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Abstract. We predict and describe a new collective mode in rotating Bose–Einstein condensates, which is very similar to Rossby waves in geophysics. In the regime of fast rotation, the Coriolis force dominates the dynamics and acts as a restoring force for acoustic-drift waves along the condensate. We derive a nonlinear equation that includes the effects of both zero-point pressure and inhomogeneity of the gas. It is shown that such waves have negative phase speed, propagating in the opposite sense of the rotation. We discuss different equilibrium configurations and compare them to those resulting from the Thomas–Fermi approximation.

The rotation of Bose–Einstein condensates (BECs) has attracted much attention recently, both theoretically and experimentally [1, 2]. Due to the superfluid character of the BEC, the effects of rotation are quite different from those observed in a normal fluid and its properties strongly depend on the effects of confinement. The pioneering experiments based on both the phase imprinting [3] and the rotating laser beam [4, 5] techniques independently confirmed the nucleation of quantized vortices, which is a clear manifestation of the superfluid properties of the condensate. Since then, much effort has been made to understand the dynamics of the rotating BEC [6] and, in particular, the mechanisms of vortex nucleation [7, 8]. In particular, interesting features of quantized vortices in BECs of alkali atoms are related to the formation of vortex arrays, where singly quantized vortices are typically arranged in highly regular triangular lattices, similar to the Abrikosov lattice for superconductors. Such a configuration is possible only when a sufficient amount of angular momentum is effectively transferred to the system,

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corresponding to the situation of a rapid rotation. The acquired angular velocity then tends to enlarge the rotating cloud, and the centrifugal force is responsible for the flattening of the density profile towards a two-dimensional configuration. In the limit where the rotation frequency Ω approaches the transverse trapping frequency $\omega_\perp$, the quadratic centrifugal and the harmonic trapping potentials cancel out and the system is no longer bounded. The possibility of reaching high angular velocities is therefore provided by the addition of anharmonic terms to the trapping potential, making the investigation of new equilibrium configurations with different vortex states and new collective modes worthwhile [9]. In the present work, we take advantage of such an interesting medium to predict a different hydrodynamical mode in rotating BECs, in complete analogy to the Rossby waves observed in geophysics. Rossby waves, also known as planetary waves, have been recognized for a long time as the main pattern of long period variability in the upper tropospheric winds [10]. These are responsible for the well-known cyclonic and anticyclonic systems that characterize the day-to-day weather systems in mid-latitudes, and can be observed both in the upper troposphere and in the oceans. The waves exist because of the variation of the Coriolis parameter f with latitude, which acts as a restoring force for an air particle that is disturbed from its equilibrium latitude. In a rotating BEC, the Coriolis parameter is replaced by twice the angular rotation frequency, 2Ω. We show that such waves are dispersive, their phase speed being always negative, which means that these oscillations always propagate westward with respect to the BEC rotation. We derive here a new equation for the Rossby waves in a rotating condensate, which accounts for the vortex lattice and the anharmonicity of the trap.

Cozzini and Stringari [11] showed, in the presence of a large number of vortices, that it is possible to average the velocity field over regions containing many vortex lines and assume that vorticity is spread continuously in the superfluid. This approximation is known as the diffused vorticity approach [11] and simply corresponds to assuming a rigid-body rotation $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$, where the angular velocity is $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ where $\Omega = \pi\hbar n_{\mathrm{v}}/m$, $n_{\mathrm{v}}(\mathbf{r})$ is the average vortex density in the vicinity of $\mathbf{r}$ and m stands for the atomic mass. Therefore, the usual irrotationality condition $\nabla \times \mathbf{v} = 0$ is no longer valid and should be replaced by $\nabla \times \mathbf{v}(\mathbf{r}) = 2\boldsymbol{\Omega}$. In that case, the macroscopic dynamics of the rotating BEC is provided by the rotational hydrodynamical equations in the rotating frame

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\frac{g\nabla n}{m} - \frac{\nabla V}{m} + \frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right) - 2\boldsymbol{\Omega}\times\mathbf{v}, \tag{2}$$

where $\mathbf{v}\cdot\nabla\mathbf{v} = \nabla(v^2)/2 - \mathbf{v}\times(\boldsymbol{\Omega}\times\mathbf{v})$. The usual hydrodynamical calculations are based on the Thomas–Fermi approximation, which neglects the quantum pressure proportional to $\hbar^2$. In this work, however, we include this quantum term, since we may be interested in Bogoliubov-like waves. Note that this procedure does not contradict the diffused vorticity approximation, since the quantum features will be included only in the dynamics of the fluctuations and do not affect the equilibrium configuration of the system. This allows one to cast quantum features that may be relevant for describing the so-called quantum turbulence [12], where the healing length sets a lower scale for the Kolmogorov cascade. Here, $V(\mathbf{r}, \Omega) = V_{\mathrm{trap}}(\mathbf{r}) - m\Omega^2 r^2/2$, where $r = (x^2 + y^2)^{1/2}$ stands for the effective trapping potential, which reads

$$V(\mathbf{r}, \Omega) = \frac{\hbar\omega_\perp}{2}\left[\left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)\frac{r^2}{a_{\mathrm{ho}}^2} + \beta\frac{r^4}{a_{\mathrm{ho}}^4}\right], \tag{3}$$

with $a_{\rm ho} = \sqrt{\hbar/m\omega_\perp}$ being the characteristic harmonic oscillator length and β the dimensionless anharmonicity parameter. The term $2\mathbf{\Omega}\times\mathbf{v}$ in equation (2) represents the Coriolis force, which will act as the restoring force for the oscillations considered here. We consider perturbations around the equilibrium configuration, making $n = n_0 + \delta n$ and $\mathbf{v} = \delta\mathbf{v}$. In that case, the system can be described by the following set of perturbed equations:

$$\frac{\partial}{\partial t}\delta n + \nabla\cdot(n_0\delta\mathbf{v}) = 0, \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \delta\mathbf{v}\cdot\nabla\right)\delta\mathbf{v} = -g\nabla\delta n - 2\mathbf{\Omega}\times\delta\mathbf{v} + \frac{\hbar^2}{4m^2}\nabla\left(\frac{\nabla^2\delta n}{n_\infty}\right), \tag{5}$$

where n_∞ is the peak density. Here, we have considered the quantum pressure to be important only for the fluctuations,

$$\frac{\hbar^2}{4m^2}\nabla\left(\frac{\nabla^2(n_0+\delta n)}{n_0}\right) \approx \frac{\hbar^2}{4m^2}\nabla\left(\frac{\nabla^2\delta n}{n_\infty}\right), \tag{6}$$

where the local density n_0 is assumed to be slowly varying with respect to δn. The rotational velocity field can be split into two parts, $\boldsymbol{\delta}\mathbf{v} \approx \delta\mathbf{v}_0 + \delta\mathbf{v}_{\rm p}$, where

$$\delta\mathbf{v}_0 = \frac{1}{2\Omega}\hat{\mathbf{z}}\times\mathbf{S} \tag{7}$$

is the zeroth-order drift velocity, resulting from taking $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \delta\mathbf{v}\cdot\nabla = 0$ in equation (5) and

$$\mathbf{S} = -g\nabla\delta n + \frac{\hbar^2}{4m^2}\nabla\left(\frac{\nabla^2\delta n}{n_\infty}\right). \tag{8}$$

The polarization velocity $\delta\mathbf{v}_{\rm p}$ is the first-order correction to the drift velocity (8) and satisfies the following equation:

$$\left(\frac{\partial}{\partial t} + \delta\mathbf{v}_0\cdot\nabla\right)\delta\mathbf{v}_0 = -2\mathbf{\Omega}\times\delta\mathbf{v}_{\rm p}, \tag{9}$$

which yields

$$\delta\mathbf{v}_{\rm p} = -\frac{1}{4\Omega^2}\frac{\partial\mathbf{S}_\perp}{\partial t} - \frac{1}{8\Omega^3}\left(\hat{\mathbf{z}}\times\mathbf{S}\right)\cdot\nabla_\perp\mathbf{S}, \tag{10}$$

where $\mathbf{S}_\perp = (S_x, S_y)$ is the transverse component of $\mathbf{S}$. The continuity equation (4) can be written in the following fashion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln n + \nabla\cdot\delta\mathbf{v}_{\rm p} = 0, \tag{11}$$

where the material derivative can be approximated as $\mathrm{d}/\mathrm{d}t \approx \partial/\partial t + \delta\mathbf{v}_0\cdot\nabla$. Using the fact that $\ln n \approx \ln n_0 + \phi$, where $\phi = \delta n/n_\infty$, and putting equations (7), (10) and (11) together, one should obtain

$$\left(1 - r_0^2\nabla_\perp^2 + \frac{1}{2}r_0^2\xi^2\nabla_\perp^4\right)\frac{\partial\phi}{\partial t} + 2\Omega\left\{\psi, \phi - \nabla^2\psi + \ln n_0\right\} = 0. \tag{12}$$

This equation is formally similar and generalizes Charney's equation [13], also referred to in the literature as the Charney–Hasegawa–Mima (CHM) equation. Here, $r_0 = c_{\rm s}/2\Omega$ represents the generalized Rossby radius, $c_{\rm s} = \sqrt{gn_\infty/m}$ is the sound speed, $\xi = \hbar/\sqrt{2mgn_\infty}$ is the healing length [14] and $\psi = r_0^2\phi - r_0^2\xi^2\nabla^2\phi/2$. The operator $\{a, b\} = r^{-1}(\partial_r a\partial_\theta b - \partial_r b\partial_\theta a)$ is simply the Poisson brackets in polar coordinates and θ represents the angular coordinate. The latter equation describes hydrodynamical drift waves in a rapidly rotating BEC and includes new features relative to the CHM equation, widely used in the study of the dynamics of waves and turbulence in plasmas and in the atmosphere. Namely, the terms proportional to ξ^2 cast the effects of zero-point pressure, which are known to play no role in geophysics. According to typical experimental conditions, we estimate the sound speed to be $c_s \sim 1\,\mathrm{mm\,s^{-1}}$, $\Omega \sim \omega_\perp \approx 2\pi \times 65\,\mathrm{Hz}$ [15], a transverse harmonic oscillator radius of $a_{\mathrm{ho}} \sim 1.7\,\mu\mathrm{m}$ and a Rossby radius around $r_0 \sim 1.2\,\mu\mathrm{m}$. The Rossby number, Ro, defines the ratio of the inertial to the Coriolis force:

$$Ro = \frac{c_s}{Lf}, \tag{13}$$

where L is the typical length of the system and f is the Coriolis parameter [10, 13]. It characterizes the importance of Coriolis accelerations arising from planetary rotation and typically ranges from 0.1–1 in the case of large-scale low-pressure atmospheres to 10^3 in the case of tornados. Making $L = a_{\mathrm{ho}}$ in a BEC and $f = 2\Omega$, we have $Ro = r_0/a_{\mathrm{ho}} \sim 0.7$, which suggests that the BEC can be regarded as a low-pressure atmosphere. Before proceeding, we should remark that in the most general case, the dynamics along the vertical (axial) direction couples to the transversal direction. In such a situation, equation (12) should be replaced by an expression accounting for the z-direction, describing the analogue of the Poincaré (or gyroscopic) waves in the atmosphere. The corresponding shallow atmosphere approximation, which allows us to neglect the propagation along the axial direction, is envisaged in the case of a rotating BEC by the condition $\omega_z \gg \omega_\perp$. Such a restriction does not compromise the following result, however.

We now show that a rotating BEC can sustain a new hydrodynamic mode corresponding to a drift-acoustic wave. For that purpose, we keep only the linear terms in equation (12) and look for perturbations of the form $\phi \sim \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. The respective dispersion relation is then readily obtained and reads

$$\omega = -v_{\mathrm{R}} k_\theta \frac{1+\xi^2k^2/2}{1+r_0^2k^2(1+\xi^2k^2/2)}, \tag{14}$$

where $k_\theta = \mathbf{k}\cdot\mathbf{e}_\theta$ is the polar (or zonal) component of the wave vector $\mathbf{k} = (k_x, k_y)$. The description of the dynamics in terms of local wave vectors is valid (and consequently the dispersion relation (14)) provided the inequality $k \ll 1/a_{\mathrm{ho}}$ holds. If the finiteness of the system is considered, one must quantize the modes and the dispersion relation must be changed. For the remainder of this paper we only address the former case. The term $v_{\mathrm{R}} = -2\Omega r_0^2\partial_r \ln n_0$ is the generalized Rossby (drift) velocity. Because the equilibrium profile is generally very smooth, we expect v_{R} to be small (compared to the Bogoliubov speed c_s), which suggests that these waves appear as a low-frequency oscillation (compared to both $\omega_\perp$ and Ω). The dispersion relation (14) is similar to the expression for barotropic Rossby waves in the atmosphere [10] and to the dispersion relation obtained for drift waves in a magnetized plasma [16]. For long wavelengths $r_0^2k^2 \ll 1$ (and consequently $\xi^2k^2 \ll 1$), equation (14) reduces to the *zonal flow* dispersion relation $\omega \approx -k_\theta v_{\mathrm{R}}$. One of the remarkable features of the zonal, transverse acoustic waves is that of having negative zonal phase and group velocities, $c_\theta^{(\mathrm{ph})} = c_\theta^{(\mathrm{g})} \approx -v_{\mathrm{R}}$. It means that they always propagate 'westward' in comparison to the rotation of the condensate (which explains the negative values for the frequency in

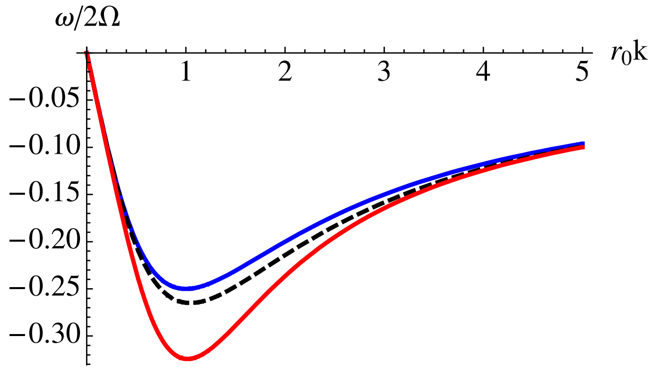

Figure 1. Dispersion relation of the Rossby waves in a rapidly rotating BEC, for $v_R = 0.1c_s$. It is clearly shown that phase speed is generally negative. The blue full line corresponds to the Thomas–Fermi case, $\xi = 0$. The black dashed and red full lines, respectively, correspond to $\xi = 0.7r_0$ and $\xi = 1.3r_0$.

equation (14)). For short wavelengths, one obtains the dispersion relation for the (actual) Rossby waves $\omega \approx -v_R k_\theta/(r_0^2 k^2)$, with phase and group velocities given approximately by

$$\begin{aligned} c_\theta^{(\mathrm{ph})} &\approx -\frac{v_R}{r_0^2 k^2} \\ c_\theta^{(\mathrm{g})} &\approx v_R \frac{2k_\theta/k - 1}{r_0^2 k^2}. \end{aligned} \tag{15}$$

In figure 1, we plot the dispersion relation (14) for different values of the healing length ξ, using $\langle v_R \rangle = 0.1c_s$, where $\langle v_R \rangle$ is the mean Rossby velocity inside the cloud. This procedure is similar to a local density approximation, which is valid for sufficiently smooth equilibrium profiles.

Although a single Rossby wave of arbitrary amplitude is a solution of equation (14), a superposition of waves is generally not. The nonlinear interaction between the waves leads to a mechanism of energy transfer. To study the interaction properties, one decomposes the solution into its Fourier series, $\phi_{\mathbf{k}} = \sum_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$, which, after plugging into equation (12), yields the following nonlinear equation:

$$\frac{\partial \tilde{\phi}_{\mathbf{k}}}{\partial t} + \mathrm{i}\omega_k \tilde{\phi}_{\mathbf{k}} = \sum_{\mathbf{k}_1,\mathbf{k}_2} \Lambda_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k}} \tilde{\phi}_{\mathbf{k}_1} \tilde{\phi}_{\mathbf{k}_2}, \tag{16}$$

where

$$\Lambda_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k}} = 2r_0^2 \delta\left(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}\right)\left(\mathbf{k}_2 \times \mathbf{k}_1\right) \cdot \mathbf{\Omega} \frac{\left(1 + \xi^2 k_1^2/2\right)\left(1 + r_0^2 k_2^2 + r_0^2 \xi^2 k_2^4/2\right)}{1 + r_0^2 k^2 + r_0^2 \xi^2 k^4/2} \tag{17}$$

is the nonlinear coupling operator and ω_k is given by equation (14). Only the waves that satisfy the condition $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}$ interact nonlinearly. The set of waves satisfying this condition is known in the literature as the *resonant triad*. This resonance mechanism is able to transfer energy between different length scales, being one of the sources of classical turbulence in plasmas and in the atmosphere [16, 17]. Here, due to the existence of additional terms that properly account for the quantum hydrodynamical features of the system, i.e when large variations of the density profile are present, we believe that equation (16) may be used to describe turbulence in rotating BECs, opening the stage to explore the similarities and differences between classical and quantum turbulence.

Another interesting feature of Rossby waves in the BEC is the possibility of finding localized structures, which may result, for example, from the saturation of the triad resonance mechanism mentioned above. Such purely nonlinear solitary structures can be obtained from the stationary solutions of equation (12), which readily yields

$$\left(1+\frac{\xi^2}{2}\nabla_\perp^2\right)\{\phi,\nabla_\perp^2\phi\}-\frac{\xi^2}{2}\{\phi,\nabla_\perp^4\phi\}=0. \tag{18}$$

In the Thomas–Fermi limit, the latter expression reduces simply to $\{\phi, \nabla_\perp^2\phi\}=0$, which is satisfied for a family of functions $\nabla_\perp^2\phi=\mathcal{F}(\phi)$, where $\mathcal{F}(x)$ is an arbitrary function of its argument. The different choices for $\mathcal{F}$ will lead to different structures, which describe many physical nonlinear stationary solutions. For example, for the choice $\mathcal{F}(\phi)\propto\exp(-2\phi)$, Stuart [18] showed that the so-called 'cat-eye' solution describes a vortex chain in a magnetized plasma sheet, which has been observed experimentally in mixing layer experiments [19]. However, in the present case there are physical limitations that impose specific constraints on the choice of the solutions. In particular, as discussed in [9], the equilibrium profile associated with the potential in equation (3), which is given by $n_0(r)=n_\infty(R_+^2-r^2)(r^2-R_-^2)$ (the peak density is $n_\infty=\beta\hbar\omega_\perp/2g$), must vanish at the Thomas–Fermi radii defined as follows:

$$\frac{R_\pm^2}{a_{\text{ho}}^2}=\frac{\Omega^2-\omega_\perp^2}{2\beta\omega_\perp^2}\pm\sqrt{\left(\frac{\Omega^2-\omega_\perp^2}{2\beta\omega_\perp^2}\right)^2+\frac{2\mu}{\beta\hbar\omega_\perp}}, \tag{19}$$

where μ represents the chemical potential. For $\mu>0$, the radius R_- is purely imaginary and the density vanishes at $R=R_+$, whereas for $\mu<0$ both R_- and R_+ are present. This reflects the transition occurring at $\mu=0$, where a hole forms at the centre of the condensate and the equilibrium profile assumes an annular shape. The simplest nonlinear structure that verifies such constraints is obtained for $\mathcal{F}(\phi)=-\kappa\phi$, and the respective radial solution, for $\ell=0$, yields $\phi(r)=AJ_0(\kappa r)+BY_0(\kappa r)$. The values of κ, A and B are such that the solution vanishes at the radii $R_\pm$. In figure 2, we plot two possible solitary structures in the overcritical rotation regime $\Omega>\omega_\perp$, obtained for both $\mu>0$ and $\mu<0$. It is interesting to observe that, even for the same set of parameters, the resulting solitary structures may differ from the usual Thomas–Fermi equilibrium profiles discussed above.

A short discussion on the relation between the present waves and the well-known Tkachenko waves in rotating BECs [20] is in order. Tkachenko waves are related to transverse oscillations of the lattice, where the rigidity of the lattice is due to the quantization of the vortices. The proposed Rossby waves, however, are inertial drift waves and the restoring force is not due to the lattice itself, but to the diffused vorticity induced by the Coriolis force. The latter can be formally confused with Tkachenko waves since they possess a transverse character, as well. The main difference (for low k) is that the Tkachenko speed c_{T} must be replaced here by $-v_{\text{R}}$. Firstly, one similarity between these two modes lies in the fact that the Rossby velocity modulus $v_{\text{R}}\ll c_{\text{s}}$, which happens to also be the case for the Tkachenko velocity c_{T}; secondly and most strikingly, the Rossby waves are westward waves, i.e. they possess negative phase velocities, which makes them wholly distinguishable from the Tkachenko waves.

In the spirit of a mean field description, we derived an equation that governs the dynamics of a new drift mode in rotating BECs, which is in close analogy to the Rossby waves in geophysics. Our equation, which has been established for the first time in the context of

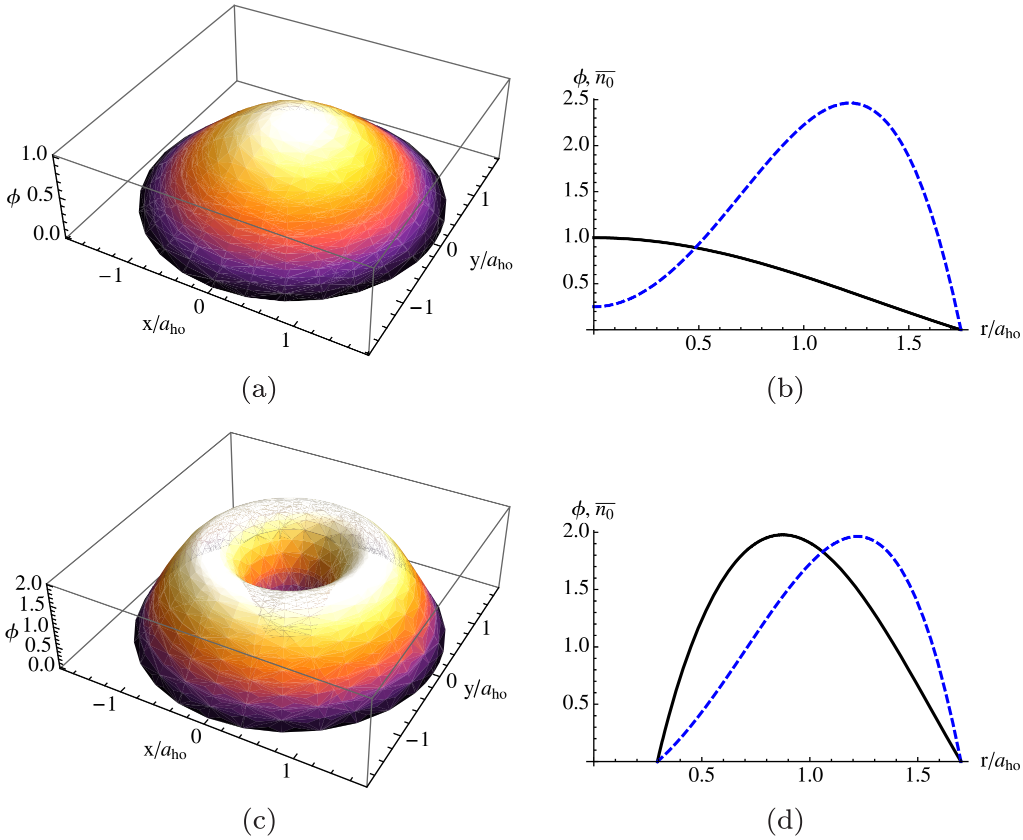

Figure 2. Nonlinear stationary solitary wave resulting from the saturation of the triad resonant decay of Rossby waves, obtained for $\Omega = 2.4\omega_\perp$ and $\beta = 1.6$: (a) $\mu = 0.2\hbar\omega_\perp$ and (c) $\mu = -0.2\hbar\omega_\perp$. Plots (b) and (d), respectively, compare the radial structures (full lines) of (a) and (c) with the corresponding Thomas–Fermi equilibria (dashed lines) discussed in the text, obtained for the same set of parameters.

superfluids (to the best of our knowledge), casts the effect of anharmonicity of the trap and can thus be extended to the overcritical rotating regime. After linearization, we derived the dispersion relation for the Rossby waves and showed that they propagate in the opposite sense to that defined by the angular rotating frequency $\mathbf{\Omega}$. A particular feature of these waves is that they decay resonantly in the form of triads, which is a clear manifestation of a three-wave mixing mechanism in BEC. A very recent and interesting work by Bludov *et al* [21] also established the connection between BECs and geophysics, where the authors reported on the occurrence of rogue waves, a well-known phenomenon in deep oceans. We therefore believe that future numerical and experimental work would reveal interesting features concerned with the nonlinear dynamics in these systems, with special emphasis on the issue of the turbulence spectrum, where we believe that the numerical integration of equation (16) would be relevant. A more detailed investigation of the temporal evolution of the Rossby wave turbulence, as well as the stability of solitary structures, certainly deserves further attention in the future.

Acknowledgments

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