) carried over to $g(p,q)$ in time t by the classical evolution. A rigorous proof of this reasoning exists so far for times smaller than the Ehrenfest time; however, we shall assume its validity for all times comparable with the Heisenberg times. This turns out to be equivalent to the usage of the Gutzwiller trace formula in the form factor estimate, to which we may then apply the usual approximations. In this section, for simplicity, we shall limit ourselves to the diagonal approximation and we refer to the simple example given above for refinements.

To this end, let us start by looking at the quantity $\mathrm{Tr}(P_{m_1}\mathrm{e}^{-\mathrm{i}Ht/\hbar}P_{m_1})$. Let us define as a generalized periodic orbit of period t any segment that connects a phase space point (p,q) with one of its symmetric partners $g(p,q)$ over time t. We call such orbits g-orbits, making explicit reference to the corresponding group element. Let λ be a g-orbit. One then additionally defines

g_λ to be the minimal element of the symmetry group G such that $g_\lambda(p,q)$ lies on the orbit λ going through (p,q) and such that no other element g' of G has the property that $g'(p,q)$ lies on the orbit λ between (p,q) and $g_\lambda(p,q)$. Note that in general $g_\lambda \neq g$. There is, however, as shown in [11], always an integer n such that $g = g_\lambda^n$.

Let (p,q) lie on a g-orbit λ. One then has

$$\langle p, q|\, \mathrm{e}^{-\mathrm{i}Ht/\hbar}\, |g(p,q)\rangle = A_\lambda \mathrm{e}^{\mathrm{i}S_\lambda/\hbar}, \tag{16}$$

where A_λ is related to the stability of the orbit in the usual way and the phase $\mathrm{e}^{\mathrm{i}S_\lambda/\hbar}$ contains all the information on the action of the g-orbit as well as its Maslov phases.

Integrating the lhs of (16) over p, q yields

$$\mathrm{Tr}\left(P_{m_2}\mathrm{e}^{-\mathrm{i}Ht/\hbar}P_{m_2}\right) = \frac{1}{|G|}\sum_{g\in G}\frac{t}{(2\pi)^d}\sum_{n=1}^{\infty}\sum_{\lambda} A_\lambda \mathrm{e}^{\mathrm{i}S_\lambda/\hbar}\chi_{m_2}(g)\delta_{g,g_\lambda^n}, \tag{17}$$

where n is the number of iterations of the primitive element leading to g. The factor t arises from the fact that an integration over the whole segment from (p,q) to $g(p,q)$ is possible and yields the same result as starting with (p,q). The sum still runs independently over the various symmetric 'copies' of the g-orbit λ, in particular over its iterations, which lead eventually to a bona fide periodic orbit. Neglecting repetitions, that is, $n \neq 1$, yields

$$\mathrm{Tr}\left(P_{m_2}\mathrm{e}^{-\mathrm{i}Ht/\hbar}P_{m_2}\right) = \frac{t}{(2\pi)^d|G|}\sum_{\lambda} A_\lambda \mathrm{e}^{\mathrm{i}S_\lambda/\hbar}\chi_{m_2}(g_\lambda). \tag{18}$$

Let us now multiply this with the complex conjugate of the corresponding expression for m_1. This gives

$$K^{(m_1,m_2)}(\tau) = \frac{t^2}{(2\pi)^{2d}|G|^2}\sum_{\lambda,\mu} A_\lambda A_\mu^\star \mathrm{e}^{\mathrm{i}(S_\lambda - S_\mu)/\hbar}\chi_{m_1}^\star(g_\mu)\chi_{m_2}(g_\lambda). \tag{19}$$

This formula generalizes (5) to arbitrary symmetries.

One now introduces the diagonal approximation, which involves keeping in this sum only those terms that have a phase equal to one: that is, only to sum over those pairs of λ and μ such that $S_\lambda = S_\mu$. Since we consider the different iterations of a g orbit to be different and since we have already discarded orbits that are non-primitive, and since, in the same vein, we discard those orbits that have a non-trivial symmetry (that is, we assume that the only $h \in G$ such that $h(p,q) = (p,q)$ is the identity), it is clear that the total number of such pairs is $|G|^2$, independent of λ and μ: there are $|G|$ degenerate λ and the same number of degenerate μ. The final result is hence

$$K^{(m_1,m_2)}(\tau) = \frac{t^2}{(2\pi)^{2d}}\sum_{\lambda}|A_\lambda|^2\chi_{m_1}^\star(g_\lambda)\,\chi_{m_2}(g_\lambda), \tag{20}$$

where summation is carried out now over one representative g-orbit λ among the $|G|$ degenerate ones. In other terms, the sum corresponds to a desymmetrized problem. Keeping all g-orbits would require a compensating factor of $|G|^{-1}$ in front of the rhs of (20).

One now rewrites this as

$$K^{(m_1,m_2)}(\tau) = \frac{t^2}{(2\pi)^{2d}}\sum_{g}\chi_{m_1}^\star(g)\chi_{m_2}(g)\sum_{\lambda}|A_\lambda|^2\delta_{g,g_\lambda} \tag{21}$$

and we now make a final assumption altogether similar to those we have been making throughout this paper: namely, we assume that sufficiently long g-orbits are equidistributed on all elements of the group G; that is, an approximately equal number of large orbits corresponds to every element g of the group G. While this is surely a reasonable assumption, it should be pointed out that actually proving it seems altogether hopeless. To the best of our knowledge, no results (apart from numerical work) are known concerning this issue. This leads us to state that the sum over λ is in fact independent of g. This is, of course, sufficient to guarantee that $K^{(m_1,m_2)}(\tau)=0$ whenever $m_1\neq m_2$. To obtain the corrections due to encounters, we proceed essentially as in the simple example treated above.

4. Continuous symmetry

In the presence of continuous symmetry, the problem has one or several integrals of motion due to Noether's theorem. We focus on one of them, which can be chosen as an action and whose eigenvalues in the semiclassical limit are given by $\hbar m$, where m is an integer. Our question is whether correlation may exist between the energy spectra of two operators, H_m and H_{m+1}. Here we denote by H_m the Hamiltonian H reduced to the appropriate symmetry subspace. In the following, we limit ourselves to the case in which the dynamics of these Hamiltonians are all overwhelmingly chaotic in the energy range of interest.

A simple example that we shall use to illustrate our argument is a system with axial symmetry in a 3d space. Introducing cylindric coordinates, we can separate the angular part of the eigenfunctions writing $\Psi=\mathrm{e}^{im\phi}\psi(\rho,z)$ where $\psi(\rho,z)$ is an eigenfunction of the Hamiltonian $H_m(\rho,z)$ in the subspace of states with the given z-component of the angular momentum $L_z=\hbar m$. As stated above, we assume that the 2d classical motion described by this Hamiltonian is chaotic. A moment's reflection will show that this example contains all the relevant features of the more general situation described at the beginning of this section.

In this case, the Hamiltonians H_m and H_{m+1} correspond to the radial Hamiltonians with magnetic quantum number m and $m+1$, respectively. The magnetic quantum number enters these Hamiltonians as a parameter, and the well-known theory of parametric correlations can be applied. Special treatment is necessary when $m=0$ or $m\sim 1$. The difference between the two Hamiltonians due to the centrifugal term $W=L_z^2/2M\rho^2$ (M is the mass of the particle) is then of the formal order $\hbar^2$. However, the energy shifts associated with the change of m by 1 are of the order of $\hbar$; this discrepancy is due to singularity of the centrifugal term at $\rho=0$. The appropriate Gutzwiller expansions in this case use the set of classical periodic orbits calculated for $L_z=0$, i.e. m-independent. The m-dependence of the spectra is taken into account via an appropriate phase added to the Maslov phase; the phase increment occurs each time the orbit hits the axis $\rho=0$; see the reviews [28, 29]. Since this additional phase is pseudo-random, the cross-correlation functions between the spectra with different $m\sim 1$ vanish due to the same mechanism as that between the spectra of different parity in the discrete case.

Let us now turn to the case of $|m|\gg 1$; more precisely consider the semiclassical limit $\hbar\to 0$, $m\to\infty$ such that the angular momentum projection $L_z=\hbar m$ remains constant. In this case the centrifugal barrier W would not let the particle approach the z-axis; the singularity at $\rho=0$ will thus be deep in the classical shadow. The change in m by 1 brings about a change in the Hamiltonian of the order of $\hbar$, namely by $\Delta H_m=\Delta W=\hbar\partial W/\partial L_z$. Before comparing the eigenvalues at different m, one must subtract the overall drift of the spectrum upwards caused by an increase in the centrifugal energy W. Note that this issue is similar to the one caused by the short periodic orbits in the discrete case: it arises from a difference in the one-particle densities of the two systems and is an issue that can also be settled by means of an appropriate unfolding procedure. This quantity can be evaluated semiclassically as the derivative of its average over the energy shell,

$$\langle W\rangle = \frac{\int d\Gamma\,\delta\,(E-H_m)\,W}{\int d\Gamma\,\delta\,(E-H_m)},$$

where $d\Gamma$ is the element of the phase space. Denoting by S_0 a typical classical action within the given (highly excited) energy range, we can introduce a dimensionless small parameter $\xi = \hbar/S_0$ and write

$$\begin{aligned} H_{m+1} &= H_m + \hbar\left(\frac{\partial W}{\partial L_z} - \left\langle\frac{\partial W}{\partial L_z}\right\rangle\right) \equiv H_m + \xi H', \\ H' &= S_0\left(\frac{\partial W}{\partial L_z} - \left\langle\frac{\partial W}{\partial L_z}\right\rangle\right). \end{aligned} \tag{22}$$

To appreciate the impact of the perturbation associated with $H_m \to H_{m+1}$, consider the general problem of parametric correlation in the family of Hamiltonians $H_\xi = H_0 + \xi H'$, where ξ is small and dimensionless; the Hamiltonians H_0, H' are assumed to have finite classical counterparts. Let us consider the cross form factor between the spectra of H_m and H_{m+1} in the semiclassical domain. We shall assume that ξ is so small that the periodic orbits of H_m, H_{m+1} practically coincide; only their action difference has to be taken into account since it is referred to as $\hbar$.

The additional classical action due to the perturbation H' over a stretch of the orbit with duration from t_1 to t_2 would be

$$\Delta S \sim \xi \int_{t_1}^{t_2} H'\,dt.$$

Now the well-established approach is to model the accumulation of ΔS on a trajectory of a chaotic system by a Gaussian stochastic process [30]; below, $\langle\langle\ldots\rangle\rangle$ denotes averaging over such a process. This can be understood if we mentally divide the periodic orbit into pieces of duration larger than the Lyapunov time $T_L = \lambda^{-1}$, where λ is the Lyapunov constant; the action differences accumulated at two such stretches would be uncorrelated. Let $t = t_2 - t_1 \gg T_L$ and assume that there is no systematic action increment, $\langle\langle\Delta S\rangle\rangle = 0$, which is guaranteed in our case by subtraction of the average in definition (22) of the perturbation H'. The averaged second moment of ΔS will grow proportional to t,

$$\langle\langle\Delta S^2\rangle\rangle = a\xi^2 t,$$

where a is a purely classical, system-specific, parameter; its dimensionality is squared action over time. Recalling that in a Gaussian process $\langle\langle\exp(i\Phi)\rangle\rangle = \exp(-\langle\langle\Phi^2\rangle\rangle/2)$, we obtain the average of the exponential of the phase difference accumulated on a periodic orbit with the period T_γ,

$$\langle\langle e^{i/\hbar\,\Delta S_\gamma}\rangle\rangle = e^{-a\frac{\xi^2}{2\hbar^2}T_\gamma}.$$

Now let us use the semiclassical expression (4) for the cross-correlation form factor, which we shall now denote by $K_\xi(\tau)$. Limiting ourselves to the diagonal approximation, we would obtain, for the time $T = \tau T_{\mathrm{H}}$,

$$K_{\xi,\mathrm{diag}}(\tau) \sim \frac{1}{\delta} \sum_{T<T_\gamma<T+\delta} A_\gamma^2 \langle\langle \mathrm{e}^{\mathrm{i}\Delta S_\xi/\hbar} \rangle\rangle \tag{23}$$

$$= \mathrm{e}^{-a\frac{\xi^2}{2\hbar^2}T} K_{\mathrm{diag}}(\tau). \tag{24}$$

The quantity $K_{\mathrm{diag}}(\tau)$ is the autocorrelation form factor in the diagonal approximation, i.e. 2τ or τ depending on the presence or absence of the time reversal. The perturbation $\xi H'$ leads thus to the decay of the form factor with the characteristic time $T_{\mathrm{decay}} = 2\hbar^2/a\xi^2$.

Applying this result to the correlation between the spectra belonging to two neighboring values of the quantum number m when $\xi = \hbar/S_0$, we obtain the decay time

$$T_{\mathrm{decay}} = \frac{2S_0^2}{a}.$$

This time is classical, i.e. finite in the semiclassical limit; however, it could be much larger than the ballistic time scale, i.e. the period of the shortest periodic orbits (time on which correlations are expected anyway).

The corresponding dimensionless $\tau_{\mathrm{decay}} = T_{\mathrm{decay}}/T_{\mathrm{H}}$ turns into zero in the limit $\hbar \to 0$. We therefore find that the cross-correlation between spectra corresponding to different symmetries vanishes in the universal regime. Considering that the periodic orbit expansions of the type (4)—and above all the averaging procedure over large numbers of periodic orbits having nearly the same action—are justified only for times exceeding the Ehrenfest time $T_{\mathrm{E}} \sim \lambda^{-1} \ln S_0/\hbar$, the vanishing of the cross-correlation only holds for times larger than T_{E}. For times shorter than T_{E}, and hence in particular for times of the order of classical times, correlations are expected to arise, based on the fact that both symmetry sectors have the same periodic orbits. Such correlations will depend on the specific features of the periodic orbits and will thus be highly system dependent. A possible connection with a recent discussion of the transition between short-time effects and the universal regime [31] is left for future investigation.

5. Conclusions

In this paper, we have given a justification for the common assumption that spectra corresponding to different IRs of a symmetry group of a given quantum chaotic problem are independent in the semiclassical limit. More precisely we have shown in the semiclassical limit that the two-point cross-correlation vanishes, a property also sometimes called weak independence. For the specific case of a mirror symmetry, we showed that the cross-correlation function is zero throughout the universal regime. The result is not limited to the diagonal approximation and short times: the non-diagonal terms of the trace formula were also evaluated and the semiclassical form of the spectral determinant was used to show that our result holds also beyond the Heisenberg time. For arbitrary discrete symmetries, and for continuous symmetries, we limited our argument to the diagonal approximation, but the arguments given for mirror symmetries carry over.

A potentially interesting line for future work appears in continuous symmetries, where the trace formulae reveal an unusual correlation on a classical time scale. The approximation used is clearly not valid on this time scale, but the result might still be a hint that such correlations exist and are different from the obvious ones resulting from short orbits.

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References

[1] Haake F 2010 *Quantum Signatures of Chaos* (Berlin: Springer)
[2] Whitney R S, Schomerus H and Kopp M 2009 Semiclassical transport in nearly symmetric quantum dots I. Symmetry breaking in the dot *Phys. Rev.* E **80** 056209
[3] Papenbrock T and Weidenmüller H A 2006 Two-body random ensemble in nuclei *Phys. Rev.* C **73** 014311
[4] Flores J, Kun Yu S and Seligman T H 2005 Slow phase relaxation as a route to quantum computing beyond the quantum chaos border *Phys. Rev.* E **72** 017201
[5] Papenbrock T and Weidenmüller H A 2008 Abundance of ground states with positive parity *Phys. Rev.* C **78** 054305
[6] Veble G, Prosen T and Robnik M 2007 Expanded boundary integral method and chaotic time-reversal doublets in quantum billiards *New J. Phys.* **9** 15
[7] Leyvraz F, Schmit C and Seligman T H 1996 Anomalous spectral statistics in a symmetrical billiard *J. Phys. A: Math. Gen.* **29** L575
[8] Gutkin B 2007 Dynamical 'breaking' of time reversal symmetry *J. Phys. A: Math. Theor.* **40** F761
[9] Bogomolny E B, Georgeot B, Giannoni M J and Schmit C 1997 Arithmetical chaos *Phys. Rep.* **291** 219–324
[10] Bogomolny E, Leyvraz F and Schmit C 1996 Distribution of eigenvalues for the modular group *Commun. Math. Phys.* **176** 577–617
[11] Seligman T H and Weidenmuller H A 1994 Semi-classical periodic-orbit theory for chaotic hamiltonians with discrete symmetries *J. Phys. A: Math. Gen.* **27** 7915
[12] Faas M, Simons B D, Zotos X and Altshuler B L 1993 Universality in the spectra of strongly correlated systems *Phys. Rev.* B **48** 5439–43
[13] Robnik M 1983 Classical dynamics of a family of billiards with analytic boundaries *J. Phys. A: Math. Gen.* **16** 3971
[14] Bäcker A, Steiner F and Stifter P 1995 Spectral statistics in the quantized cardioid billiard *Phys. Rev.* E **52** 2463–72
[15] Bäcker A and Dullin H R 1997 Symbolic dynamics and periodic orbits for the cardioid billiard *J. Phys. A: Math. Gen.* **30** 1991
[16] Pavloff N 1994 Discrete symmetries in the Weyl expansion for quantum billiards *J. Phys. A: Math. Gen.* **27** 4317
[17] Braun P and Haake F 2010 Level statistics in arithmetical and pseudo-arithmetical chaos *J. Phys. A: Math. Theor.* **43** 262001
[18] Berry M V 1985 Semiclassical theory of spectral rigidity *Proc. R. Soc. A: Math. Phys. Sci.* **400** 229–51
[19] Sieber M and Richter K 2001 Correlations between periodic orbits and their rôle in spectral statistics *Phys. Scr.* **90** 128
[20] Müller S, Heusler S, Braun P, Haake F and Altland A 2005 Periodic-orbit theory of universality in quantum chaos *Phys. Rev.* E **72** 046207
[21] Bogomolny E B and Keating J P 1996 Gutzwiller's trace formula and spectral statistics: beyond the diagonal approximation *Phys. Rev. Lett.* **77** 1472–5
[22] Heusler S, Müller S, Altland A, Braun P and Haake F 2007 Periodic-orbit theory of level correlations *Phys. Rev. Lett.* **98** 044103

[23] Müller S, Heusler S, Altland A, Braun P and Haake F 2009 Periodic-orbit theory of universal level correlations in quantum chaos *New J. Phys.* **11** 103025

[24] Berry M V and Keating J P 1990 A rule for quantizing chaos? *J. Phys. A: Math. Gen.* **23** 4839

[25] Keating J P 1992 Periodic orbit resummation and the quantization of chaos *Proc. R. Soc.* A*: Math. Phys. Sci.* **436** 99–108

[26] Berry M V and Keating J P 1992 A new asymptotic representation for $\zeta(\frac{1}{2}+\mathrm{i}t)$ and quantum spectral determinants *Proc. R. Soc.* A*: Math. Phys. Sci.* **437** 151–73

[27] Hannay J H and Ozorio de Almeida A M 1984 Periodic orbits and a correlation function for the semiclassical density of states *J. Phys. A: Math. Gen.* **17** 3429

[28] Friedrich H and Wintgen D 1989 The hydrogen atom in a uniform magnetic field—an example of chaos *Phys. Rep.* **183** 37–79

[29] Friedrich H and Wintgen D 1989 Classical and quantal chaos in the diamagnetic Kepler problem *Prog. Theor. Phys. Suppl.* **98** 198–286

[30] Richter K 2000 *Semiclassical Theory of Mesoscopic Quantum Systems (Springer Tracts in Modern Physics* vol 161*)* (Berlin: Springer) (War: Habilitationsschrift, 1997)

[31] Smith A M and Kaplan L 2010 Integrating random matrix theory predictions with short-time dynamical effects in chaotic systems *Phys. Rev.* E **82** 016214