Scale-free network topology and multifractality in a weighted planar stochastic lattice

To cite this article: M K Hassan et al 2010 New J. Phys. 12 093045

View the article online for updates and enhancements.

Related content
- Scale-free coordination number disorder and multifractal size disorder in weighted planar stochastic lattice
  M K Hassan, M Z Hassan and N I Pavel
- Weighted tunable clustering in local-world networks with increment behavior
  Ying-Hong Ma, Huijia Li and Xiao-Dong Zhang
- The topology of an accelerated growth network
  Xiaoling Yu, Zhihao Li, Duanming Zhang et al.

Recent citations
- Dynamic scaling, data-collapse and self-similarity in Barabási–Albert networks
  M Kamrul Hassan et al
- Scale-free coordination number disorder and multifractal size disorder in weighted planar stochastic lattice
  M K Hassan et al
Scale-free network topology and multifractality in a weighted planar stochastic lattice

M K Hassan\textsuperscript{1,3}, M Z Hassan\textsuperscript{2} and N I Pavel\textsuperscript{1}

\textsuperscript{1} Theoretical Physics Group, Department of Physics, University of Dhaka, Dhaka 1000, Bangladesh
\textsuperscript{2} Institute of Computer Science, Bangladesh Atomic Energy Commission, Dhaka 1000, Bangladesh
E-mail: khassan@univdhaka.edu

Received 2 June 2010
Published 27 September 2010
Online at \url{http://www.njp.org/}
doi:10.1088/1367-2630/12/9/093045

Abstract. We propose a weighted planar stochastic lattice (WPSL) formed by the random sequential partition of a plane into contiguous and non-overlapping blocks and we find that it evolves following several non-trivial conservation laws, namely $\sum_{i}^{N} x_i^{n-1} y_i^{4/n-1}$ is independent of time $\forall n$, where $x_i$ and $y_i$ are the length and width of the $i$th block. Its dual on the other hand, obtained by replacing each block with a node at its centre and the common border between blocks with an edge joining the two vertices, emerges as a network with a power-law degree distribution $P(k) \sim k^{-\gamma}$, where $\gamma = 5.66$ reveals scale-free coordination number disorder since $P(k)$ also describes the fraction of blocks having $k$ neighbours. To quantify the size disorder, we show that if the $i$th block is populated with $p_i \sim x_i^3$, then its distribution in the WPSL exhibits multifractality.

Contents

1. Introduction 2
2. Algorithm of the model 4
3. Topological properties of a weighted planar stochastic lattice (WPSL) 5
4. Geometric properties of WPSL 8
5. Summary 12
Acknowledgments 12
References 12

\textsuperscript{3} Author to whom any correspondence should be addressed.
1. Introduction

Planar cellular structures formed by tessellation, tiling or subdivision of a plane into contiguous and non-overlapping cells have always generated interest among scientists in general and physicists in particular because cellular structures are ubiquitous in nature. Examples include acicular texture in martensite growth, tessellated pavement on ocean shores, agricultural land division according to ownership, grain texture in polycrystals, cell texture in biology, soap froths and so on [1]–[4]. For instance, the Voronoi lattice formed by partitioning a plane into convex polygons and Apollonian packing generated by tiling a plane into contiguous and non-overlapping disks have found widespread applications [5, 6]. There are some theoretical models, both random and deterministic, developed either to directly mimic natural cellular structures or to serve as a tool on which one can study various physical problems [1]. However, cells in most of the existing structures do not have sides that share with more than one side of other cells. In reality, the sides of a cell can share a part or a whole side of another cell. As a result, the number of neighbours of a cell can be higher than the number of sides it has. Moreover, a cellular structure may emerge through evolution where cells can be of different sizes and have different numbers of neighbours since nature favours these properties as a matter of rule rather than exception. A lattice with such properties can be of great interest because it can mimic a disordered medium on which one can study percolation or random walk problems.

In this article, we propose a weighted planar stochastic lattice (WPSL) as a space-filling cellular structure where annealed coordination number disorder and size disorder are introduced in a natural way. The definition of the model is trivially simple. It starts with an initiator, say a square of unit area, and a generator that divides it randomly into four blocks. The generator thereafter is sequentially applied over and over again to only one of the available blocks picked preferentially with respect to their areas. It results in the partitioning of the square into ever smaller mutually exclusive rectangular blocks. A snapshot of the WPSL at the late stage (figure 1) provides an awe-inspiring perspective on the emergence of an intriguing and rich pattern of blocks. We intend to investigate its topological and geometrical properties in an attempt to find some order in this seemingly disordered lattice. If the blocks of the WPSL are regarded as isolated fragments, then the model can also describe the fragmentation of a two-dimensional (2D) object by the random sequential nucleation of seeds from which two orthogonal cracks parallel to the sides of the parent object are grown until intercepted by existing cracks [7, 8]. In reality, fragments produced in the fracture of solids by the propagation of interacting cracks is a formidable mathematical problem. The model in question can, however, be considered as the minimum model that should be capable of capturing the essential features of the underlying mechanism. The WPSL can also describe martensite formation because we find its definition to be remarkably similar to the model proposed by Rao et al, which is also reflected in the similarity between figures 1 and 2 of [1, 9]. Yet another application, perhaps a little exotic, is the random search tree problem in computer science [10].

Searching for an order in the disorder is always an attractive proposition in physics. To this end, we invoke the concept of complex network topology to quantify the coordination number disorder and the idea of multifractality to quantify the size disorder of the blocks in WPSL. It is interesting to note that the dual of the WPSL (DWPSS), obtained by replacing each block with a node at its centre and the common border between blocks with an edge joining the two corresponding vertices, emerges as a network. The area of the respective blocks is assigned to the corresponding nodes to characterize them as their fitness parameter. Nodes in the
Figure 1. A snapshot of the weighted planar stochastic lattice.

DWPSL are therefore characterized by their respective fitness parameter and the corresponding degree $k$ is defined as the number of links a node has. For a decade, there has been a surge of interest in finding the degree distribution $P(k)$ triggered by the work of A.-L. Barabasi and his co-workers. They revolutionized the notion of network theory by recognizing that real networks are not static but rather grow by the addition of new nodes establishing links preferentially, known as the preferential attachment (PA) rule, to the nodes that are already well connected [11]. Incorporating both the ingredients, growth and the PA rule, Barabasi and Albert (BA) presented a simple theoretical model and showed that such a network self-organizes into a power-law degree distribution $P(k) \sim k^{-\gamma}$ with $\gamma = 3$ [12]. The phenomenal success of the BA model lies in the fact that it can capture, at least qualitatively, the key features of many real-life networks. Interestingly, we find that the DWPSL has all the ingredients of the BA model and its degree distribution $P(k)$ follows a heavy-tailed power law, but with exponent $\gamma = 5.66$ showing that the coordination number of the WPSL is scale free in character.

In addition to characterizing the blocks of the WPSL by the coordination number $k$, the blocks can also be characterized by their respective length $x$ and width $y$. We then find that the dynamics of the WPSL are governed by infinitely many conservation laws, namely the quantity $\sum_i x_i^{n-1} y_i^{A/n-1}$ remains independent of time $\forall n$ where blocks are labelled by index $i = 1, 2, \ldots, N$. For instance, total area is one obvious conserved quantity obtained by setting $n = 2$, sum of the cubic power of the length (or width) of all the existing blocks $\sum_i x_i^3$ is a non-trivial conserved quantity obtained by setting $n = 1$ (or $n = 4$). Interestingly, we find that when the $i$th block is populated with the fraction of the measure $p_i \sim x_i^3$, then the distribution of the population in the WPSL emerges as multifractal, indicating further development towards gaining deeper insight into the complex nature of the WPSL that we proposed. Multifractal analysis was initially proposed to treat turbulence, but was later successfully applied in a...
Figure 2. Schematic illustration of the first few steps of the algorithm.

This paper is organized as follows. In section 2, we give the exact algorithm of the model. In section 3, various structural topological properties of the WPSL and its dual are discussed in order to quantify the annealed coordination number disorder. In section 4, we discuss the geometric properties of the WPSL in an attempt to quantify annealed size disorder. Finally, section 5 gives a short summary of our results.

2. Algorithm of the model

Perhaps an exact algorithm can provide a better description of the model than the mere definition. In step one, the generator divides the initiator, say a square of unit area, randomly into four smaller blocks. We then label the four newly created blocks by their respective areas $a_1$, $a_2$, $a_3$ and $a_4$ in a clockwise fashion, starting from the upper left block (see figure 2). In each step thereafter, only one block is picked preferentially with respect to the respective area (which we also refer to as the fitness parameter) and it is then divided randomly into four blocks. In general, the $j$th step of the algorithm can be described as follows. (i) Subdivide the interval $[0, 1]$ into $(3^j - 2)$ subintervals of size $[0, a_1]$, $[a_1, a_1 + a_2]$, $\ldots$, $[\sum_{i=1}^{3^j-3} a_i, 1]$, each of which represents blocks labelled by their areas $a_1, a_2, \ldots, a_{(3^j-2)}$, respectively. (ii) Generate a random number $R$ from the interval $[0, 1]$ and find which of the $(3^j - 2)$ subinterval contains this $R$. The corresponding block it represents, say the $p$th block of area $a_p$, is picked. (iii) Calculate the length $x_p$ and the width $y_p$ of this block and keep note of the coordinate of the lower left corner of the $p$th block, say $(x_{\text{low}}, y_{\text{low}})$. (iv) Generate two random numbers $x_R$ and $y_R$ from $[0, x_p]$ and $[0, y_p]$, respectively, and hence the point $(x_R + x_{\text{low}}, y_R + y_{\text{low}})$ mimics a random point chosen in the block $p$. (v) Draw two perpendicular lines through the point $(x_R + x_{\text{low}}, y_R + y_{\text{low}})$ parallel to the sides of the $p$th block in order to divide it into four smaller blocks. The label $a_p$ is now redundant and hence it can be reused. (vi) Label the four newly created blocks according to their areas $a_p$, $a_{(3^j-1)}$, $a_{3j}$ and $a_{(3^j+1)}$, respectively, in a clockwise fashion starting from the upper left corner. (vii) Increase the time by one unit and repeat steps (i)–(vi) ad infinitum.
3. Topological properties of a weighted planar stochastic lattice (WPSL)

We first focus our analysis on the blocks of the WPSL and their coordination numbers. Note that for a square lattice, the deterministic counterpart of the WPSL, the coordination number is a constant equal to 4. However, the coordination number in the WPSL neither is a constant nor has a typical mean value; rather the coordination number that each block assumes in the WPSL is random. Moreover, it is allowed to evolve with time, and hence the coordination number disorder in the WPSL can be regarded as of the annealed type. Defining each step of the algorithm as one time unit and imposing periodic boundary conditions in the simulation, we find that the number of blocks $N_k(t)$ that have coordination number $k$ (or $k$ neighbours) continues to grow linearly with time $N_k(t) = m_k t$ (see figure 3). On the other hand, the number of total blocks $N(t) = \sum_k N_k(t)$ at time $t$ in the lattice also grows linearly with time $N(t) \sim 3t$, which we can write as $N(t) \sim 3t$ in the asymptotic regime. The ratio of the two quantities $\rho_k(t) = N_k(t)/N(t)$ that describes the fraction of the total blocks that have coordination number $k$ is $\rho_k(t) = m_k/3$. This implies that $\rho_k(t)$ becomes a global property because it is independent of time and the size of the lattice. We now take the WPSL of fixed size or time and look into its structural properties. For instance, we want to find out what fraction of the total blocks of the WPSL of a given size has coordination number $k$. For this, we collect data for $\rho_t$ as a function of $k$ and obtain the coordination number distribution function $\rho_t(k)$, where subscript $t$ indicates fixed time. Interestingly, the same WPSL can be interpreted as a network if the blocks of the
Figure 4. Degree distribution $P(k)$ for the DWPSL network where data points represent the average of 50 independent realizations. The line has a slope exactly equal to 5.66, revealing a power-law degree distribution with exponent $\gamma = 5.66$.

lattice are regarded as nodes and the common borders between blocks are regarded as their links, which is topologically identical to the DWPSL.

In fact, the data for the degree distribution $P(k)$ of the DWPSL network is exactly the same as the coordination number distribution function $\rho_t(k)$ of the WPSL, i.e. $\rho_t(k) \equiv P(k)$. The plot of $\ln(P(k))$ versus $\ln(k)$ in figure 4, using data obtained after an ensemble average over 50 independent realizations, clearly suggests that fitting a straight line to the data is possible. This implies that the degree distribution decays obeying power law

$$P(k) \sim k^{-\gamma}.$$  (1)

However, note that figure 4 has a heavy or fat tail, a benchmark of the scale-free network, which represents highly connected hub nodes. The presence of a messy tail end complicates the process of fitting the data into power-law forms, estimating its exponent $\gamma$ and identifying the range over which the power law holds [15]. One way of reducing the noise at the tail end of the degree distribution is to plot cumulative distribution $P(k' > k)$. We therefore plot $\ln(P(k' > k))$ versus $\ln(k)$ in figure 5 using the same data as figure 4, and find that the heavy tail smooths out naturally where no data are obscured. The straight line fit of figure 5 has a slope $\gamma - 1 = 4.66$, which indicates that the degree distribution (figure 4) decays following the power law with exponent $\gamma = 5.66$. It is worthwhile mentioning that the mean coordination number $\sum_k k P(k)$ asymptotically reaches a constant value equal to 5.333.

We thus find that in the large-size limit the WPSL develops some order in the sense that its annealed coordination number disorder or the degree distribution of its dual is scale free in character. This is in sharp contrast to the quenched coordination number disorder found in

Figure 5. Cumulative degree distribution $P(k' > k)$ using the same data as for figure 4. The dotted line with slope equal to $\gamma - 1 = 4.66$ is drawn to guide our eyes.

The Voronoi lattice, where it is almost impossible to find cells that have significantly more or fewer neighbours than the mean value $k = 6$ [16]. In fact, it has been shown that the degree distribution of the dual of the Voronoi lattice is Gaussian. The square lattice, on the other hand, is self-dual and its degree distribution is $P(k) = \delta(k - 4)$. Further, it is interesting to point out that the exponent $\gamma = 5.66$ is significantly bigger than that usually found in most real-life networks, which is typically $2 < \gamma \leq 3$. This suggests that in addition to the PA rule, the network in question has to obey some constraints. For instance, nodes in the WPSL are spatially embedded in Euclidean space, and links gained by the incoming nodes are constrained by the spatial location and the fitting parameter of the nodes. Owing to its unique dynamics, this was not unexpected. Perhaps it is worth mentioning that the degree distribution of the electric power grid, whose nodes such as WPSL are also embedded in the spatial position, is shown to exhibit a power law but with exponent $\gamma_{\text{power}} = 4$ [11].

The power-law degree distribution $P(k)$ has been found in many seemingly unrelated real-life networks. This implies that there must exist some common underlying mechanisms for which disparate systems behave in such a remarkably similar fashion [12]. Barabasi and Albert argued that growth and the PA rule represent the main essence behind the emergence of such a power law. Indeed, the DWPSL network too grows with time, but in sharp contrast to the BA model. The BA network grows by the addition of one single node with $m$ edges per unit time, whereas the DWPSL network grows by the addition of a group of three nodes that are already linked by two edges. It also differs in the way in which incoming nodes establish links with existing nodes.
To understand the growth mechanism of the DWPSL network, let us look into the $j$th step of the algorithm. First, a node (e.g. labelled $a_p$) is picked from the $(3j - 2)$ nodes preferentially with respect to the fitness parameter of a node (i.e. according to their respective areas). Next, the node $a_p$ is connected with two new nodes $(3j - 1)$ and $(3j + 1)$ in order to establish their links with the existing network. At the same time, at least two or more links of $a_p$ with other nodes are removed (although the exact number depends on the number of neighbours that $a_p$ already has) in favour of linking them among the three incoming nodes in a self-organized fashion. In the process, the degree $k_p$ of the node $a_p$ will either decrease (may turn into a node with marginally low degree in case it is a highly connected node) or at best remain the same, but will never increase. It may therefore appear that the PA rule is not followed here. A closer look into the dynamics, however, reveals otherwise. It is interesting to note that an existing node during the process gains links only if one of its neighbours is picked and not itself. This implies that the more links (or degree) a node has, the greater its chance of gaining more links because they can be reached in a larger number of ways. It essentially embodies the intuitive idea of the PA rule. Therefore, the DWPSL network can be seen to follow the PA rule but in disguise.

4. Geometric properties of WPSL

We again focus on the blocks of the WPSL, but this time we characterize them by their length $x$ and width $y$ instead of the number of neighbours they have. Then the evolution of their distribution function $C(x, y; t)$ can be described by the following kinetic equation [7, 8],

$$\frac{\partial C(x, y; t)}{\partial t} = -xyC(x, y; t) + 4 \int_x^\infty \int_y^\infty C(x_1, y_1; t) \, dx_1 \, dy_1.$$  

(2)

Incorporating the two-tuple Millen transform

$$M(m, n; t) = \int_0^\infty \int_0^\infty x^{m-1} y^{n-1} C(x, y; t) \, dx \, dy$$  

(3)

in the kinetic equation yields

$$\frac{dM(m, n; t)}{dt} = \left(\frac{4}{mn} - 1\right) M(m + 1, n + 1; t).$$  

(4)

Iterating it to get all the derivatives of $M(m, n; t)$, and then substituting them into the Taylor series expansion of $M(m, n; t)$ about $t = 0$, one can write its solution in terms of the generalized hypergeometric function [17],

$$M(m, n; t) = _2F_2(a_+, a_-; m, n; -t),$$  

(5)

where $M(m, n; t) = M(n, m; t)$ for symmetry reasons and

$$a_{\pm} = \frac{m + n}{2} \pm \left[\left(\frac{m - n}{2}\right)^2 + 4\right]^{1/2}.$$  

(6)

One can immediately see that (i) $M(1, 1; t) = 1 + 3t$ is the total number of blocks $N(t)$ and (ii) $M(2, 2; t) = 1$ is the total area of all the blocks, which is obviously a conserved quantity. The behaviour of $M(m, n; t)$ in the long time limit is

$$M(m, n; t) \sim t^{-a_-}.$$  

(7)
This implies that the system is in fact governed by several conservation laws, namely

\[ M(n, 4/n; t) = \sum_{i}^{N} x_{i}^{n-1} y_{i}^{4/(n)-1} \]

are independent of time for all \( n \), which has been confirmed by numerical simulation (see figure 6).

We now focus on the distribution function \( n(x, t) = \int_{0}^{\infty} C(x, y, t) \, dy \) that describes the concentration of blocks of length \( x \) at time \( t \) regardless of the size of their widths \( y \). Then the \( q \)th moment of \( n(x, t) \) is defined as

\[ M_{q}(t) = \int_{0}^{\infty} x^{q} n(x, t) \, dx. \]  

(8)

Appreciating the fact that \( M_{q}(t) = M(q + 1, 1; t) \) and using equation (7), one can immediately write that

\[ M_{q}(t) \sim t^{\left\{ \sqrt{q^{2}+16}-(q+2) \right\}/2}. \]  

(9)

Note that had we focused on \( n(y, t) \) instead of \( n(x, t) \), we would have exactly the same result because their \( q \)th moments are identical, \( M(q + 1, 1; t) = M(1, q + 1; t) \), due to symmetry reasons. We find that the quantity \( M_{3}(t) \) and hence \( \sum_{i}^{N} x_{i}^{3} \) or \( \sum_{i}^{N} x_{i}^{3} \) is a conserved quantity. However, yet another interesting fact is that although \( M_{3}(t) \) remains constant against time in every independent realization, the exact numerical value fluctuates from sample to sample (see figure 7). This clearly indicates the lack of self-averaging or wild fluctuation. Nonetheless, during each realization we can use \( M_{3}(t) \) as a measure to populate the \( i \)th block with the fraction of the total population \( p_{i} = x_{i}^{3} / \sum_{j} x_{j}^{3} \). The corresponding ‘partition function’ is then

\[ Z_{q}(t) = \sum_{i} p_{i}^{q} \sim M_{3q}(t), \]  

(10)

Figure 6. Plots of \( \sum_{i}^{N} x_{i}^{n-1} y_{i}^{4/(n)-1} \) versus \( N \) for \( n = 3, 4, 5 \) drawn using data collected from one realization.
whose solution can be written immediately from equation (9) to give
\[ Z_q(t) \sim t \left\{ \sqrt{9q^2 + 16 - (3q + 2)^2} \right\} / 2. \] (11)

We find it instructive to express \( Z_q \) in terms of the square root of the mean area
\( \delta(t) = \sqrt{M(2, 2; t)} / M(1, 1; t) \sim t^{-1/2} \) because it gives the weighted number of squares \( N(q, \delta) \) needed to cover the measure that scales as
\[ N(q, \delta) \sim \delta^{-\tau(q)}, \] (12)
where the mass exponent
\[ \tau(q) = \sqrt{9q^2 + 16 - (3q + 2)}. \] (13)

The nonlinear nature of \( \tau(q) \) suggests that an infinite hierarchy of exponents is required to specify how the moments of the probabilities \( \{p\} \) scale with \( \delta \). Note that \( \tau(0) = 2 \) is the Hausdorff–Besicovitch (H-D) dimension of the WPSL since the bare number \( N(q = 0, \delta) \sim \delta^{-\tau(0)} \) and \( \tau(1) = 0 \) follows from the conservation laws (or normalization of the probabilities \( \sum p_i = 1 \)).

We now perform the Legendre transformation of \( \tau(q) \) by using the Lipschitz–Hölder exponent
\[ \alpha = -\frac{d\tau(q)}{dq} \] (14)
as an independent variable to obtain the new function
\[ f(\alpha) = q\alpha + \tau(q). \] (15)
Substituting this in equation (12), we find that
\[ N(q, \delta) \sim \delta^{q \alpha - f(\alpha)}, \] (16)
which is the dominant value of the integral
\[ N(\alpha, \delta) \sim \int \rho(\alpha) \, d\alpha \delta^{-f(\alpha)} \delta^{q \alpha} \] (17)
obtained by extremal conditions. We can hence infer that the number of squares \( dN(\alpha, \delta) \) needed to cover the WPSL with \( \alpha \) in the range \( \alpha \) to \( \alpha + d\alpha \) scales as
\[ \rho(\alpha) \, d\alpha \, \delta^{-f(\alpha)}, \] (18)
where \( \rho(\alpha) \, d\alpha \) is the number of times the subdivided regions are indexed by \( \alpha \) [18]. This implies that a spectrum of spatially intertwined fractal dimensions,
\[ f(\alpha(q)) = \frac{16}{\sqrt{9q^2 + 16}} - 2, \] (19)
is needed to characterize the measure, which is always concave in character (see figure 8). This implies that the size disorder of the blocks is multifractal in character because the measure \( \{p_\alpha\} \) is related to the size of the blocks. That is, the distribution of \( \{p_\alpha\} \) in the WPSL can be subdivided into a union of fractal subsets, each with fractal dimension \( f(\alpha) \leq 2 \) in which the measure \( p_\alpha \) scales as \( \delta^\alpha \). Note that \( f(\alpha) \) is always concave in character (see figure 8), with a single maximum at \( q = 0 \) corresponding to the dimension of the WPSL with empty blocks.

On the other hand, we find that the entropy \( S(\delta) = -\sum_i p_i \ln p_i \) associated with the partition of the measure on the support (WPSL) by using the relation \( \sum_i p_i^q \sim \delta^{-\tau(q)} \) in the definition of \( S(\delta) \). Then a few steps of algebraic manipulation show that \( S(\delta) \) exhibits scaling
\[ S(\delta) = \ln \delta^{-\alpha_1}, \] (20)
where the exponent \( \alpha_1 = \frac{6}{5} \) is obtained from
\[ \alpha_q = -\frac{d\tau(q)}{dq} \big|_{q}. \] (21)
It is interesting to note that $\alpha_1$ is related to the generalized dimension $D_q$, also related to the Rényi entropy $H_q(p) = \frac{1}{q-1} \ln \sum_i p_i^q$ in information theory, given by

$$D_q = \lim_{\delta \to 0} \frac{H_q(p)}{\ln \delta} = \frac{\tau(q)}{1-q},$$

which is often used in the multifractal formalism because it can also provide insightful interpretation. For instance, $D_0 = \tau(0)$ is the dimension of the support, $D_1 = \alpha_1$ is the Renyi information dimension and $D_2$ is the correlation dimension [19, 20].

5. Summary

We have proposed and studied a WPSL that has annealed coordination number and block size disorder. We have shown that coordination number disorder is scale free in character because the degree distribution of its DWPSL, which is topologically identical to the network obtained by considering blocks of the WPSL as nodes and the common border between blocks as links, exhibits a power law. However, the novelty of this network is that it grows by the addition of a group of already linked nodes that then establish links with the existing nodes following the PA rule, although in disguise in the sense that an existing node gains links only if one of its neighbours is picked and not itself. In addition, we have shown that if the blocks of the WPSL are characterized by their respective length $x$ and width $y$, we find that $\sum_i x_i^n y_i^{-1} y_i^A/n^{-1}$ remains constant regardless of the size of the lattice. However, the numerical values of the conserved quantities except for $n = 2$ vary from sample to sample, revealing the absence of self-averaging—an indication of wild fluctuation. We have shown that if the blocks are occupied with a fraction of the measure equal to the cubic power of their respective length or width, then distribution on the WPSL is multifractal in nature. Such a multifractal lattice with scale-free coordination disorder can be of great interest because it has the potential to mimic a disordered medium on which one can study various physical phenomena, such as percolation and random walk problems.

Acknowledgments

MKH and NIP acknowledge financial support from the Bose Centre for Advanced Study and Research in Natural Sciences of Dhaka University, Bangladesh.

References
