Additivity and non-additivity of multipartite entanglement measures

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Corrigendum

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In the appendix, equations (A.2), (A.3) and (A.4) are incorrect. In (A.2) and (A.3), $\rho'_{N}$ on the right hand side of the equal sign should be replaced by its complex conjugate $\rho'_{N}^{*}$. In (A.4), $\rho'_{N}$ in the last three lines should be replaced by $\rho'_{N}^{*}$.

There are three consequences of this error in the main text. Firstly, on the right hand side of the last equal sign in equation (60) and equation (61), $\rho'_{N}$ should be replaced by $\rho'_{N}^{*}$. Fortunately, this replacement does not affect any of the results in section 4.

Secondly, proposition 23 should be replaced by the following.

Proposition 23. Suppose $\rho_{N}$ and $\rho'_{N}$ are two $N$-partite states on the Hilbert space $\bigotimes_{j=1}^{N} H_{j}$ with $\dim H_{j} = d_{j}$; define $d_{T} = \prod_{j=1}^{N} d_{j}$; then $G(\rho_{N} \otimes \rho'_{N}) \leq \log d_{T} - \log \text{tr}(\rho_{N} \rho'_{N})$. In particular, $G(\rho_{N} \otimes \rho'_{N}^{*}) \leq \log d_{T} - \log \text{tr}(\rho_{N}^{2})$; $G^{\infty}(\rho_{N}) \leq \frac{1}{2} G(\rho_{N}^{2}) \leq \frac{1}{2} \log d_{T} - \frac{1}{2} \log \text{tr}(\rho_{N} \rho'_{N})$.

The implications are also changed as follows. If the GM of $\rho_{N}$ is strong additive and thus $G(\rho_{N} \otimes \rho'_{N}) = G(\rho_{N}) + G(\rho'_{N}) \leq 2G(\rho_{N})$, proposition 23 implies that $G(\rho_{N}) \leq \frac{1}{2} \log d_{T} - \frac{1}{2} \log \text{tr}(\rho_{N}^{2})$. In other words, GM cannot be strong additive (as opposed to ‘additive’ in the published version) if the states are too entangled with respect to GM. For states with real entries in the computational basis, proposition 23 sets a universal upper bound for $G(\rho_{N}^{2})$ and $G^{\infty}(\rho_{N})$; that is, $G^{\infty}(\rho_{N}) \leq \frac{1}{2} G(\rho_{N}^{2}) \leq \frac{1}{2} \log d_{T} - \frac{1}{2} \log \text{tr}(\rho_{N}^{2})$. As a result, GM cannot be additive if the states are too entangled with respect to GM.

Thirdly, as a result of the revision in proposition 23, theorem 24 must be weakened as follows.

Theorem 24. Suppose pure states are drawn according to the Haar measure from the Hilbert space $\bigotimes_{j=1}^{N} H_{j}$ with $N \geq 3$ and $\dim H_{j} = d_{j}$ ($d_{j} \geq 2$, $\forall j$); define $d_{T} = \prod_{j=1}^{N} d_{j}$ and $d_{S} = \sum_{j=1}^{N} d_{j}$. The fraction of pure states whose GM is strong additive is smaller than $\exp\left[-\frac{1}{3} \sqrt{d_{T}} + d_{S} \ln(59Nd_{T})\right]$; the fraction of pure states $|\psi\rangle$ such that $[\log d_{T} - \log(d_{S} \ln d_{T}) - \log \frac{2}{3}] \leq G(|\psi\rangle) \leq G(|\psi\rangle \otimes |\psi^{*}\rangle) \leq \log d_{T}$ is larger than $1 - d_{T}^{-d_{S}}$. For pure states with real entries in the computational basis, the fraction of pure states whose GM is additive is smaller than $\exp\left[-\frac{1}{3} \sqrt{d_{T}} + d_{S} \ln(59Nd_{T})\right]$; the fraction of pure states $|\psi\rangle$ such that
\[ \log d_T - \log(d_5 \ln d_T) - \log 9 \leq G(|\psi\rangle) \leq G(|\psi\rangle^{\otimes 2}) \leq \log d_T \text{ is larger than } 1 - d^{-d_S}. \]

The second part of the new version of the theorem, which is concerned with pure states with real entries in the computational basis, is derived in the same way as the first part; more details can be found in the latest version of [1]. Now, theorem 24 only implies that GM is not strong additive (as opposed to ‘non-additive’ in the published version) for almost all pure multipartite states, provided that the number of parties is sufficiently large and the dimensions of the local Hilbert spaces are comparable. However, concerning pure states with real entries in the computational basis, theorem 24 does imply that GM is non-additive for almost all pure multipartite states, and the GM of one copy and two copies of identical states, respectively, are almost equal. Finally, the implications of additivity property for one-way quantum computation and for asymptotic state transformation remain the same as in the published version.

References

Additivity and non-additivity of multipartite entanglement measures

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Abstract. We study the additivity property of three multipartite entanglement measures, i.e. the geometric measure of entanglement (GM), the relative entropy of entanglement and the logarithmic global robustness. Firstly, we show the additivity of GM of multipartite states with real and non-negative entries in the computational basis. Many states of experimental and theoretical interest have this property, e.g. Bell diagonal states, maximally correlated generalized Bell diagonal states, generalized Dicke states, the Smolin state and the generalization of Dür’s multipartite bound entangled states. We also prove the additivity of the other two measures for some of these examples. Secondly, we show the non-additivity of GM of all antisymmetric states of three or more parties. We also provide a unified explanation of the non-additivity of the three measures of the antisymmetric projector states. In addition, we derive analytical formulae for the three measures of one copy and two copies of the antisymmetric projector states, respectively. Thirdly, we show, with a statistical approach, that almost all multipartite pure states with a sufficiently large number of parties are nearly maximally entangled with respect to GM and relative entropy of entanglement, and they have non-additive GM. Hence, more states may be suitable for universal quantum computation if measurements can be performed on two copies of the resource states. We also show that almost all the multipartite pure states cannot

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be produced reversibly with the combination of multipartite GHZ states under asymptotic LOCC, unless the relative entropy of entanglement is non-additive for generic multipartite pure states.

1. Introduction

Quantum entanglement has attracted intensive attention due to its intriguing properties and potential applications in quantum information processing [1]–[3]. Some geometrically motivated entanglement measures have been providing us with new insights into quantum entanglement, e.g. entanglement of formation [4], relative entropy of entanglement (REE) [5, 6], geometric measure of entanglement (GM) [7, 8], global robustness (GR)
and squashed entanglement. Besides providing a simple geometric picture, they are closely related to some operationally motivated entanglement measures, e.g. entanglement of distillation and entanglement cost. Their additivity property for the bipartite case has been studied by many researchers as a central issue in quantum information theory, because this property is closely related to operational meanings. However, concerning the multipartite setting, only the additivity of squashed entanglement has been proved, while the additivity problem on other measures has largely remained open.

In this paper, we focus on the additivity property of three main entanglement measures in the multipartite case, i.e. REE, GM and logarithmic global robustness (LGR). These entanglement measures and their additivity property are closely related to operational concepts in the multipartite case, as mentioned below. Our results may improve the understanding of multipartite entanglement and stimulate more research work on the three entanglement measures as well as others, such as the tangle and generalized concurrence.

REE is a lower bound to entanglement of formation and an upper bound to entanglement of distillation in the bipartite case. It has a clear statistical meaning as the minimal error rate of mistaking an entangled state for a closest separable state. It has also been employed by Linden et al to study the conditions on reversible state transformation and by Acín et al to study the structure of reversible entanglement generating sets in the tripartite scenario. In addition, Brandão and Plenio have shown that the asymptotic REE equals an asymptotic smooth modification of LGR and a modified version of entanglement of distillation and entanglement cost, which means that the asymptotic REE quantifies the entanglement resources under asymptotic non-entangling operations. In condensed matter physics, REE is also useful for characterizing multipartite thermal correlations and macroscopic entanglement, such as that in high-temperature superconductors.

GM is closely related to the construction of optimal entanglement witnesses and discrimination of quantum states under LOCC. The GM of tripartite pure states is closely related to the maximum output purity of the quantum channels corresponding to these states. Recently, GM has been utilized to determine the universality of resource states for one-way quantum computation. It has also been applied to show that most entangled states are too entangled to be useful as computational resources. Furthermore, a connection between the GM defined via the convex roof and a distant like measure has also been proposed. In condensed matter physics, GM is useful for studying quantum many-body systems, such as characterizing ground state properties and detecting phase transitions.

GR is closely related to state discrimination under LOCC and entanglement quantification with witness operators. It is best suited to studying the survival of entanglement in thermal states and to determining the noise thresholds in the generation of resource states for measurement-based quantum computation.

On the other hand, the additivity property of the three measures REE, GM and LGR greatly affects the utility of multipartite states. For example, in state discrimination under LOCC, the additivity property of these measures may affect the advantage offered by joint measurements on multiple copies of input states over separate measurements. The additivity of GM of generic multipartite states is closely related to their universality as resource states for one-way quantum computation, as we shall see in section 5.3.

The additivity property of the three measures REE, GM and LGR is also closely related to the calculation of their asymptotic or regularized entanglement measures, which are the...
asymptotic limit of the regularized quantities with the \( n \)-copy state. These asymptotic measures will be referred to as asymptotic GM, REE and LGR, and are abbreviated to AGM, AREE and ALGR, respectively. They are useful in the study of the classical capacity of quantum multi-terminal channels [31]. AREE can be used as an invariant when we build the minimal reversible entanglement generating set (MREGS) under asymptotic LOCC. MREGS is a finite set of pure entangled states from which all pure entangled states can be produced reversibly in the asymptotic sense, which is an essential open problem in quantum information theory [24, 25]. AREE also determines the rate of state transformation under asymptotic non-entangling operations [26]. In the bipartite case, AREE provides a lower bound to entanglement cost and an upper bound to entanglement of distillation. Hence, it is essential to compute the regularized entanglement measures. However, the problem is generally very difficult. One main approach to computing these asymptotic measures is to prove their additivity, which is also a focus of the present paper. In this case, the asymptotic measures equal the respective one-shot measures.

Our main approach is the following. Under some group theoretical condition, Hayashi et al [30] showed a relation among REE, LGR and GM. Due to this relation, we can treat the additivity problem of REE and LGR from that of GM in this special case. Hence, we can concentrate on the additivity problem of GM.

Firstly, we derive a novel and general additivity theorem for GM of multipartite states with real and non-negative entries in the computational basis. Applying this theorem, we show the additivity of GM for many multipartite states of either practical or theoretical interest, such as (i) two-qubit Bell diagonal states; (ii) maximally correlated generalized Bell diagonal states, which are closely related to local copying [42]; (iii) isotropic states, which are closely related to depolarization channel [43]; (iv) generalized Dicke states [44], which are useful for quantum communication and quantum networking, and can already be realized using current technologies [45]–[48]; (v) the Smolin state [49], which is useful for remote information concentration [50], super-activation [51], quantum secret sharing [52], etc; and (vi) Dür’s multipartite entangled states, which include bound entangled states that can violate the Bell inequality [53]. By means of the relation among the three measures GM, REE and LGR, we also show the additivity of REE of these examples and the additivity of LGR of the generalized Dicke states and the Smolin state. As a direct application, we obtain AGM and AREE of the above-mentioned examples and ALGR of the generalized Dicke states and the Smolin state.

Our approach is also able to provide a lower bound to AREE and ALGR for generic multipartite states with non-negative entries in the computational basis, such as isotropic states [43], mixtures of generalized Dicke states. In the bipartite scenario, our lower bound to AREE is also a lower bound to entanglement cost. For non-negative tripartite pure states, the additivity of GM implies the multiplicativity of the maximum output purity of the quantum channels related to these states according to the Werner–Holevo recipe [32].

Secondly, we show the non-additivity of GM of antisymmetric states shared over three or more parties and many bipartite antisymmetric states. We also quantify how additivity of GM is violated in the case of antisymmetric projector states, which include antisymmetric basis states and antisymmetric Werner states as special examples, and treat the same problem for REE and LGR. For the antisymmetric projector states, while the three one-shot entanglement measures are generally non-additive, we obtain a relation among AREE, AGM and ALGR. Generalized antisymmetric states [54] are also treated as further counterexamples of the additivity of GM.
Table 1. Additive cases: GM, REE and LGR of multipartite entangled states. All states listed here have additive GM and REE; generalized Dicke states and the Smolin state also have additive LGR; hence the asymptotic measures are equal to the one-shot measures in these cases. REE of the Bell diagonal states was calculated in [5, 16]. REE of the maximally correlated generalized Bell diagonal states and isotropic states as well as their additivity was obtained in [16]. GM of the generalized Dicke states was calculated in [8]; REE and LGR of the generalized Dicke states were calculated in [30, 55, 56]. REE of the Smolin state was calculated in [50, 57]; REE of Dür’s multipartite entangled states was calculated in [56, 57].

<table>
<thead>
<tr>
<th>States</th>
<th>Symbol</th>
<th>GM</th>
<th>REE</th>
<th>LGR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bell diagonal states</td>
<td>$\rho_{BD}(p)$</td>
<td>$1 - \log (p_0 + p_1)$</td>
<td>$1 - H(p_0, 1 - p_0)$</td>
<td>$\log (2p_0)$</td>
</tr>
<tr>
<td>Maximally correlated</td>
<td>$\rho_{MCB}(p)$</td>
<td>$\log d$</td>
<td>$\log d - H(p)$</td>
<td>$-$</td>
</tr>
<tr>
<td>generalized Bell</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>diagonal states</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Isotropic states</td>
<td>$\rho_{I, \lambda}$</td>
<td>$\log \frac{d(d+1)}{\lambda d+1}$</td>
<td>$\log d + \lambda \log \lambda$</td>
<td>$\log(d\lambda)$</td>
</tr>
<tr>
<td>Generalized</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dicke states</td>
<td>$</td>
<td>N, \vec{k}\rangle$</td>
<td>$\log \left[ \prod_{j=0}^{d-1} \left( \frac{N}{\lambda!} \right)^{k_j} \right]$</td>
<td>GM</td>
</tr>
<tr>
<td>Smolin state</td>
<td>$\rho_{ABCD}$</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Dür’s multipartite</td>
<td>$\rho_N(x)$</td>
<td>$\log \frac{2N}{1-x}$</td>
<td>$0 \leq x \leq \frac{1}{N+1}$</td>
<td>$x$</td>
</tr>
<tr>
<td>entangled states</td>
<td>$N \geq 4$</td>
<td>$\log \frac{2}{x}$</td>
<td>$\frac{1}{N+1} \leq x \leq 1$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

Thirdly, we show, with a statistical approach, that almost all multipartite pure states are nearly maximally entangled with respect to GM and REE and, meanwhile, have non-additive GM. Our results have a great impact on the universality of resource states for one-way quantum computation and on asymptotic state transformation. As a twist to the assertion of Gross et al. [35] that most quantum states are too entangled to be useful as computational resources, we show that more states may be suitable for universal quantum computation if measurements can be performed on two copies of the resource states. In addition, we show that almost all multipartite pure states cannot be prepared reversibly with multipartite GHZ states (with different number of parties) under LOCC even in the asymptotic sense, unless REE is non-additive for generic multipartite pure states.

For the convenience of the readers, we summarize the main results for GM, REE and LGR of the states studied in this paper in tables 1 and 2. More details can be found in the relevant sections of the main text.

The paper is organized as follows. Section 2 is devoted to reviewing the preliminary knowledge and terminology, and to showing the relation among the three measures REE, GM and LGR. In section 3, we prove a general additivity theorem for GM of multipartite states with non-negative entries in the computational basis, and apply it to many multipartite
Table 2. Non-additive cases: GM, REE and LGR of some antisymmetric and generalized antisymmetric states. All states listed satisfy $E_R(\rho) = R_L(\rho) = G(\rho) - S(\rho)$, except two copies of generalized antisymmetric states, where it is not known. When $N = 2$, the antisymmetric projector state reduces to the antisymmetric Werner state. The GM of a single copy of the antisymmetric basis state and generalized antisymmetric state was calculated in [30, 54]. The REE and LGR of a single copy of the antisymmetric basis state and generalized antisymmetric state were calculated in [30, 55, 56].

<table>
<thead>
<tr>
<th>States</th>
<th>Symbol</th>
<th>GM</th>
<th>REE</th>
<th>LGR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antisymmetric basis state</td>
<td>$</td>
<td>\psi_{N-}\rangle$</td>
<td>$\log N!$</td>
<td>$\log N!$</td>
</tr>
<tr>
<td>(Slater determinant state)</td>
<td>$</td>
<td>\psi_{N-}\rangle \otimes^2$</td>
<td>$N\log N$</td>
<td>$N\log N$</td>
</tr>
<tr>
<td>Antisymmetric projector state</td>
<td>$\rho_{d,N}$</td>
<td>$\log \frac{d!}{(d-N)!}$</td>
<td>$\log \frac{d!}{(d-N)!}$</td>
<td>$\log \frac{d!}{(d-N)!}$</td>
</tr>
<tr>
<td></td>
<td>$\rho_{d,2} \otimes \rho_{d,N}$</td>
<td>$\log \frac{d!}{N(d-d-N)!}$</td>
<td>$\log \frac{d!}{N(d-d-N)!}$</td>
<td>$\log \frac{d!}{N(d-d-N)!}$</td>
</tr>
<tr>
<td></td>
<td>$(N \leq d_1 \leq d_2)$</td>
<td>$\log \frac{d!}{N(d-d-N)!}$</td>
<td>$\log \frac{d!}{N(d-d-N)!}$</td>
<td>$\log \frac{d!}{N(d-d-N)!}$</td>
</tr>
<tr>
<td>Generalized antisymmetric state</td>
<td>$</td>
<td>\psi_{d,p,d_p}\rangle$</td>
<td>$\log [(d^p)!]$</td>
<td>$\log [(d^p)!]$</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>\psi_{d,p,d_p}\rangle \otimes^2$</td>
<td>$d^p \log d^p$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

states, e.g. Bell diagonal states, maximally correlated generalized Bell diagonal states, isotropic states, generalized Dicke states, mixture of Dicke states, the Smolin state and Dür’s multipartite entangled states. Also, we treat the additivity problem of REE and LGR of these examples and discuss the implications of these results for state transformation. In section 4, we focus on the antisymmetric subspace, and show the non-additivity of GM of states in this subspace when there are three or more parties. We also establish a simple relation among the three measures for the tensor product of antisymmetric projector states, and compute GM, REE and LGR for one copy and two copies of antisymmetric projector states. Generalized antisymmetric states are also treated as further counterexamples to the additivity of GM. In section 5, we show that GM is non-additive for almost all multipartite pure states and discuss the impact of our results on the universality of resource states for one-way quantum computation and on asymptotic state transformation. We conclude with a summary and some open problems.

2. Preliminary knowledge and terminology

In this section, we recall the definitions and basic properties of the three main multipartite entanglement measures, that is, REE, GM and the global robustness of entanglement, and introduce the additivity problem on these entanglement measures. We also present a few known results concerning the relations among these measures, which will play an important role later. The impact of permutation symmetry on GM and the connection between the GM of tripartite pure states and the maximum output purity of quantum channels are also discussed briefly.
2.1. Geometric measure of entanglement (GM), relative entropy and global robustness of entanglement

Consider an $N$-partite state $\rho$ shared over the parties $A_1, \ldots, A_N$ with joint Hilbert space $\bigotimes_{j=1}^N \mathcal{H}_j$. REE measures the minimum distance in terms of relative entropy between the given state $\rho$ and the set of separable states and is defined as \[ E_R(\rho) := \min_{\sigma \in \text{SEP}} S(\rho \parallel \sigma), \] where $S(\rho \parallel \sigma) = \text{tr} \rho (\log \rho - \log \sigma)$ is the quantum relative entropy, and the logarithm has base 2 throughout this paper. SEP denotes the set of fully separable states, which are of the form $\sigma = \sum_j \sigma_j^1 \otimes \cdots \otimes \sigma_j^N$, such that $\sigma_j^k$ is a single-particle state of the $k$th party. For a pure state $\rho = |\psi\rangle \langle \psi|$,

\[ E_R(|\psi\rangle) := E_R(\rho), \]

is used to denote $E_R(\rho)$ throughout this paper; similarly for other entanglement measures to be introduced. Any state $\sigma$ minimizing (1) is a closest separable state of $\rho$. As its definition involves the minimization over all separable states, REE is known only for a few examples, such as bipartite pure states [5, 6, 58], Bell diagonal states [5, 16], some two-qubit states [19], Werner states [17, 18, 59], maximally correlated states, isotropic states [16], generalized Dicke states [30, 55, 56], antisymmetric basis states [30, 54], some graph states [31], the Smolin state and Dür’s multipartite entangled states [56, 57]. A numeric method for computing REE of bipartite states has been proposed in [6].

REE with respect to the set of states with positive partial transpose (PPT) $E_{R,PPT}$, which is obtained by replacing the set of separable states in (1) with the set of PPT states, has also received much attention [16, 17, 60]. However, in this paper, we shall follow the definition in (1).

GM measures the closest distance in terms of overlap between a given state and the set of separable states, or, equivalently, the set of pure product states, and is defined as \[ \Lambda^2(\rho) := \max_{\sigma \in \text{SEP}} \text{tr}(\rho \sigma) = \max_{|\psi\rangle \in \text{PRO}} \langle \psi|\rho|\psi\rangle, \] \[ G(\rho) := -2 \log \Lambda(\rho). \] Here, PRO denotes the set of fully pure product states in the Hilbert space $\bigotimes_{j=1}^N \mathcal{H}_j$. Any pure product state maximizing (2) is a closest product state of $\rho$. It should be emphasized that, for mixed states, GM defined in (3) is not an entanglement measure proper, and there are alternative definitions of GM through the convex roof construction [8]. However, GM of $\rho$ defined in (3) is closely related to GM of the purification of $\rho$ [61], and also to REE and LGR of $\rho$, as we shall see later. Meanwhile, this definition is also useful in the construction of optimal entanglement witnesses [8], and in the study of state discrimination under LOCC [29, 30]. Thus, we shall follow the definition in (3) in this paper. GM is known only for a few examples, such as bipartite pure states, GHZ-type states, generalized Dicke states [8], antisymmetric basis states [30, 54], pure symmetric three-qubit states [62]–[64], some other pure three-qubit states [8, 62, 65] and some graph states [31].

Different from the above two entanglement measures, GR [9, 10] of entanglement measures how sensitive an entangled state is to the mixture of noise, and is defined as \[ R_g(\rho) := \min \left\{ t : t \geq 0, \exists \text{ a state } \Delta, \frac{\rho + t\Delta}{1 + t} \in \text{SEP} \right\}. \]
LGR of entanglement is defined as

\[ R_L(\rho) := \log(1 + R_g). \quad (5) \]

LGR is known for even fewer examples, such as bipartite pure states \([9, 10]\), generalized Dicke states, antisymmetric basis states \([30, 56]\) and some graph states \([31]\). A numerical method for computing LGR has been proposed in \([66, 67]\).

2.2. Additivity problem of multipartite entanglement measures

In quantum information processing, it is generally more efficient to process a family of quantum states together rather than process each one individually. In this case, entanglement measures can still serve as invariants under reversible LOCC transformation, provided that we consider the family of states as a whole. A fundamental problem in entanglement theory is whether the entanglement of the tensor product of states is the sum of that of each individual. First we need to make it clear what the entanglement of the tensor product of states means. Take two states as an example: let \(\rho\) be an \(N\)-partite state shared over the parties \(A_1, \ldots, A_N\), and \(\sigma\) be another \(N\)-partite state shared over the parties \(A_1', \ldots, A_N'\), where we have added superscripts to the names of the parties to distinguish the two states. Now there are \(2N\) parties involved in the tensor product state \(\rho \otimes \sigma\). However, in most scenarios that concerned us, the pair of parties \(A_j, A_j'\) for each \(j = 1, \ldots, N\) will be in the same lab, and can be taken as a single party \(A_j\). In this sense, \(\rho \otimes \sigma\) can be seen as an \(N\)-partite state shared over the parties \(A_1, \ldots, A_N\). The definition of any entanglement measure, such as GM, REE and LGR of the tensor product state \(\rho \otimes \sigma\), follows this convention throughout this paper; similarly for the tensor product of more than two states, except when stated otherwise.

A particularly important case is the entanglement of the tensor product of multiple copies of the same state. In the limit of a large number of copies, we obtain the regularized or asymptotic entanglement measure, which reads

\[ E^\infty(\rho) := \lim_{n \to \infty} \frac{1}{n} E(\rho^\otimes n), \quad (6) \]

where \(E\) is the entanglement measure under consideration. When \(E\) is taken as \(E_R\), \(G\) or \(R_L\), respectively, the resulting regularized measures are referred to as asymptotic REE (AREE) \(E_R^\infty\), asymptotic GM (AGM) \(G^\infty\) and asymptotic LGR (ALGR) \(R_L^\infty\), respectively.

The entanglement \(E\) of an \(N\)-partite state \(\rho\) is called additive if \(E^\infty(\rho) = E(\rho)\) and strong additive if the equality \(E(\rho \otimes \sigma) = E(\rho) + E(\sigma)\) holds for any \(N\)-partite state \(\sigma\). Obviously, strong additivity implies additivity. An entanglement measure itself is called (strong) additive if it is (strong) additive for any state. Similarly, the entanglement of the two states \(\rho, \sigma\) is called additive if the equality \(E(\rho \otimes \sigma) = E(\rho) + E(\sigma)\) holds.

Historically, both GM and REE had been conjectured to be additive until counterexamples were found. The first counterexample to the additivity of REE is the antisymmetric Werner state found by Vollbrecht and Werner \([18]\). The first counterexample to the additivity of GM is the tripartite antisymmetric basis state found by Werner and Holevo \([32]\). Coincidentally, these counterexamples are both antisymmetric states, and the tripartite antisymmetric basis state is exactly a purification of the two-qutrit antisymmetric Werner state. We shall reveal the reason behind this coincidence in section 4.

For bipartite pure states, REE is equal to the von Neumann entropy of each reduced density matrix \([5, 6, 58]\); GM is equal to the logarithm of the inverse of the largest eigenvalue.
of each reduced density matrix [8]; and LGR is equal to one half of the logarithm of the trace of the positive square root of each reduced density matrix [9, 10], thus, REE, GM and LGR are all additive. GM and REE are also additive for any multipartite pure states with generalized Schmidt decomposition, such as the GHZ state. More generally, REE (GM, LGR) of a multipartite pure state is additive if it is equal to the same measure under some bipartite cut. For example, some graph states have additive REE, GM and LGR for this reason [31]. In general, it is very difficult to prove the additivity or non-additivity of GM, REE and LGR of a given state or to compute AGM, AREE and ALGR. The additivity of REE is known to hold for a few other examples, such as maximally correlated states, isotropic states [16], two-qubit Werner states [17, 59] and some other two-qubit states [16, 19]. Little is known about the additivity property of GM and LGR.

2.3. Relations among the three measures

There is a simple inequality among the three measures REE, GM and LGR [29, 56],

\[ R_L(\rho) \geq E_R(\rho) \geq G(\rho) - S(\rho), \]  

where \( S(\rho) \) is the von Neumann entropy. So the inequality \( R_L(\rho) \geq E_R(\rho) \geq G(\rho) \) holds when \( \rho \) is a pure state. The same is true if the three measures are replaced by their respective regularized measures. This inequality and its equality condition are crucial in translating our results for GM to that for REE and LGR in the later sections.

A sufficient condition on the equality is given as lemma 9 in appendix C of [30]. For convenience, we reproduce it in the following proposition.

**Proposition 1.** Assume that a projector state \( P_{tr} P \) satisfies the following. There exist a compact group \( H \), its unitary representation \( U \) and a product state \( |\phi_N\rangle \) such that (1) \( U(g) \) is a local unitary for all \( g \in H \). (2) \( U(g)PU(g)^\dagger = P \). (3) The state \( |\phi_N\rangle \) is one of the closest product states of \( P \). (4) \( \int_H U(g)|\phi_N\rangle\langle\phi_N|U(g)^\dagger \mu(dg) \geq \frac{|\langle\phi_N|P|\phi_N\rangle|}{\text{tr} P} P \), where \( \mu \) is the invariant probability measure on \( H \). Then,

\[ R_L \left( \frac{P_{tr} P}{\text{tr} P} \right) = E_R \left( \frac{P_{tr} P}{\text{tr} P} \right) = G \left( \frac{P_{tr} P}{\text{tr} P} \right) - \log \text{tr} P. \] (8)

Under condition (1), conditions (2)–(4) are satisfied if (5) the range of \( P \) is an irreducible representation of \( H \) whose multiplicity is one in the representation \( U \).

For example, generalized Dicke states, antisymmetric basis states [30, 55] and some graph states [30, 31] satisfy the conditions (1)–(4), so they satisfy (8). In this case, if GM is additive, then both LGR and REE are additive, which follows from proposition 2 below. If, in addition, condition (5) is satisfied, then LGR, REE and GM are simultaneously additive or simultaneously non-additive, which follows from proposition 3 below.

**Proposition 2.** Assume that two multipartite states \( \rho, \sigma \) satisfy \( R_L(\rho) = E_R(\rho) = G(\rho) - S(\rho) \), \( R_L(\sigma) = E_R(\sigma) = G(\sigma) - S(\sigma) \) and \( G(\rho \otimes \sigma) = G(\rho) + G(\sigma) \), then the following relations hold:

\[ R_L(\rho \otimes \sigma) = R_L(\rho) + R_L(\sigma), \quad E_R(\rho \otimes \sigma) = E_R(\rho) + E_R(\sigma). \] (9)
Proof.

\[
R_L(\rho) + R_L(\sigma) \geq R_L(\rho \otimes \sigma) \geq G_R(\rho \otimes \sigma) \geq G(\rho \otimes \sigma) - S(\rho \otimes \sigma)
\]

\[
= G(\rho) + G(\sigma) - S(\rho) - S(\sigma) = R_L(\rho) + R_L(\sigma).
\]

(10)

\[
\]

Proof. Assume that \( n \) projector states \( \frac{P_j}{trP_j} \) for \( j = 1, \ldots, n \) satisfy conditions (1) and (5) of proposition 1; then

\[
R_L \left( \bigotimes_{j=1}^{n} \frac{P_j}{trP_j} \right) = E_R \left( \bigotimes_{j=1}^{n} \frac{P_j}{trP_j} \right) = G \left( \bigotimes_{j=1}^{n} \frac{P_j}{trP_j} \right) - \sum_{j=1}^{n} \log trP_j.
\]

(11)

Proposition 3. Assume that \( n \) projector states \( \frac{P_j}{trP_j} \) for \( j = 1, \ldots, n \) satisfy conditions (1) and (5) of proposition 1; then

\[
R_L \left( \bigotimes_{j=1}^{n} \frac{P_j}{trP_j} \right) = E_R \left( \bigotimes_{j=1}^{n} \frac{P_j}{trP_j} \right) = G \left( \bigotimes_{j=1}^{n} \frac{P_j}{trP_j} \right) - \sum_{j=1}^{n} \log trP_j.
\]

(11)

Proof. Let \( H_j \) and \( U_j \) be the group and the local unitary representation satisfying conditions (1) and (5) of proposition 1 concerning the projector state \( \frac{P_j}{trP_j} \) for \( j = 1, \ldots, n \). Define the representation \( \prod_{j=1}^{n} \times U_j \) of the direct product group \( \prod_{j=1}^{n} G_j \) by \( (\prod_{j=1}^{n} \times U_j)(g_1, \ldots, g_n) := \prod_{j=1}^{n} \otimes U_j(g_j) \). This satisfies conditions (1) and (5) of proposition 1 concerning the projector state \( \frac{G_j^{P_j}}{\prod_{j=1}^{n} trP_j} \), which implies (11).

Next, we present two known results concerning the relation between a given entanglement measure of a pure multipartite state and that of its reduced states after tracing out one party. Let \( |\psi\rangle \) be an \( N \)-partite pure state, and \( \rho \) one of its \( (N-1) \)-partite reduced states. Firstly, Jung et al \[61\] have proved the following equality holds:

\[
G(\rho) = G(|\psi\rangle). \tag{12}
\]

So the additivity problem on an \( N \)-partite pure state is equivalent to that on its \( (N-1) \)-partite reduced states.

Secondly, Plenio and Vedral \[58\] have proved a useful inequality concerning REE,

\[
E_R(|\psi\rangle) \geq E_R(\rho) + S(\rho), \tag{13}
\]

which means that the reduction in entanglement is no less than the increase in entropy due to deletion of a subsystem. If \( G(|\psi\rangle) = E_R(|\psi\rangle) \) (this is true if, for example, proposition 1 is satisfied), combining (7), (12) and (13), we obtain an interesting equality,

\[
G(|\psi\rangle) = G(\rho) = E_R(|\psi\rangle) = E_R(\rho) + S(\rho). \tag{14}
\]

In this case, the total entanglement \( E_R(|\psi\rangle) \) is the sum of the remaining entanglement \( E_R(\rho) \) after losing a subsystem and the increase in entropy \( S(\rho) \). Moreover, if GM of \( |\psi\rangle \) is additive, then GM of \( \rho \), REE of \( |\psi\rangle \) and that of \( \rho \) are all additive.

2.4. GM and permutation symmetry

Permutation symmetry plays an important role in the study of multipartite entanglement. A multipartite state is called (permutation) symmetric (antisymmetric) if its support is contained in the symmetric (antisymmetric) subspace, and permutation invariant if it is invariant under permutation of the parties. Note that both symmetric states and antisymmetric states are permutation invariant. Hayashi et al \[68\] and Wei and Severini \[69\] have shown that the closest
product state to a symmetric pure state with non-negative amplitudes in the computational basis can be chosen to be symmetric. Hübener et al [70] have shown this fact for general symmetric states (corollary 5). In addition, if $\rho$ is a pure state shared over three or more parties, the closest product state is necessarily symmetric (lemma 1). Here we present a stronger result for general symmetric states shared over three or more parties.

**Proposition 4.** The closest product state to any $N$-partite pure or mixed symmetric state with $N \geq 3$ is necessarily symmetric.

**Proof.** Let $\rho$ be an $N$-partite symmetric state with $N \geq 3$. Assume that $\rho$ is mixed, otherwise the proposition is already proved as lemma 1 in [70]. Suppose $|\psi\rangle$ is a purification of $\rho$, and $|\varphi_N\rangle$ a closest product state to $\rho$. According to theorem 1 in [61], there exists a single-particle state $|a\rangle$, such that $|\varphi_N\rangle \otimes |a\rangle$ is a closest product state to $|\psi\rangle$; thus $|\varphi_N\rangle$ is a closest product state to the unnormalized state $\langle a|\psi\rangle$. Since the purification has the form $|\psi\rangle = \sum_j |\psi_j\rangle \otimes |j\rangle$ with each $|\psi_j\rangle$ a symmetric $N$-partite state, $\langle a|\psi\rangle$ is an unnormalized $N$-partite pure symmetric state with $N \geq 3$. According to lemma 1 in [70], $|\varphi_N\rangle$ is necessarily symmetric too. $\square$

We shall prove an analogue of proposition 4 for antisymmetric states in section 4.1.

### 2.5. GM of tripartite pure states and maximum output purity of quantum channels

Finally, we mention an interesting connection between GM of tripartite pure states and the maximum output purity of quantum channels established by Werner and Holevo [32]. Let $\Phi$ be a CP map with the Kraus form $\Phi(\rho) = \sum_k A_k \rho A_k^\dagger$. The maximum output purity of the map $\Phi$ is defined as

$$\nu_p(\Phi) := \max_{\rho} \| \Phi(\rho) \|_p,$$

where $\| \rho \|_p = (\text{tr} \rho^p)^{1/p}$, and the maximum is taken over all quantum states. From the Kraus representation of the map $\Phi$, one can construct a tripartite state $|\Phi\rangle$ (not necessarily normalized) with components $\langle h_j|A_k|e_l\rangle$ and vice versa, where $|h_j\rangle$s and $|e_j\rangle$s are orthonormal bases in the appropriate Hilbert spaces, respectively. Note that, as far as entanglement measures are concerned, it does not matter which Kraus representation of the map $\Phi$ is chosen, because different representations lead to tripartite states that are equivalent under local unitary transformations. It should be emphasized that the map constructed from a generic tripartite pure state according to the above correspondence may not be trace preserving.

The maximum output purity of the channel $\Phi$ and GM of the tripartite state $|\Phi\rangle$ are related to each other through the following simple formula [32]:

$$\nu_\infty(\Phi) = \Lambda^2(|\Phi\rangle).$$

According to this result, we can compute GM of a tripartite pure state by computing the maximum output purity $\nu_\infty$ of the corresponding map and vice versa. Generally speaking, the computation of the maximum output purity involves far fewer optimization parameters. Moreover, we can translate the multiplicativity property about the maximum output purity to the additivity property about GM and vice versa. Actually, the non-additivity of GM of the tripartite antisymmetric basis state corresponds exactly to the non-multiplicativity of the maximum output purity $\nu_\infty$ of the Werner–Holevo channel [32].
3. Additivity of GM of non-negative multipartite states

A density matrix is called non-negative if all its entries in the computational basis are non-negative. Many states of either theoretical or practical interest can be written as non-negative states, with an appropriate choice of basis, such as (i) two-qubit Bell diagonal states, (ii) maximally correlated generalized Bell diagonal states, (iii) isotropic states, (iv) generalized Dicke states, (v) the Smolin state and (vi) Dür’s multipartite entangled states.

In this section, we prove a general theorem of the strong additivity of GM for non-negative states and show the additivity of REE and LGR for many states mentioned in the last paragraph. For general non-negative states, our additivity result for GM can provide a lower bound to AREE and ALGR. These results can be used to study state discrimination under LOCC [29, 30], the classical capacity of quantum multi-terminal channel [31]. The result for AREE can be utilized to determine the possibility of reversible transformation among certain multipartite states under asymptotic LOCC and determine the transformation rate under asymptotic non-entangling operations. For non-negative bipartite states, our results also provide a lower bound to entanglement of formation and entanglement cost. For non-negative pure tripartite states, the additivity of GM implies the multiplicativity of the maximum output purity of the quantum channels related to these states according to the Werner–Holevo recipe [32].

In section 3.1, we prove the strong additivity of GM of arbitrary non-negative states and provide a nontrivial lower bound to AREE and ALGR, which translates to a lower bound to entanglement of formation and entanglement cost in the bipartite case. In section 3.2, we prove the strong additivity of GM for Bell diagonal states, maximally correlated generalized Bell diagonal states, isotropic states, and the additivity of REE of Bell diagonal states, maximally correlated generalized Bell diagonal states. In section 3.3, we prove the strong additivity of GM and additivity of REE of generalized Dicke states, and their reduced states after tracing out one party, as well as the additivity of LGR of generalized Dicke states. The implication of these results on asymptotic state transformation is also discussed briefly. In section 3.4, we give a lower bound to AREE of the mixture of Dicke states. In section 3.5, we prove the strong additivity of GM and the additivity of REE and LGR of the Smolin state. In section 3.6, we prove the strong additivity of GM and additivity of REE of Dür’s multipartite entangled states.

3.1. General additivity theorem for GM of non-negative states

We start by proving our main theorem of this section.

**Theorem 5.** GM of any non-negative N-partite state $\rho$ is strong additive; that is, for any other N-partite state $\sigma$, the following equalities hold: $\Lambda(\rho \otimes \sigma) = \Lambda(\rho) \Lambda(\sigma)$, $G(\rho \otimes \sigma) = G(\rho) + G(\sigma)$.

**Proof.** Assume that $|\varphi_N\rangle$ is a closest product state to $\rho \otimes \sigma$; we can write it in the following form:

$$|\varphi_N\rangle = \bigotimes_{l=1}^N |a_l\rangle_{A_l} |a_l\rangle_{\bar{A}_l} = \bigotimes_{l=1}^{d_l-1} \sum_{j=0}^{d_l-1} a_{lj} |j\rangle_{A_l} |c_{lj}\rangle_{\bar{A}_l},$$

(17)
where $|j_l\rangle_{A_l}$s for given $l$ form an orthonormal basis, $|c_{l,j_l}\rangle_{A_l}$s are normalized states, and $a_{l,j_l} \geq 0$, $\sum_{j_l=0}^{d_{l}-1} a_{l,j_l}^2 = 1$.

$$\Lambda^2(\rho \otimes \sigma) = \langle \varphi_N | \rho \otimes \sigma | \varphi_N \rangle$$

$$= \left| \sum_{k_1,j_1,...,k_N,j_N} \left\{ \left( \prod_{l=1}^{N} a_{l,j_l}a_{l,k_l} \right) \left[ \left( \bigotimes_{l=1}^{N} |k_l\rangle \right) \rho \left( \bigotimes_{l=1}^{N} |j_l\rangle \right) \right] \left[ \left( \bigotimes_{l=1}^{N} |c_{l,k_l}\rangle \right) \sigma \left( \bigotimes_{l=1}^{N} |c_{l,j_l}\rangle \right) \right] \right\} \right|$$

$$\leq \sum_{k_1,j_1,...,k_N,j_N} \left\{ \left( \prod_{l=1}^{N} a_{l,j_l}a_{l,k_l} \right) \left[ \left( \bigotimes_{l=1}^{N} |k_l\rangle \right) \rho \left( \bigotimes_{l=1}^{N} |j_l\rangle \right) \right] \right\} \Lambda^2(\sigma) \leq \Lambda^2(\rho) \Lambda^2(\sigma). \quad (18)$$

In the above derivation, the next to last inequality is due to the assumption that $\rho$ is non-negative and the following inequality,

$$\left| \left( \bigotimes_{l=1}^{N} |c_{l,k_l}\rangle \right) \sigma \left( \bigotimes_{l=1}^{N} |c_{l,j_l}\rangle \right) \right|$$

$$\leq \left[ \left( \bigotimes_{l=1}^{N} |c_{l,k_l}\rangle \right) \sigma \left( \bigotimes_{l=1}^{N} |c_{l,j_l}\rangle \right) \left( \bigotimes_{l=1}^{N} |c_{l,k_l}\rangle \right) \sigma \left( \bigotimes_{l=1}^{N} |c_{l,j_l}\rangle \right) \right]^{1/2} \leq \Lambda^2(\sigma), \quad (19)$$

which follows from the Schwarz inequality and the definition of $\Lambda^2(\sigma)$ (Wei T-C 2010, private communication). Combining with the opposite inequality $\Lambda^2(\rho \otimes \sigma) \geq \Lambda^2(\rho)\Lambda^2(\sigma)$, which follows from the subadditivity of GM, we conclude that $\Lambda^2(\rho \otimes \sigma) = \Lambda^2(\rho)\Lambda^2(\sigma)$, $G(\rho \otimes \sigma) = G(\rho) + G(\sigma)$. Evidently, the closest product state to $\rho \otimes \sigma$ can be chosen as the tensor product of the closest product states to $\rho$ and $\sigma$, respectively.

For a non-negative $N$-partite pure state $\rho = |\Phi_N\rangle \langle \Phi_N|$, theorem 5 says that $G(|\Phi_N\rangle \otimes |\Gamma_N\rangle) = G(|\Phi_N\rangle) + G(|\Gamma_N\rangle)$ for any $N$-partite pure state $|\Gamma_N\rangle$. Theorem 5 provides a new way to compute GM of the tensor product of multipartite states, when GM of each member is known. In particular, it enables us to calculate AGM of non-negative states, which are a large family of multipartite states.

**Corollary 6.** For any non-negative state $\rho$, $G^\infty(\rho) = G(\rho)$.

For a non-negative pure tripartite state, the additivity of GM translates immediately to the multiplicativity of the maximum output purity $\nu_{int}$ of the corresponding quantum channel constructed according to the Werner–Holevo recipe [32]. Thus, theorem 5 may also be useful in the study of the additivity problem concerning quantum channels.

In addition, theorem 5 gives a lower bound to AREE and ALGR for non-negative states. This lower bound is often nontrivial, as we shall see later. According to (7), for non-negative states...
states $\rho_j$s, $R_L(\otimes_j \rho_j) \geq E_R(\otimes_j \rho_j) \geq G(\otimes_j \rho_j) - S(\otimes_j \rho_j) = \sum_j G(\rho_j) - \sum_j S(\rho_j)$, where we have employed the additivity of von Neumann entropy. In particular, $R_L^\infty(\rho) \geq E_R^\infty(\rho) \geq G(\rho) - S(\rho)$.

For a generic bipartite state $\rho$, recall that $E_F(\rho) \geq E_R(\rho)$ and $E_c(\rho) \geq E_R^\infty(\rho)$, where $E_F$ and $E_c$ denote entanglement of formation and entanglement cost, respectively. Therefore, when $\rho$ is non-negative, $G(\rho) - S(\rho)$ also gives a lower bound to the entanglement of formation and entanglement cost.

**Theorem 7.** Both ALGR and AREE of a non-negative state $\rho$ are lower bounded by the difference between GM and the von Neumann entropy of the state, $R_L^\infty(\rho) \geq E_R^\infty(\rho) \geq G(\rho) - S(\rho)$. The bound to AREE is tight if $E_R(\rho) = G(\rho) - S(\rho)$, and the bound to ALGR is tight if $R_L(\rho) = E_R(\rho) = G(\rho) - S(\rho)$. If $\rho$ is a non-negative bipartite state, $G(\rho) - S(\rho)$ is also a lower bound to entanglement of formation and entanglement cost, $E_F(\rho) \geq E_c(\rho) \geq G(\rho) - S(\rho)$.

Next, we prove a useful lemma concerning the closest product states of non-negative states.

**Lemma 8.** The closest product state to a non-negative state $\rho$ can be chosen to be non-negative.

**Proof.** Represent $\rho$ in the computational basis,

$$\rho = \sum_{k_1, j_1, \ldots, k_N, j_N} \rho_{k_1, \ldots, k_N; j_1, \ldots, j_N} |k_1, \ldots, k_N\rangle \langle j_1, \ldots, j_N|,$$

(20)

where $\rho_{k_1, \ldots, k_N; j_1, \ldots, j_N} \geq 0$. Assume that $|\varphi_N\rangle$ is a closest product state to $\rho$, which reads

$$|\varphi_N\rangle := |a_1\rangle \otimes \cdots \otimes |a_N\rangle, \quad \text{with} \quad |a_l\rangle := (b_{0,l}, \ldots, b_{h-1,l})^T, \ \forall l.$$

(21)

$$\langle \varphi_N | \rho | \varphi_N \rangle = \sum_{k_1, j_1, \ldots, k_N, j_N} \rho_{k_1, \ldots, k_N; j_1, \ldots, j_N} \langle a_1, \ldots, a_N | k_1, \ldots, k_N \rangle \langle j_1, \ldots, j_N | a_1, \ldots, a_N \rangle$$

$$\leq \sum_{k_1, j_1, \ldots, k_N, j_N} \rho_{k_1, \ldots, k_N; j_1, \ldots, j_N} \prod_{l=1}^N |b_{k_l}^* b_{j_l}|,$$

(22)

the inequality is saturated when $b_{j_l}$s are all non-negative, that is, $|\varphi_N\rangle$ is non-negative. $\square$

In the rest of this section, we illustrate the power of theorems 5 and 7 and lemma 8 with many concrete examples. In particular, we prove the strong additivity of GM of the following states: Bell diagonal states, maximally correlated generalized Bell diagonal states, isotropic states, generalized Dicke states, mixture of Dicke states, the Smolin state and Dür’s multipartite entangled states. Moreover, we prove the additivity of REE of Bell diagonal states, maximally correlated generalized Bell diagonal states, generalized Dicke states, generalized Dicke states with one party traced out, the Smolin state and Dür’s multipartite entangled states. The additivity of LGR of generalized Dicke states and the Smolin state is also shown. The implications of these results to state transformation under asymptotic LOCC and asymptotic non-entangling operations, respectively, are also discussed briefly.
3.2. Bipartite mixed states and tripartite pure states

In the bipartite scenario, for any pure states, REE, GM and LGR can be easily calculated and their additivity has been shown [5, 6], [8]–[10]. Note that any bipartite pure state is non-negative in the Schmidt basis. Hence, its GM is strong additive according to theorem 5. The same is true for any multipartite state with a generalized Schmidt decomposition. However, even in the bipartite scenario, the calculation of REE, GM and LGR is not so trivial for mixed states. Moreover, the additivity problem for generic mixed states is notoriously difficult. Due to (12), the difficulty in GM for bipartite mixed states is equivalent to that for tripartite pure states.

As one of the most simple examples of bipartite mixed states, we focus on maximally correlated generalized Bell diagonal states. Maximally correlated states are known as a typical example, where REE is known to be additive \( \Psi_1 \). Actually, this is true for all Bell diagonal states. Let

\[
\rho_{\text{MCB}}(p) := \sum_{k=0}^{d-1} p_k |\Psi_k\rangle\langle\Psi_k|, \quad \text{with} \quad |\Psi_k\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{(2\pi i k / d) j} |jj\rangle,
\]

(23)

where \( p = (p_0, \ldots, p_{d-1}) \) is a probability distribution. It is easy to see that \( \Lambda^2(\rho_{\text{MCB}}(p)) = \max_{|\phi\rangle} |\phi\rangle\langle\phi|\rho_{\text{MCB}}(p)|\phi\rangle\langle\phi| \leq \max_{|\phi\rangle} |\phi\rangle\langle\phi|\Psi_k\rangle\langle\Psi_k|\langle\phi| \leq \frac{2}{d} \), and the upper bound is achievable by setting \( |\phi\rangle = |j\rangle \), \( \forall j \). In addition, the state \( \rho_{\text{MCB}}(p) \) can be converted into a non-negative state via local unitary transformations, such as the simultaneous local Fourier transformation. According to theorem 5, we obtain

**Proposition 9.** The maximally correlated generalized Bell diagonal state in (23) has strong additive GM and thus \( G^\infty(\rho_{\text{MCB}}(p)) = G(\rho_{\text{MCB}}(p)) = \log d \).

Let \( \sigma = \sum_{j} \frac{1}{d} |jj\rangle\langle jj| \); we have \( E^\infty_{R}(\rho_{\text{MCB}}(p)) \leq E_{R}(\rho_{\text{MCB}}(p)) \leq S(\rho_{\text{MCB}}(p) \parallel \sigma) = \log d - H(p) \), where \( H(p) \) is the Shannon entropy of the distribution \( p \). Applying the inequality \( (7) \) and its asymptotic version to the maximally correlated generalized Bell diagonal state \( \rho_{\text{MCB}}(p) \), we obtain the additivity of REE for \( \rho_{\text{MCB}}(p) \):

\[
E^\infty_{R}(\rho_{\text{MCB}}(p)) = E_{R}(\rho_{\text{MCB}}(p)) = \log d - H(p).
\]

(24)

The same result has been obtained by Rains [16] with a different method.

In the two-qubit system, any rank-two Bell diagonal state, a mixture of two orthogonal Bell states, can always be converted into the form in (23) with a suitable local unitary transformation. So, any two-qubit rank-two Bell diagonal state has strong additive GM and additive REE. Actually, this is true for all Bell diagonal states. Let \( \rho_{\text{BD}} \) be any Bell diagonal state,

\[
\rho_{\text{BD}}(p) := \sum_{j=0}^{3} p_j |\Psi_j\rangle\langle\Psi_j|,
\]

(25)

where \( p = (p_0, p_1, p_2, p_3) \) is a probability distribution, and \( |\Psi_j\rangle \)s are the standard Bell basis. \( |\Psi_0\rangle \) and \( |\Psi_1\rangle \) are already defined in (23). The other two states are defined as

\[
|\Psi_2\rangle := \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle), \quad |\Psi_3\rangle := \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle).
\]

(26)

Since local unitary transformations can realize all 24 permutations of the four Bell states, without loss of generality, we may assume \( p_0 \geq p_1 \geq p_2 \geq p_3 \). Then \( \rho_{\text{BD}} \) is clearly a non-negative state, and its GM is strong additive according to theorem 5. Meanwhile, according
to lemma 8, its closest product state can be chosen to be non-negative. Let $|\varphi_2\rangle = (\cos \theta_1|0\rangle + \sin \theta_1|1\rangle) \otimes (\cos \theta_2|0\rangle + \sin \theta_2|1\rangle)$, with $0 < \theta_1, \theta_2 \leq \frac{\pi}{2}$.

$$\Lambda^2(\rho_{BD}(p)) = \max_{\{|\varphi_2\rangle\}} \langle \varphi_2 | \rho_{BD}(p) | \varphi_2 \rangle$$

$$= \frac{1}{2} \max_{\theta_1, \theta_2} \left[ p_0 \cos^2(\theta_1 - \theta_2) + p_1 \cos^2(\theta_1 + \theta_2) + p_2 \sin^2(\theta_1 + \theta_2) + p_3 \sin^2(\theta_1 - \theta_2) \right]$$

$$= \frac{1}{2} \max_{\theta_1, \theta_2} \left[ p_2 + p_3 + (p_0 - p_2) \cos^2(\theta_1 - \theta_2) + (p_1 - p_2) \cos^2(\theta_1 + \theta_2) \right] = \frac{p_0 + p_1}{2}. \quad (27)$$

The maximum in the above equation can be obtained at $\theta_1 = \theta_2 = 0$, that is, $|\varphi_2\rangle = |00\rangle$.

REE of the Bell diagonal states have been computed by Vedral et al [5] and by Rains [16], with the result $E_R(\rho_{BD}(p)) = 0$ if $p_0 \leq \frac{1}{2}$ and

$$E_R(\rho_{BD}(p)) = 1 + p_0 \log p_0 + (1 - p_0) \log(1 - p_0) = 1 - H(p_0, 1 - p_0), \quad (28)$$

if $p_0 \geq \frac{1}{2}$, where $H(p_0, 1 - p_0)$ is the binary Shannon entropy. Although, in general, $E_R(\rho_{BD}(p)) > G(\rho_{BD}(p)) - S(\rho_{BD}(p))$, except for rank-two Bell diagonal states, REE of Bell diagonal states is also additive. This can be shown as follows, with a suitable local unitary transformation and twirling, $\rho_{BD}(p)$ can be turned into a Werner state with the same maximal eigenvalue $p_0$ and thus with the same REE according to (28). Recall that REE of any two-qubit Werner state is additive [17, 59]. It follows from the monotonicity of AREE under LOCC that REE of any Bell diagonal state is also additive.

**Proposition 10.** The Bell diagonal state in (25) has strong additive GM and additive REE; thus $G^\infty(\rho_{BD}(p)) = G(\rho_{BD}(p)) = 1 - \log(p_0 + p_1)$ and $E_R^\infty(\rho_{BD}(p)) = E_R(\rho_{BD}(p)) = 1 - H(p_0, 1 - p_0)$.

To compute LGR of the Bell diagonal state $\rho_{BD}(p)$, let $\rho'$ be an unnormalized separable state with the minimal trace such that $\rho' \geq \rho_{BD}$; then $R_L(\rho_{BD}(p)) = \log[\text{tr}(\rho')]$. In addition, $\rho'$ can also be chosen to be a Bell diagonal state. Since a Bell diagonal state is separable if and only if its largest eigenvalue is not larger than one half of its trace, $\rho'$ can be chosen to be $\rho' = \rho_{BD}(p) + \frac{2p_0 - 1}{2} (I - |\Psi_0\rangle \langle \Psi_0|)$. We thus obtain

$$R_L(\rho_{BD}(p)) = \log(2p_0) \quad \text{for} \quad p_0 \geq \frac{1}{2}. \quad (29)$$

Next, we consider the isotropic state $\rho_{1,\lambda} := \frac{1 - \lambda}{2(d-1)} (I - |\Psi_0\rangle \langle \Psi_0|), \quad \lambda \leq 1$. It is easy to see that $\Lambda^2(\rho_{1,\lambda}) = \frac{d(d+1)}{d^2+1}$, and the state $|jj\rangle$ for each $j = 0, 1, \ldots, d - 1$ is a closest product state. Since $\rho_{1,\lambda}$ is a non-negative state, its GM is strong additive according to theorem 5. So we obtain

**Proposition 11.** The isotropic state $\rho_{1,\lambda}$ with $\frac{1}{d^2} \leq \lambda \leq 1$ has strong additive GM and thus $G^\infty(\rho_{1,\lambda}) = G(\rho_{1,\lambda}) = \log\frac{d(d+1)}{d^2+1}$.

REE and AREE of the isotropic state were calculated by Rains [16] with the result

$$E_R^\infty(\rho_{1,\lambda}) = E_R(\rho_{1,\lambda}) = \begin{cases} 0, & 0 \leq \lambda \leq \frac{1}{d}, \\ \log d + \lambda \log(1 - \lambda) \log \frac{1 - \lambda}{d - 1}, & \frac{1}{d} \leq \lambda \leq 1. \end{cases} \quad (30)$$

To compute LGR of the isotropic state $\rho_{I,\lambda}$, let $\rho'$ be an unnormalized separable state with the minimal trace such that $\rho' \geq \rho_{I,\lambda}$; then $R_L(\rho_{I,\lambda}) = \log(\text{tr}(\rho'))$. In addition, $\rho'$ can also be chosen to be an isotropic state. Since the isotropic state $\rho_{I,\lambda}$ is separable if $0 \leq \lambda \leq \frac{1}{d}$, and entangled otherwise, $\rho'$ can be chosen to be $\rho' = \rho_{I,\lambda} + \frac{d-1}{d^2} (I - |\Psi_0\rangle\langle\Psi_0|)$. We thus obtain

$$R_L(\rho_{I,\lambda}) = \begin{cases} 0, & 0 \leq \lambda \leq \frac{1}{d}, \\ \log(d\lambda), & \frac{1}{d} \leq \lambda \leq 1. \end{cases} \quad (31)$$

Now, we focus on pure three-qubit states as the most simple multipartite pure states. Recall that any pure three-qubit state can be turned into the following form via a suitable local unitary transformation [71]:

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|110\rangle + \lambda_3|111\rangle,$$

$$\lambda_j \geq 0, \quad \sum_j \lambda_j^2 = 1, \quad 0 \leq \phi \leq \pi.$$  

(32)

If $\phi = 0$, the resulting four-parameter family of states are all non-negative. In that case, according to theorems 5 and 7, their GM is strong additive and gives a lower bound to their AREE. The bound to AREE is tight for the W state, as we shall see in section 3.3.

For generic two-qubit states, the previous numerical calculation in [6] found no counterexample to the additivity of REE, while our numerical calculation found no counterexample to the additivity of GM. We thus conjecture that both REE and GM are additive for generic two-qubit states. Note that each bipartite reduced state of a pure three-qubit state is a rank-two two-qubit state. According to (12), the GM of pure three-qubit states would be additive if the GM of two-qubit states were additive.

3.3. Generalized Dicke states

Generalized Dicke states are also called symmetric basis states. They are defined in $\mathcal{H} = (\mathbb{C}^d)^\otimes N$ as [8, 44]

$$|N, \vec{k}\rangle := \frac{1}{\sqrt{C_{N,\vec{k}}}} \sum_{\{P\}} P(\underbrace{0, \ldots, 0}_{k_0}, \underbrace{1, \ldots, 1}_{k_1}, \ldots, \underbrace{d-1, \ldots, d-1}_{k_{d-1}}),$$

$$\vec{k} := (k_0, k_1, \ldots, k_{d-1}), \quad \sum_{j=0}^{d-1} k_j = N.$$  

(33)

Here, $\{P\}$ denotes the set of all distinct permutations of the spins, and $C_{N,\vec{k}} = \frac{N!}{\prod_{j=0}^{d-1} k_j!}$ is the normalization factor. When $d \geq N$, the state

$$|N, (1, \ldots, 1, 0, \ldots, 0)\rangle$$

is sometimes referred to as the totally symmetric basis state and written as $|\psi_{N+}\rangle$ [30]. When $d = 2$, $|N, (k_0, k_1)\rangle$ is called a Dicke state and denoted as $|N, k_0\rangle$. Dicke states are useful for quantum communication and quantum networking [45, 46]. Some typical Dicke states have been realized in trapped atomic ions [47]. Recently, the multiqubit Dicke state with half-excitations $|N, N/2\rangle$ has been employed to implement a scalable quantum search based on

Grover’s algorithm by using adiabatic techniques [48]. In view of the fast progress made in experiments, further theoretical study is required to explore the full potential of Dicke states.

GM, REE and LGR of the generalized Dicke states have been computed in [8, 30, 56] with the result

\[ R_L \left( |N, \tilde{k} \rangle \right) = E_R \left( |N, \tilde{k} \rangle \right) = G \left( |N, \tilde{k} \rangle \right) = - \log \left[ \frac{N!}{\prod_{j=0}^{d-1} k_j} \prod_{j=0}^{d-1} \left( \frac{k_j}{N} \right)^{k_j} \right]. \] (34)

In addition, the generalized Dicke states have been proved to satisfy conditions (1)–(4) of proposition 1 in section III B of [30]. Since the generalized Dicke states have non-negative amplitudes, theorem 5 and proposition 2 imply that

\[ R_L \left( \bigotimes_a |N, \tilde{k}_a \rangle \right) = E_R \left( \bigotimes_a |N, \tilde{k}_a \rangle \right) = G \left( \bigotimes_a |N, \tilde{k}_a \rangle \right) \]

\[ = \sum_a R_L(|N, \tilde{k}_a \rangle) = \sum_a E_R(|N, \tilde{k}_a \rangle) = \sum_a G(|N, \tilde{k}_a \rangle) \]

\[ = - \sum_a \log \left[ \frac{N!}{\prod_{j=0}^{d_a-1} k_{a,j}} \prod_{j=0}^{d_a-1} \left( \frac{k_{a,j}}{N} \right)^{k_{a,j}} \right]. \] (35)

In particular, when all states \( |N, \tilde{k}_a \rangle \) are identical, we obtain

**Proposition 12.** Generalized Dicke states have strong additive GM, additive REE and LGR; hence

\[ R^\infty_L (|N, \tilde{k} \rangle) = E^\infty_R (|N, \tilde{k} \rangle) = G^\infty (|N, \tilde{k} \rangle) = - \log \left[ \frac{N!}{\prod_{j=0}^{d-1} k_j} \prod_{j=0}^{d-1} \left( \frac{k_j}{N} \right)^{k_j} \right]. \] (36)

Let \( \tilde{\rho}_{N, \tilde{k}} \) be the \((N-1)\)-partite reduced state of the \(N\)-partite generalized Dicke state \( |N, \tilde{k} \rangle \). Since \( E_R(|N, \tilde{k} \rangle) = G(|N, \tilde{k} \rangle) \), equation (14) implies that \( E_R(\tilde{\rho}_{N, \tilde{k}}) = E_R(|N, \tilde{k} \rangle) - S(\tilde{\rho}_{N, \tilde{k}}) = E_R(|N, \tilde{k} \rangle) - H(\tilde{k}/N) \), where \( H(\tilde{k}/N) \) is the Shannon entropy. This equality has already been proved in [56] with explicit calculation. In contrast, our derivation is much simpler and more general. Finally, since REE of \( \tilde{\rho}_{N, \tilde{k}} \) is also additive, we get AREE as follows:

\[ E^\infty_R (\tilde{\rho}_{N, \tilde{k}}) = E_R(\tilde{\rho}_{N, \tilde{k}}) = - \log \left[ \frac{N!}{\prod_{j=0}^{d-1} k_j} \prod_{j=0}^{d-1} \left( \frac{k_j}{N} \right)^{k_j} \right] - H(\tilde{k}/N). \] (37)

In the case \( N = 3 \), the above result gives a lower bound to the entanglement cost of the following two states, respectively: the two-qubit state \( \frac{1}{2}(|01\rangle + |10\rangle) (|01\rangle + |10\rangle) + \frac{1}{2}|00\rangle \langle 00| \) and the two-qutrit state \( \frac{1}{6}(|01\rangle + |10\rangle)(|01\rangle + |10\rangle) + \frac{1}{6}|02\rangle \langle 02| + |20\rangle \langle 02| + |02\rangle \langle 20| + |21\rangle \langle 21| + |12\rangle \langle 21| + |21\rangle \langle 12| \).

Another application of our result is to help determine whether two multipartite pure states can be inter-converted reversibly under asymptotic LOCC and to help solve the long-standing problem about MREGS [24, 25]. Consider two tripartite states \( |\psi_1\rangle, |\psi_2\rangle \) over the three parties \( A_1, A_2 \) and \( A_3 \). According to the result of Linden et al [23], reversible transformation between the two states under asymptotic LOCC would mean that the ratio of AREE

Table 3. Bipartite (across the cut $A_1 : A_2 A_3$) and tripartite AREE of the GHZ state, W state, totally symmetric basis state $|\psi_{3+}\rangle = |3, (1,1,1)\rangle$ and antisymmetric basis state $|\psi_{3-}\rangle$, respectively. The ratio of the bipartite AREE to the tripartite AREE listed in the last column of the table decreases monotonically down the column, which implies that there is no reversible transformation between any two of the four states under asymptotic LOCC.

<table>
<thead>
<tr>
<th>States</th>
<th>$E^\infty_R(A_1 : A_2 A_3)$</th>
<th>$E^\infty_R$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>GHZ</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>W</td>
<td>$\frac{1}{4} \log^2 \frac{27}{4}$</td>
<td>$\log \frac{5}{2}$</td>
<td>0.7849</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{3+}\rangle$</td>
<td>$\log^3$</td>
<td>$\log \frac{5}{2}$</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{3-}\rangle$</td>
<td>$\log 3$</td>
<td>$\geq \log 5$</td>
</tr>
</tbody>
</table>

$E^\infty_R(A_1 : A_2 A_3)$ across the cut $A_1 : A_2 A_3$ to the tripartite AREE $E^\infty_R$ is conserved. Table 3 shows the bipartite and tripartite AREE of the GHZ state, W state, tripartite totally symmetric and antisymmetric basis states $|\psi_{3\pm}\rangle$ ($|\psi_{3-}\rangle$ is defined in (55), in section 4.1, respectively. The inequality $E^\infty_R(\psi_{3-}) \geq \log 5$ in the table follows from (13) and the result $E^\infty_R(\text{tr}_{A_1} |\psi_{3-}\rangle \langle \psi_{3-}|) \leq E^\infty_{\text{R, PPT}}(\text{tr}_{A_1} |\psi_{3-}\rangle \langle \psi_{3-}|) = \log \frac{5}{3}$ [71]. With these results, it is immediately clear that there is no reversible transformation between any two states among the four states.

A similar argument can be used to show that the transformation between the $N$-partite GHZ state and any $N$-partite symmetric basis state is not reversible. Also, the transformation between two symmetric basis states is generally not reversible if they cannot be converted into each other by a permutation of the vectors in the computational basis.

3.4. Mixture of Dicke states

Next, we consider the mixture of Dicke states

$$\rho(\{p_k\}) := \sum_k p_k |N, k\rangle \langle N, k|.$$  

REE of these states has been derived by Wei [55, 56]. We shall give a lower bound to AREE of these states based on the relation between REE and GM. Similar techniques can also be applied to the mixture of generalized Dicke states. The lower bound can often be improved if the convexity of AREE is taken into account, as we shall see shortly. For simplicity, we illustrate our method with the mixture of two Dicke states. Following [55, 56], define

$$\rho_{N; k_1, k_2}(s) := s |N, k_1\rangle \langle N, k_1| + (1-s) |N, k_2\rangle \langle N, k_2|, \quad 0 \leq s \leq 1, \quad k_1 < k_2.$$  

Since the mixture of Dicke states is both symmetrical and non-negative, corollary 5 in [70] (see also proposition 4) and lemma 8 imply that the closest product state to $\rho_{N; k_1, k_2}(s)$ can be chosen to be of the form $|\varphi_N\rangle = (\cos \theta |0\rangle + \sin \theta |1\rangle)^\otimes N$ with $0 \leq \theta \leq \frac{\pi}{2}$.

$$\Lambda^2(\rho_{N; k_1, k_2}(s)) = \max_{\theta} \langle \varphi_N |\rho_{N; k_1, k_2}(s)| \varphi_N\rangle,$$

$$= \max_{\theta} \left[ s \binom{N}{k_1} \cos^{2k_1} \theta \sin^{2N-2k_1} \theta + (1-s) \binom{N}{k_2} \cos^{2k_2} \theta \sin^{2N-2k_2} \theta \right].$$  

The maximization over $\theta$ is easy to carry out. For example, let $x = \cos^2\theta$, the extremal condition will lead to a $(k_2 - k_1 + 1)$-order polynomial equation in $x$, which can be solved straightforwardly. In particular, this equation can be solved analytically if $k_2 - k_1 \leq 3$. Since $\rho_{N;k_1,k_2}(s)$ is non-negative, according to theorems 5 and 7, $G(\rho_{N;k_1,k_2}(s))$ is strong additive, and $E(\rho_{N;k_1,k_2}(s))$ is lower bounded by $G(\rho_{N;k_1,k_2}(s)) - S(\rho_{N;k_1,k_2}(s))$.

Figure 1 illustrates $E_R$ (REE is given by theorem 1 of Wei [56]) and $G - S$ for the following three families of states:

$$\rho_{2;0,1}(s) = s |11\rangle\langle 11| + (1 - s) |\Psi_2\rangle\langle \Psi_2|,$$

$$\rho_{3;1,2}(s) = s |\tilde{W}\rangle\langle \tilde{W}| + (1 - s) |W\rangle\langle W|,$$

$$\rho_{3;0,2}(s) = s |111\rangle\langle 111| + (1 - s) |W\rangle\langle W|,$$

where $|\Psi_2\rangle = \frac{\sqrt{2}}{2} (|01\rangle + |10\rangle)$, $W = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle)$ and $|\tilde{W}\rangle = \frac{1}{\sqrt{3}} (|011\rangle + |101\rangle + |110\rangle)$. For the first family of states $\rho_{2;0,1}(s)$ (upper plot of figure 1), $G - S$ gives a very good lower bound to AREE when $0 \leq s \leq 0.4$. The bound is tight at $s = \frac{1}{2}$, since $\rho_{2;0,1}(\frac{1}{2})$ is the bipartite reduced state of the Dicke state $|\tilde{W}\rangle\langle \tilde{W}|$. Taking the convexity of AREE into account, we can raise the lower bound for $s > \frac{1}{2}$ to the one represented by the dotted line, which is the tangent line to both the curve $E_R(s)$ and the curve $G(s) - S(s)$ at $s = \frac{1}{2}$. In addition, $G - S$ is a lower bound to entanglement cost. For $\rho_{3;1,2}(s)$ (middle plot), the bound is very good in the whole parameter region. The bound is tight at $s = \frac{1}{2}$, since $\rho_{3;1,2}(\frac{1}{2})$ is the tripartite reduced state of the Dicke state $|4, 2\rangle\langle 4, 2|$. For $\rho_{3;0,2}(s)$ (lower plot), REE is obtained through convex roof construction from the dotted curve, as described in [56]. The lower bound to AREE given by $G(s) - S(s)$ does not look very good at first glance. However, taking the convexity of AREE into account, we can obtain a lower bound to AREE that is very close to REE for almost the entire family of states $\rho_{3;0,2}(s)$.
3.5. The Smolin state

The Smolin state is a four-qubit unlockable bound entangled state, from which no pure entanglement can be distilled under LOCC. However, if any two of the four parties come together, they can create a singlet between the other two parties [49]. The Smolin state can be expressed in several equivalent forms, one of which is

$$\rho_{ABCD} = \frac{1}{4} \sum_{j=0}^{3} (|\Psi_j\rangle\langle \Psi_j|)_{AB} \otimes (|\Psi_j\rangle\langle \Psi_j|)_{CD}, \quad (42)$$

where $|\Psi_j\rangle$s are the four Bell states $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. It can also be written in a more symmetric form,

$$\rho_{ABCD} = \frac{1}{16} \left( I^4 \otimes \sum_{j=1}^{3} \sigma_j^A \otimes \sigma_j^B \otimes \sigma_j^C \otimes \sigma_j^D \right), \quad (43)$$

which clearly shows that it is permutation invariant and non-negative.

Since its discovery, the Smolin state has found many applications, such as remote information concentration [50], superactivation [51] and multiparty secret sharing [52]. It can maximally violate a two-setting Bell inequality similar to the CHSH inequality [72]. It was also used to show that four orthogonal Bell states cannot be discriminated locally even probabilistically [73]. Recently, Amselem and Bourennane [74] have realized the Smolin state in experiments with polarized photons and characterized its entanglement properties. Several improved experiments were performed later by other groups [75, 76]. Hence, it is desirable to quantify the amount of entanglement in the Smolin state.

The multipartite REE of the Smolin state has been derived by Murao and Vedral [50] and Wei et al [57], with the result $E_R(\rho_{ABCD}) = 1$. The derivation in [57] relies on the following alternative representation of the Smolin state, which again shows that it is non-negative:

$$\rho_{ABCD} = \frac{1}{4} \sum_{j=0}^{3} |X_j\rangle\langle X_j|, \quad (44)$$

with

$$|X_0\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \quad |X_1\rangle = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle),$$

$$|X_2\rangle = \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle), \quad |X_3\rangle = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle).$$

They also give a closest separable state to $\rho_{ABCD}$, which reads

$$\rho_{\text{sep}} = \frac{1}{8} (|0000\rangle\langle 0000| + |1111\rangle\langle 1111| + |0011\rangle\langle 0011| + |1100\rangle\langle 1100| + |0101\rangle\langle 0101| + |1010\rangle\langle 1010| + |0110\rangle\langle 0110| + |1001\rangle\langle 1001|). \quad (45)$$

Note that $\rho_{\text{sep}} = \frac{1}{2}(\rho_{ABCD} + \rho_\perp)$, where $\rho_\perp$ is orthogonal to $\rho_{ABCD}$, hence $R_L(\rho_{ABCD}) \leq 1$ according to (4) and (5). Since $R_L(\rho_{ABCD}) \geq E_R(\rho_{ABCD}) = 1$, we obtain $R_L(\rho_{ABCD}) = 1$. 

To compute GM of the Smolin state, note that the closest product state to \( \rho_{ABCD} \) can be chosen to be non-negative, according to lemma 8. Suppose that \( |\varphi_4\rangle = \otimes_{j=1}^4 (c_j |0\rangle + s_j |1\rangle) \) is a closest product state, where \( c_j = \cos \theta_j \), \( s_j = \sin \theta_j \) with \( 0 \leq \theta_j \leq \pi/2 \) for \( j = 1, 2, 3, 4 \).

\[
\Lambda^2(\rho_{ABCD}) = (\varphi_4 | \rho_{ABCD} | \varphi_4),
\]

\[
= \frac{1}{8} \left[ (c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4)^2 + (c_1 c_2 s_3 s_4 + s_1 s_2 c_3 c_4)^2 
+ (c_1 s_2 s_3 s_4 + s_1 c_2 s_3 c_4)^2 + (c_1 s_2 c_3 s_4 + s_1 c_2 s_3 c_4)^2 \right] \leq \frac{1}{8},
\]

(46)

where the last inequality was derived in [57]. The same result can also be obtained with the approach presented in [75]. Since \( \Lambda^2(\rho_{ABCD}) \geq (0000| \rho_{ABCD} | 0000) = \frac{1}{8} \), we thus obtain \( \Lambda^2(\rho_{ABCD}) = \frac{1}{8} \) and \( G(\rho_{ABCD}) = 3 \). Note that \( S(\rho_{ABCD}) = 2 \), \( R_L(\rho_{ABCD}) = E_R(\rho_{ABCD}) = G(\rho_{ABCD}) - S(\rho_{ABCD}) \), and \( \rho_{ABCD} \) is non-negative. According to theorems 5 and 7, we have

Proposition 13. The Smolin state has strong additive GM, additive REE and LGR, and thus

\[
G^\infty(\rho_{ABCD}) = G(\rho_{ABCD}) = 3,
\]

\[
E^\infty_R(\rho_{ABCD}) = E_R(\rho_{ABCD}) = 1,
\]

(47)

\[
R^\infty_L(\rho_{ABCD}) = R_L(\rho_{ABCD}) = 1.
\]

The additivity of REE of the Smolin state can also be derived in an alternative way by first considering REE under the bipartite cut \( A: BCD \) [57]. Since every pure state in support of \( \rho_{A:BCD} \) is maximally entangled, the entanglement of formation of the state is given by \( E_F(\rho_{A:BCD}) = 1 \). On the other hand, \( E_D(\rho_{A:BCD}) \geq 1 \), where \( E_D \) denotes entanglement of distillation, because a singlet can be distilled from the Smolin state when any two of the four parties come together. From the chain of inequalities, \( E_F(\rho_{A:BCD}) \geq E_c(\rho_{A:BCD}) \geq E^\infty_R(\rho_{A:BCD}) \geq E_D(\rho_{A:BCD}) \), it follows that \( E_c(\rho_{A:BCD}) = E^\infty_R(\rho_{A:BCD}) = E_D(\rho_{A:BCD}) = 1 \). The additivity of REE of the Smolin state then follows from the following chain of inequalities:

\[
1 \leq E^\infty_R(\rho_{ABCD}) \leq E^\infty_c(\rho_{ABCD}) \leq E_R(\rho_{ABCD}) = 1.
\]

Recall that, under asymptotic non-entangling operations, state transformation can be made reversible, and AREE determines the transformation rate [26]. Hence, the Smolin state and the four-qubit GHZ state can be converted into each other reversibly under these operations.

3.6. Dür’s multipartite entangled states

Dür’s multipartite bound entangled state \( \rho_N \) was found in the search for the relation between distillability of multipartite entangled states and violation of Bell’s inequality [53].

\[
\rho_N = \frac{1}{N + 1} \left[ |\Psi_G\rangle \langle \Psi_G| + \frac{1}{2} \sum_{k=1}^N (P_k + \tilde{P}_k) \right],
\]

(48)

where \( |\Psi_G\rangle = \frac{1}{\sqrt{2}} (|0^{\otimes N}\rangle + e^{i\omega_N} |1^{\otimes N}\rangle) \) is the \( N \)-partite GHZ state, \( P_k \) is the projector onto the product state \( |u_k\rangle = |0\rangle_A_1 |0\rangle_A_2 \cdots |1\rangle_A_n \cdots |0\rangle_A_N \), and \( \tilde{P}_k \) is the projector onto the product state \( |v_k\rangle = |1\rangle_A_1 |1\rangle_A_2 \cdots |0\rangle_A_n \cdots |1\rangle_A_N \). Dür has shown that, for \( N \geq 4 \), the state in (48) is bound entangled and for \( N \geq 8 \) it violates the two-setting Mermin–Klyshko–Bell inequality [53]. Since the phase factor \( e^{i\omega_N} \) can be absorbed by redefining the computational basis, we may assume
\(e^{i\omega_n} = 1\) without loss of generality. It is then clear that \(\rho_N\) is non-negative. In the following discussion, we assume \(N \geq 4\).

Wei et al [57] have generalized Dür’s multipartite bound entangled state to the following family of states,

\[
\rho_N(x) = x|\Psi_G\rangle\langle\Psi_G| + \frac{1-x}{2N} \sum_{k=1}^{N} (P_k + \tilde{P}_k),
\]

(49)

and shown that the state is bound entangled if \(0 \leq x \leq \frac{1}{N+1}\) and free entangled if \(\frac{1}{N+1} < x \leq 1\). Moreover, they had conjectured REE of this state to be

\[
E_R[\rho_N(x)] = x \quad \text{for} \quad N \geq 4,
\]

(50)

which was later proved in [56].

We shall show that REE of \(\rho_N(x)\) is additive by first showing that REE of \(\rho_N = \rho_N(\frac{1}{N+1})\) is additive and then extending the result to the whole family of states via the convexity of AREE. Note that \(\rho_N(x)\) is a convex combination of \(\rho_N(0)\) and \(\rho_N(1)\), that is, \(\rho_N(x) = x\rho_N(1) + (1-x)\rho_N(0)\).

Since \(\rho_N(x)\) is non-negative, its closest product state can be chosen to be non-negative, according to lemma 8. Let \(|\varphi_N\rangle = \otimes_{j=1}^{N} (c_j |0\rangle + s_j |1\rangle)\) be a closest product state, where \(c_j = \cos \theta_j\), \(s_j = \sin \theta_j\), with \(0 \leq \theta_j \leq \frac{\pi}{2}\) for \(j = 1, 2, \ldots, N\).

\[
\Lambda^2(\rho_N(x)) = \max_{|\varphi_N\rangle} \langle |\varphi_N\rangle |\rho_N(x)\rangle |\varphi_N\rangle
\]

\[
= \max_{\theta_1, \ldots, \theta_N} \left\{ \frac{x}{2} (c_1 \cdots c_N + s_1 \cdots s_N)^2 + \frac{1-x}{2N} \sum_{k=1}^{N} (c_1 \cdots c_k-1 s_k c_{k+1} \cdots c_N)^2 \right\}
\]

\[
+ (s_1 \cdots s_{k-1} c_k s_{k+1} \cdots s_N)^2 \right\}.
\]

(51)

When \(x = \frac{1}{N+1}\), \(\rho_N(x) = \rho_N\), according to appendix A in [57], \(\Lambda^2(\rho_N) \leq \frac{1}{2(N+1)}\), and since the product state \(|0^{\otimes N}\rangle\) achieves this bound, it follows that \(\Lambda^2(\rho_N) \leq \frac{1}{2(N+1)}\), \(G(\rho_N) = \log(2(N+1))\). In addition, \(|u_1\rangle, |v_k\rangle\), \(\forall k\) and \(|0^{\otimes N}\rangle, |1^{\otimes N}\rangle\) are also closest product states to \(\rho_N\). When \(0 \leq x \leq \frac{1}{N+1}, \{|u_k\rangle, |v_k\rangle\}, \forall k\) are still closest product states to \(\rho_N(x)\); similarly, when \(\frac{1}{N+1} \leq x \leq 1\), \(|0^{\otimes N}\rangle, |1^{\otimes N}\rangle\) are still closest product states to \(\rho_N(x)\). So, for \(N \geq 4\), we obtain

\[
\Lambda^2(\rho_N(x)) = \begin{cases} 
\frac{1-x}{2N}, & 0 \leq x \leq \frac{1}{N+1}, \\
\frac{x}{2}, & \frac{1}{N+1} \leq x \leq 1.
\end{cases}
\]

(52)

Meanwhile, according to theorems 5 and 7, GM of Dür’s multipartite entangled states is strong additive, and \(G(\rho_N(x)) - S(\rho_N(x))\) gives a lower bound to \(E_R^{\infty}(\rho_N(x))\). The bound is tight at the following three points: \(x = 0, 1, \frac{1}{N+1}\), and \(E_R^{\infty}[\rho_N(0)] = E_R[\rho_N(0)] = 0, E_R^{\infty}[\rho_N(1)] = E_R[\rho_N(1)] = 1, E_R^{\infty}(\rho_N) = E_R(\rho_N) = \frac{1}{N+1}\) (see also figure 2). Although the bound is in general not tight, from the convexity of AREE, we can already conclude that \(E_R[\rho_N(x)] = x\).

Figure 2. REE and lower bound to AREE given by $G - S$ of Dür’s multipartite entangled state $\rho_N(x)$ with $N = 4$. The lower bound is tight only at three points $x = 0, \frac{1}{N+1}, 1$, but it is enough to prove the additivity of REE due to the convexity of AREE.

Proposition 14. GM and REE of Dür’s multipartite entangled state $\rho_N(x)$ with $N \geq 4$ are strong additive and additive, respectively, and thus

$$
G^\infty[\rho_N(x)] = G[\rho_N(x)] = \begin{cases} 
- \log \frac{1-x}{2N}, & 0 \leq x \leq \frac{1}{N+1}, \\
- \log \frac{x}{2N}, & \frac{1}{N+1} \leq x \leq 1,
\end{cases}
$$

$$
E_R^\infty[\rho_N(x)] = E_R[\rho_N(x)] = x.
$$

4. Non-additivity of GM of antisymmetric states

In this section, we turn to the antisymmetric subspace and explore the connection between the permutation symmetry and the additivity property of multipartite entanglement measures. Starting from a simple observation on the closest product states to antisymmetric states and that to symmetric states, we show that GM is non-additive for all antisymmetric states shared over three or more parties and many bipartite antisymmetric states. As a typical example of antisymmetric states, antisymmetric projector states are then treated in detail. In particular, we establish a simple equality among the three measures GM, REE and LGR of the tensor product of antisymmetric projector states, and derive analytical formulae of the three measures in the case of one copy and two copies. Along the way, a unified explanation of the non-additivity of the three measures GM, REE and LGR of the antisymmetric projector states is also obtained. Our results may be found useful in the study of fermion systems, which are described by antisymmetric wave functions due to the super-selection rule.

In section 4.1, we introduce the Slater determinant states, which are analogues of product states in the antisymmetric subspace, and we give a simple criterion on when an antisymmetric
state is a Slater determinant state. Then we prove that the $N$ one-particle reduced states of each closest product state to any $N$-partite antisymmetric state are mutually orthogonal. A lower bound to the three measures GM, REE and LGR is also derived based on this observation. In section 4.2, we show that GM of antisymmetric states shared over three or more parties is non-additive. In section 4.3, we establish a simple equality among the three measures GM, REE and LGR of the tensor product of antisymmetric projector states, and compute the three measures in the case of one copy and two copies, respectively. REE and LGR of the mixture of Slater determinant states are also derived. In section 4.4, we treat generalized antisymmetric states \cite{54} as further counterexamples to the additivity of GM.

4.1. GM of antisymmetric states

We shall be concerned with antisymmetric states in the multipartite Hilbert space $\mathcal{H} = \bigotimes_{j=1}^{N} \mathcal{H}_j$ with $\text{Dim} \mathcal{H}_j = d$ and $N \leq d$. A pure state $|\psi_N\rangle$ is antisymmetric if every odd permutation of the parties induces a sign change. All unnormalized pure antisymmetric states form the antisymmetric subspace $\mathcal{H}_-$, whose dimension is $\binom{d}{N} = d!/[N!(d-N)!]$. An $N$-partite state $\rho_N$ is antisymmetric if its support is contained in the antisymmetric subspace. Let $P_{d,N}$ be the projector onto the antisymmetric subspace $\mathcal{H}_-$; then $\text{tr}(P_{d,N}) = \text{Dim} \mathcal{H}_- = \binom{d}{N}$. Any $N$-partite state $\rho_N$ is an antisymmetric state if and only if the equality $\rho_N = P_{d,N}\rho_N P_{d,N}$ holds.

A typical example of antisymmetric states is the antisymmetric projector state, $\rho_{d,N} = \frac{P_{d,N}}{\text{tr}(P_{d,N})}$, which includes the antisymmetric basis state and antisymmetric Werner state as special cases.

Given $N$ orthonormal single-particle states, $|a_1\rangle, \ldots, |a_N\rangle$, a Slater determinant state can be constructed by anti-symmetrization, a procedure routinely used in the study of fermion systems, i.e.,

$$|a_1\rangle \wedge \cdots \wedge |a_N\rangle := \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} \text{sgn}(\sigma) |a_{\sigma(1)}\rangle \wedge \cdots \wedge |a_{\sigma(N)}\rangle,$$  \hspace{1cm} (54)

where $S_N$ is the symmetry group of $N$ letters, $\text{sgn} (\sigma)$ is the signature of $\sigma$ \cite{77}, and $\frac{1}{\sqrt{N!}}$ is the normalization factor. Apparently, all Slater determinant states are locally unitarily equivalent to each other. In particular, they are locally unitarily equivalent to antisymmetric basis states, $|j_1\rangle \wedge \cdots \wedge |j_N\rangle$, with $0 \leq j_1 < \cdots < j_N \leq d - 1$, which form an orthonormal basis in the antisymmetric subspace. When $d = N$, there is only one antisymmetric basis state, $|\psi_{N-}\rangle := |0\rangle \wedge |1\rangle \wedge \cdots \wedge |N-1\rangle.$ \hspace{1cm} (55)

For convenience of the following discussion, we summarize a few useful properties of the Slater determinant states; see \cite{78} for some mathematical background. If the $N$ single-particle states $|a_1\rangle, \ldots, |a_N\rangle$ are linearly dependent, then $|a_1\rangle \wedge \cdots \wedge |a_N\rangle$ vanishes. If they are linearly independent but not mutually orthogonal, $|a_1\rangle \wedge \cdots \wedge |a_N\rangle$ is a subnormalized Slater determinant state. In addition, we can always choose $N$ orthonormal states $|a'_1\rangle, \ldots, |a'_N\rangle$ from the span of $|a_1\rangle, \ldots, |a_N\rangle$, such that $|a_1\rangle \wedge \cdots \wedge |a_N\rangle = c|a'_1\rangle \wedge \cdots \wedge |a'_N\rangle$, where $c$ is a constant with modulus between 0 and 1. The projection of a generic pure product state onto the antisymmetric subspace is a subnormalized Slater determinant state, that is,

$$P_{d,N}(|a_1\rangle \otimes \cdots \otimes |a_N\rangle) = \frac{1}{\sqrt{N!}} |a'_1\rangle \wedge \cdots \wedge |a'_N\rangle.$$ \hspace{1cm} (56)

Suppose that $|b_1\rangle, \ldots, |b_N\rangle$ are another $N$ normalized single-particle states. Then $|a_1\rangle \wedge \cdots \wedge |a_N\rangle$ and $|b_1\rangle \wedge \cdots \wedge |b_N\rangle$ are linearly independent if and only if the subspaces spanned by
The one-particle reduced states of each closest product state to any antisymmetric state is in a sense the analogue of proposition for antisymmetric states. It is a basic task to determine whether it is a Slater determinant state. Note that the one-particle reduced state of any Slater determinant state is a subnormalized projector with rank $N$. On the other hand, if the one-particle reduced state of an antisymmetric state is of rank $N$, then there is only one linearly independent Slater determinant state that can be constructed from the one-particle states in support of this one-particle reduced state. Obviously, the rank of the one-particle reduced state cannot be less than $N$; otherwise, no Slater determinant state can be constructed. So we obtain

**Proposition 15.** The one-particle reduced state of any $N$-partite antisymmetric state is of rank not less than $N$. Moreover, an antisymmetric state is a Slater determinant state if and only if its one-particle reduced state is of rank $N$.

We are now ready to study GM of antisymmetric states. Let $\rho_N$ be an $N$-partite antisymmetric state, that is, $P_{d,N}\rho_N P_{d,N} = \rho_N$. Let $\varphi_N = |a_1\rangle \otimes \cdots \otimes |a_N\rangle$,

$$\Lambda^2(\rho_N) = \max_{|\varphi_N\rangle} \langle \varphi_N | \rho_N | \varphi_N \rangle = \max_{|\varphi_N\rangle} \langle \varphi_N | P_{d,N} \rho_N P_{d,N} | \varphi_N \rangle,$$

$$= \frac{1}{N!} \max_{|a_1\rangle, \ldots, |a_N\rangle} \langle a_1 | \wedge \cdots \wedge |a_N\rangle \rho_N (|a_1\rangle \wedge \cdots \wedge |a_N\rangle). \tag{57}$$

Recall that $|a_1\rangle \wedge \cdots \wedge |a_N\rangle$ is in general a subnormalized Slater determinant state, and that it is normalized if and only if the $N$ single-particle states $|a_1\rangle, \ldots, |a_N\rangle$ are orthonormal, which is also a necessary condition for $|\varphi_N\rangle$ to be a closest product state.

**Proposition 16.** The $N$ one-particle reduced states of each closest product state to any $N$-partite antisymmetric state are mutually orthogonal.

Thus the search for the closest product state of $\rho_N$ is equivalent to the search for its closest Slater determinant state. A peculiar feature of an antisymmetric state $\rho_N$ is the high degeneracy of its closest product states. If $|a_1\rangle \otimes \cdots \otimes |a_N\rangle$ is a closest product state, then the tensor product of any $N$ orthonormal states from the span of the $N$ single-particle states $|a_1\rangle, \ldots, |a_N\rangle$ is also a closest product state. Recall that there is a one-to-one correspondence between Slater determinant states and subspaces of dimension $N$ of the single-particle Hilbert space.

Proposition 16 is in a sense the analogue of proposition 4 for antisymmetric states. It is crucial to computing GM of antisymmetric states and to proving the non-additivity of GM of antisymmetric states shared over three or more parties in section 4.2.

Suppose that $\lambda_{\max}$ is the largest eigenvalue of $\rho_N$, then $\Lambda^2(\rho_N) \leq \frac{\lambda_{\max}}{N!}$ according to (57), and the inequality is saturated if and only if there is a Slater determinant state in the eigenspace corresponding to $\lambda_{\max}$. So we obtain

$$R_L(\rho_N) \geq E_R(\rho_N) \geq G(\rho_N) - S(\rho_N) \geq -\log \frac{\lambda_{\max}}{N} - S(\rho_N). \tag{58}$$

For antisymmetric projector states, all the inequalities are saturated, as we shall see in section 4.3.
4.2. Non-additivity theorem for GM of antisymmetric states

The permutation symmetry of multipartite antisymmetric states plays a crucial role in determining the properties of their closest product states, as demonstrated in propositions 4 and 16. Moreover, it is also the origin of the non-additivity of GM of antisymmetric states\(^6\).

**Theorem 17.** When \( N \geq 3 \), GM is non-additive for any two \( N \)-partite antisymmetric states \( \rho_N \) and \( \rho_N' \), that is, \( G(\rho_N \otimes \rho_N') < G(\rho_N) + G(\rho_N') \).

**Proof.** Suppose that there exists a closest product state of \( \rho_N \otimes \rho_N' \) that is of the tensor-product form \( |\varphi_N\rangle \otimes |\varphi_N'\rangle \); then \( |\varphi_N\rangle \) and \( |\varphi_N'\rangle \) are closest product states of \( \rho_N \) and \( \rho_N' \), respectively. Since the set of one-particle reduced states of \( |\varphi_N\rangle \) \((|\varphi_N'\rangle\) are mutually orthogonal according to proposition 16, \( |\varphi_N\rangle \otimes |\varphi_N'\rangle \) cannot be symmetric. On the other hand, \( \rho_N \otimes \rho_N' \) is a symmetric state and, if \( N \geq 3 \), its closest product states are necessarily symmetric according to proposition 4, hence a contradiction would arise. In other words, no closest product state of \( \rho_N \otimes \rho_N' \) can be written as a tensor product of the closest product states of \( \rho_N \) and \( \rho_N' \), respectively, which implies that \( G(\rho_N \otimes \rho_N') < G(\rho_N) + G(\rho_N') \).

The non-additivity of GM of antisymmetric states can be understood as follows. Antisymmetric states are generally more entangled than symmetric states, as noted in [30]. However, two copies of antisymmetric states turn out to be a symmetric state. Theorem 17 establishes a simple connection between permutation symmetry and the additivity property of GM of multipartite states. In some special cases, this connection also carries over to other multipartite entanglement measures, such as REE and LGR, as we shall see in section 4.2.

For a pure tripartite antisymmetric state, the non-additivity of GM translates immediately to the non-multiplicativity of the maximum output purity \( \nu_\infty \) of the corresponding quantum channel constructed according to the Werner–Holevo recipe. For example, the non-multiplicativity of the Werner–Holevo channel is equivalent to the non-additivity of the GM of the tripartite antisymmetric basis state [32].

Theorem 17 can be generalized to cover the situation where the two states are not fully antisymmetric.

**Corollary 18.** GM is non-additive for two \( N \)-partite states if there exists a subsystem of three parties such that the respective tripartite reduced states of the two \( N \)-partite states are both antisymmetric.

**Proof.** Assume that \( N > 3 \); suppose that \( \sigma_N, \sigma_N' \) are two arbitrary \( N \)-partite states and \( \sigma_N^{A_1, A_2, A_3}, \sigma_N'^{A_1, A_2, A_3} \) their respective tripartite reduced states, which are antisymmetric. Let \( |a_1\rangle \otimes \cdots \otimes |a_N\rangle \) and \( |a'_1\rangle \otimes \cdots \otimes |a'_N\rangle \) be the closest product states to \( \sigma_N \) and \( \sigma_N' \) respectively; then \( (|a_1\rangle \otimes \cdots \otimes |a_N\rangle)\sigma_N(|a_1\rangle \otimes \cdots \otimes |a_N\rangle) \) and \( (|a'_1\rangle \otimes \cdots \otimes |a'_N\rangle)\sigma_N'(|a'_1\rangle \otimes \cdots \otimes |a'_N\rangle) \) are both antisymmetric. Applying theorem 17 to the two subnormalized antisymmetric states, we obtain \( G(\sigma_N \otimes \sigma_N') < G(\sigma_N) + G(\sigma_N') \).

In the bipartite scenario, if either \( \rho_2 \) or \( \rho_2' \) is pure, then \( G(\rho_2 \otimes \rho_2') = G(\rho_2) + G(\rho_2') \), since the GM of bipartite pure states is strong additive, as shown in section 3.2. On the other hand,
any closest product state to $\rho_2 \otimes \rho'_2$ cannot be of tensor-product form if it is symmetric and vice versa, according to the same reasoning as that in the proof of theorem 17. The additivity of $\Lambda^2$ of $\rho_2$ and that of $\rho'_2$ is related to the existence of closest product states of $\rho_2 \otimes \rho'_2$ which are not symmetric. This in turn is due to the degeneracy of the Schmidt coefficients of $\rho_2$ or $\rho'_2$ [77]. Indeed, every Schmidt coefficient of a bipartite pure antisymmetric state is at least doubly degenerate [79].

For generic bipartite antisymmetric states, we suspect that the non-additivity of $\Lambda^2$ is a rule rather than an exception, which is supported by the following observation. If both $\rho_2$ and $\rho'_2$ admit purifications that are antisymmetric, then their GM is non-additive, due to theorem 17 and (12).

Theorem 17 can also be derived in a slightly different way, which offers a new perspective. According to corollary 5 in [70] (see also proposition 4), the closest product state to $\rho_N \otimes \rho'_N$ can be chosen to be symmetric. Let

$$|\varphi_N\rangle = |a_V\rangle^{\otimes N}, \quad |a_V\rangle = \frac{1}{\sqrt{d}} \sum_{j,k=0}^{d-1} V_{jk}|j\rangle\langle k|,$$

$$V = \sum_{j,k=0}^{d-1} V_{jk}|j\rangle\langle k|, \quad \text{with } \text{tr}(VV^\dagger) = d. \quad (59)$$

According to (A.2) in the appendix,

$$\Lambda^2(\rho_N \otimes \rho'_N) = \max_{|\varphi_N\rangle} \langle \varphi_N|\rho_N \otimes \rho'_N|\varphi_N\rangle = \max_{u \in V^\perp = d} \frac{1}{d^N} \text{tr}(\rho_N^{1/2}V^{\otimes N}\rho'_N V^{\perp \otimes N} \rho_N^{1/2}) \quad (60)$$

$$= \max_{u \in V^\perp = d} \frac{1}{d^N} \text{tr}(\rho_N^{1/2}V^{\wedge N}\rho'_N V^{\perp \wedge N} \rho_N^{1/2}), \quad (61)$$

where $V^{\wedge N} = P_{d,N}V^{\otimes N}P_{d,N}$ is the restriction of $V^{\otimes N}$ onto the antisymmetric subspace, which does not vanish if and only if the rank of $V$ is no less than $N$. Since the rank of $V$ is exactly the Schmidt rank of $|a_V\rangle$, the Schmidt rank of $|a_V\rangle$ must be not less than $N$, if $|a_V\rangle^{\otimes N}$ is a closest product state. Recall that the closest product state to $\rho_N \otimes \rho'_N$ is necessarily symmetric if $N \geq 3$, according to proposition 4. It follows that each closest product state to $\rho_N \otimes \rho'_N$ must be entangled across the cut $A_1, \ldots, A_N$: $A_1^1, \ldots, A_N^2$, which implies that $G(\rho_N \otimes \rho'_N) < G(\rho_N) + G(\rho'_N)$.

In addition to providing an alternative proof of theorem 17, the second approach also enables us to compute GM of the antisymmetric projector states in section 4.3, and to derive a universal upper bound to GM of two copies of arbitrary multipartite states in section 5.2.

4.3. Antisymmetric projector states

In this section, we focus on the antisymmetric projector states, which are typical examples of antisymmetric states, and include antisymmetric basis states and antisymmetric Werner states as special cases. In particular, we establish a simple equality among the three measures GM, REE and LGR of the tensor product of antisymmetric projector states, and compute the three measures in the case of one copy and two copies, respectively. Our study provides a unified explanation of the non-additivity of the three measures of the antisymmetric projector states.
The antisymmetric projector $P_{d,N}$ is invariant under the action of the unitary group $U(d)$ with the representation $U \mapsto U^\otimes N$ for $U \in U(d)$. The range of $P_{d,N}$ is an irreducible representation with multiplicity one [30]. In other words, it satisfies conditions (1), (5) of proposition 1. Moreover, the tensor product of the antisymmetric projector states $\bigotimes_{j=1}^n \rho_{d_j,N}$ satisfies the conditions of proposition 3. So we obtain

**Proposition 19.** GM, REE and LGR of antisymmetric projector states satisfy the following equalities:

$$R_L\left(\bigotimes_{j=1}^n \rho_{d_j,N}\right) = E_R\left(\bigotimes_{j=1}^n \rho_{d_j,N}\right) = G\left(\bigotimes_{j=1}^n \rho_{d_j,N}\right) - \sum_{j=1}^n \log \operatorname{tr} P_{d_j,N},$$

where

$$R_L^\infty(\rho_{d,N}) = E_R^\infty(\rho_{d,N}) = G^\infty(\rho_{d,N}) - \log \operatorname{tr} P_{d,N}. \tag{62}$$

Combining the above result with that for symmetric basis states presented in section 3.3, we obtain

**Proposition 20.** The three measures GM, REE and LGR are equal for the tensor product of any number of symmetric basis states and antisymmetric basis states; so are AGM, AREE and ALGR.

For the single-copy antisymmetric projector state $\rho_{d,N}$, all eigenvalues are equal to $1/\operatorname{tr}(P_{d,N})$, and the eigenspace corresponding to the largest eigenvalue of $\rho_{d,N}$ is exactly the antisymmetric subspace. Hence, all the inequalities in (58) are saturated, which implies that

$$R_L(\rho_{d,N}) = E_R(\rho_{d,N}) = G(\rho_{d,N}) - \log[\operatorname{tr}(P_{d,N})] = \log(N!). \tag{63}$$

Interestingly, REE and LGR of the antisymmetric projector state $\rho_{d,N}$ do not depend on the dimension of the single-particle Hilbert space. When $d = N$, the antisymmetric projector state turns out to be an antisymmetric basis state. The result for GM reduces to that found in [30, 54], and the result for REE and LGR reduces to that found in [30, 55, 56].

When $N = 2$, there is a lower bound to AREE of $\rho_{d,N}$ found by Christandl et al [80], which reads $E_R^\infty(\rho_{d,2}) \geq \log \sqrt{\frac{d}{3}}$, from which we can get a lower bound to ALGR, $R_L^\infty(\rho_{d,2}) \geq \log \sqrt{\frac{d}{3}}$. In general, none of the three measures is easy to compute for the tensor product of antisymmetric projector states.

We now focus on two copies of antisymmetric projector states. Let $|\varphi_N\rangle$ be as defined in (59), according to (A.3) in the appendix,

$$\langle \varphi_N | P_{d,N}^{\otimes 2} | \varphi_N \rangle = \frac{1}{d^N} \operatorname{tr}(P_{d,N} V^\otimes N P_{d,N} V^\dagger \otimes N) = \frac{1}{d^N} \operatorname{tr}(P_{d,N} (V V^\dagger)^\otimes N P_{d,N}), \tag{64}$$

where, in deriving the last equality, we have used the fact that $V^\otimes N$ and $P_{d,N}$ commute due to the Weyl reciprocity (see also [78]). The trace in (64) is exactly the $N$th symmetric polynomial of the set of eigenvalues $\mu_0, \ldots, \mu_{d-1}$ of $VV^\dagger$ [78] ($\mu_j \geq 0, \sum_{j=0}^{d-1} \mu_j = \operatorname{tr}(V V^\dagger) = d$), that is,

$$\langle \varphi_N | P_{d,N}^{\otimes 2} | \varphi_N \rangle = \frac{1}{d^N} \sum_{0 \leq j_1 < \cdot \cdot \cdot < j_N \leq d-1} \mu_{j_1} \cdot \cdot \cdot \mu_{j_N}. \tag{65}$$

Recall that elementary symmetric polynomials are Schur concave functions [81], so the maximum in (65) is obtained if and only if $\mu_0 = \mu_1 = \cdot \cdot \cdot = \mu_{d-1} = 1$, that is, $V$ is unitary,
or equivalently, $|a_V\rangle$ is maximally entangled. So we obtain

$$
\max_{|\psi_N\rangle} \langle \psi_N | P_{d,N} \otimes | \psi_N \rangle = \frac{1}{d^N} \binom{d}{N} = \frac{d!}{d^N N! (d-N)!}.
$$

(66)

In conjunction with (62), (63), corollary 5 in [70] and proposition 4, we get

**Proposition 21.** GM, REE and LGR of one copy and two copies of the antisymmetric projector states are, respectively, given by

$$
R_L(\rho_{d,N}) = E_R(\rho_{d,N}) = G(\rho_{d,N}) - \log \left[ \text{tr}(P_{d,N}) \right] = \log(N!).
$$

$$
R_L(\rho_{d,N}^{\otimes 2}) = E_R(\rho_{d,N}^{\otimes 2}) = G(\rho_{d,N}^{\otimes 2}) - 2 \log \left[ \text{tr}(P_{d,N}) \right] = \log \frac{d^N N! (d-N)!}{d!}.
$$

(67)

For $\rho_{d,N}$, a state is a closest product state if and only if it is a tensor product of orthonormal single-particle states. For $\rho_{d,N}^{\otimes 2}$, any tensor product of identical maximally entangled states across the cut $A_1^j: A_2^j$ for $j = 1, \ldots, N$, respectively, is a closest product state, and each closest product state must be of this form if $N \geq 3$.

GM, REE and LGR of $\rho_{d,N}$ are all non-additive if $d \geq 3$ and $2 \leq N \leq d$. Moreover, REE (LGR) of $\rho_{d,N}^{\otimes 2}$ is almost equal to REE (LGR) of $\rho_{d,N}$ if $d \gg 1$.

When $d = N$, $\rho_{d,N} = P_{d,N} = |\psi_{N-}\rangle \langle \psi_{N-}|$, equation (67) reduces to

$$
R_L(|\psi_{N-}\rangle^{\otimes 2}) = E_R(|\psi_{N-}\rangle^{\otimes 2}) = G(|\psi_{N-}\rangle^{\otimes 2}) = N \log N.
$$

(68)

Compared with (63), GM, REE and LGR of the antisymmetric basis state are all non-additive if $N \geq 3$. Moreover, GM, REE and LGR of $|\psi_{N-}\rangle^{\otimes 2}$ are almost equal to that of $|\psi_{N-}\rangle$ if $N \gg 1$.

Recall that GM, REE and LGR are all equal to $\log N!$ for the antisymmetric basis state $|\psi_{N-}\rangle$ according to (63), and they are all equal to $N \log N - \log N!$ for the symmetric basis state $|\psi_{N+}\rangle$ according to (34). Since $|\psi_{N+}\rangle$ is non-negative, theorem 5 and proposition 2 imply that

$$
R_L(|\psi_{N+}\rangle \otimes |\psi_{N-}\rangle) = E_R(|\psi_{N+}\rangle \otimes |\psi_{N-}\rangle) = G(|\psi_{N+}\rangle \otimes |\psi_{N-}\rangle) = G(|\psi_{N+}\rangle) + G(|\psi_{N-}\rangle) = N \log N.
$$

(69)

Surprisingly, GM, REE and LGR are all equal to $N \log N$ for both $|\psi_{N-}\rangle^{\otimes 2}$ and $|\psi_{N+}\rangle \otimes |\psi_{N-}\rangle$. However, it is not known whether this is just a coincidence or there is a deep reason.

When $N = 2$, $\rho_{d,2}$ is an antisymmetric Werner state and (66) reduces to

$$
R_L(\rho_{d,2}^{\otimes 2}) = E_R(\rho_{d,2}^{\otimes 2}) = G(\rho_{d,2}^{\otimes 2}) - 2 \log \frac{d(d-1)}{2} = \log \frac{2d}{d-1}.
$$

(70)

REE of two copies of antisymmetric Werner states was derived by Vollbrecht and Werner [18], who discovered the Werner state as the first counterexample to the additivity of REE. In the case of the two-qutrit antisymmetric Werner state, the non-additivity of the three measures is in contrast with the additivity of entanglement of formation [82].

Equation (67) can also be generalized to the tensor product of two antisymmetric projector states whose respective single-particle Hilbert spaces have different dimensions, say $d_1, d_2$. 

respectively. Suppose that $N \leq d_1 \leq d_2$, with a similar reasoning that leads to (67), one can show that

$$R_L(\rho_{d_1, N} \otimes \rho_{d_2, N}) = E_R(\rho_{d_1, N} \otimes \rho_{d_2, N}) = G(\rho_{d_1, N} \otimes \rho_{d_2, N}) - \log[\text{tr}(P_{d_1, N})] - \log[\text{tr}(P_{d_2, N})] = \log \frac{d^N N!(d_1 - N)!}{d_1!}. \quad (71)$$

Interestingly, REE and LGR of $\rho_{d_1, N} \otimes \rho_{d_2, N}$ are independent of $d_2$, as long as $N \leq d_1 \leq d_2$.

The antisymmetric projector state can be seen as a uniform mixture of Slater determinant states. The above results on REE and LGR can also be generalized to an arbitrary mixture of Slater determinant states. Let $\rho_N = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, where $|\psi_j\rangle$s are $N$-partite Slater determinant states, and $\{p_j\}$ is a probability distribution. Due to the convexity of REE and LGR,

$$E_R(\rho_N^{\otimes n}) \leq E_R(|\psi_{N-1}\rangle^{\otimes n}), \quad R_L(\rho_N^{\otimes n}) \leq R_L(|\psi_{N-1}\rangle^{\otimes n}). \quad (72)$$

On the other hand, since $\rho_N$ can be turned into the antisymmetric projector state by twirling,

$$E_R(\rho_N^{\otimes n}) \geq E_R(\rho_{d, N}^{\otimes n}), \quad R_L(\rho_N^{\otimes n}) \geq R_L(\rho_{d, N}^{\otimes n}). \quad (73)$$

Combining (72), (73) and proposition 21, we obtain

**Proposition 22.** REE and LGR of any mixture $\rho_N$ of $N$-partite Slater determinant states satisfy the following equations:

$$E_R(\rho_N) = R_L(\rho_N) = \log N!, \quad \log \frac{d^N N!(d - N)!}{d_1!} \leq E_R(\rho_N^{\otimes 2}) \leq R_L(\rho_N^{\otimes 2}) \leq N \log N. \quad (74)$$

If $N \geq 3$, GM, REE and LGR of any mixture of Slater determinant states are all non-additive.

### 4.4. Generalized antisymmetric states

In all counterexamples to the additivity of GM considered so far, the dimension of the single-particle Hilbert space is at least three. However, this constraint is not necessary. We shall demonstrate this point with the example of generalized antisymmetric states.

Let $d$, $p$ and $k$ be three integers satisfying $k \leq d^p$ and $\phi$ a function from $1, 2, \ldots, d^p$ to $p$-tuples,

$$\phi(1) = (0, 0, \ldots, 0, 0),$$

$$\phi(2) = (0, 0, \ldots, 0, 1),$$

$$\vdots$$

$$\phi(d^p - 1) = (d - 1, d - 1, \ldots, d - 1, d - 2),$$

$$\phi(d^p) = (d - 1, d - 1, \ldots, d - 1, d - 1).$$

For each triple $d$, $p$ and $k$, define an $N$-partite state with $N = kp$ as follows:

$$|\psi_{d, p, k}\rangle := \frac{1}{\sqrt{k!}} \sum_{\sigma} \text{sgn}(\sigma) |\phi(\sigma(1)), \ldots, \phi(\sigma(k))\rangle. \quad (76)$$

\(|\psi_{d,p,k}\rangle\) can be seen as a \(k\)-partite antisymmetric basis state with single-particle Hilbert space of dimension \(d^p\), if we divide the \(kp\) parties into \(k\) blocks each with \(p\) parties, and view each block as a single party. The state \(|\psi_{d,p,d^p}\rangle\) is exactly the generalized antisymmetric state introduced by Bravyi [54].

By definition, \(\Lambda^2(|\psi_{d,p,k}\rangle) \leq \Lambda^2(|\psi_{k^-}\rangle) = \frac{1}{k!}\), and since \(|\langle \psi_{d,p,k}| \phi(1), \ldots, \phi(k)\rangle|^2 = \frac{1}{kp}\), we have

\[
G(|\psi_{d,p,k}\rangle) = G(|\psi_{k^-}\rangle) = \log(k!).
\] (77)

When \(k = d^p\), this result reduces to that found by Bravyi [54].

To compute REE and LGR of \(|\psi_{d,p,k}\rangle\), note that the separable state

\[
\rho_{\text{sep}} = \frac{1}{k!} \sum_{\sigma \in S_k} |\phi(\sigma(1)), \ldots, \phi(\sigma(k))\rangle \langle \phi(\sigma(1)), \ldots, \phi(\sigma(k))|
\] (78)

can be written in the form, \(\rho_{\text{sep}} = \frac{1}{kp} |\psi_{d,p,k}\rangle \langle \psi_{d,p,k}| + \rho_{\perp}\), where \(\rho_{\perp}\) (not normalized) is orthogonal to \(|\psi_{d,p,k}\rangle \langle \psi_{d,p,k}|\); hence, \(R_L(|\psi_{d,p,k}\rangle) \leq \log(k!)\) according to (4) and (5). On the other hand, \(R_L(|\psi_{d,p,k}\rangle) \geq E_K(|\psi_{d,p,k}\rangle) \geq G(|\psi_{d,p,k}\rangle) = \log(k!),\) so we obtain

\[
R_L(|\psi_{d,p,k}\rangle) = E_K(|\psi_{d,p,k}\rangle) = \log(k!).
\] (79)

When \(k = d^p\), this result reduces to that found by Wei et al [55].

Now consider two copies of generalized antisymmetric states. For the same reason as in the case of single copy, \(\Lambda^2(|\psi_{d,p,d^p}\rangle^{\otimes 2}) \leq \Lambda^2(|\psi_{d^p,d^p}\rangle^{\otimes 2}) = (d^p)^{-d^p}\). Suppose that \(|a_j\rangle\) is a \(d \otimes d\) maximally entangled state across the cut \(A^1_j : A^2_j\), for \(j = 1, \ldots, p\), then \(\otimes_{j=1}^p |a_j\rangle\) is a maximally entangled state across the cut \(A^1_1 \ldots A^1_p : A^2_1 \ldots A^2_p\). According to proposition 21 in section 4.3, we obtain

\[
\Lambda^2(|\psi_{d,p,d^p}\rangle^{\otimes 2}) = (d^p)^{-d^p}, \quad G(|\psi_{d,p,d^p}\rangle^{\otimes 2}) = d^p \log(d^p).
\] (80)

GM of generalized antisymmetric states is also non-additive when \(d^p > 2\). The eight-qubit state \(|\psi_{2,4}\rangle\) is one such example. It will be interesting to know whether there exists a multiqubit state with fewer number of parties whose GM is non-additive.

When \(k < d^p\), it is not so easy to compute GM of \(|\psi_{d,p,k}\rangle^{\otimes 2}\). Nevertheless, a good upper bound is enough to reveal the non-additivity of GM in many cases. For example, let \(|\psi\rangle = \sum_{j=0}^{d^p-1} |jj\rangle^{\otimes p}\), then \(\Lambda^2(|\psi_{d,p,k}\rangle^{\otimes 2}) \geq |\langle \psi_{d,p,k}|\psi\rangle|^2 = \frac{1}{kp}\), or equivalently, \(G(|\psi_{d,p,k}\rangle^{\otimes 2}) \leq kp \log d\). In conjunction with (77), we can discover many multiqubit states whose GM is non-additive, such as \(|\psi_{2,3,6}\rangle, |\psi_{2,3,7}\rangle\) and \(|\psi_{2,3,8}\rangle\).

5. Non-additivity of GM of generic multipartite states

Many examples of GM that were presented in previous sections invite the following question: what is the typical behavior concerning the additivity property of GM of multipartite states: additive or non-additive? In this section, we show that, if the number of parties is sufficiently large and the dimensions of the local Hilbert spaces are comparable, then GM is non-additive for almost all multipartite states. What is more surprising, GM for one copy and two copies, respectively, of these states is almost equal. This conclusion follows from the following two observations, which are of independent interest: firstly, almost all multipartite pure states are nearly maximally entangled with respect to GM and REE; secondly, there is a nontrivial universal upper bound for GM of multipartite states with tensor-product form. Our
results have a significant impact on universal one-way quantum computation, as well as on asymptotic state transformation under LOCC.

5.1. Universal upper bound for GM of multipartite states with tensor-product form

In this section, we derive a universal upper bound for GM of the tensor product of two multipartite states and discuss its implications.

**Proposition 23.** Suppose that $\rho_N$ and $\rho'_N$ are two $N$-partite states on the Hilbert space $\bigotimes_{j=1}^{N} \mathcal{H}_j$ with $\dim \mathcal{H}_j = d_j$; define $d_T = \prod_{j=1}^{N} d_j$; then $G(\rho_N \otimes \rho'_N) \leq \log d_T - \log \text{tr}(\rho_N \rho'_N)$. In particular, $G^\infty(\rho_N) \leq \frac{1}{2} G(\rho_N^{\otimes 2}) \leq \frac{1}{2} \log d_T - \frac{1}{2} \log \text{tr}(\rho_N^2)$.

**Proof.** Let $|\varphi_N\rangle = \bigotimes_{j=1}^{N} |\phi^+_j\rangle$, where $|\phi^+_j\rangle = \frac{1}{\sqrt{d_j}} \sum_{k=0}^{d_j-1} |k\rangle$ is a maximally entangled state (also a pure isotropic state) across the two copies of the $j$th party. According to (A.2) in the appendix, $\langle \varphi_N | \rho_N \otimes \rho'_N | \varphi_N \rangle = \text{tr}(\rho_N \rho'_N)/d_T$; hence, $\Lambda^2(\rho_N \otimes \rho'_N) \geq \text{tr}(\rho_N \rho'_N)/d_T$; that is, $G(\rho_N \otimes \rho'_N) \leq \log d_T - \log \text{tr}(\rho_N \rho'_N)$. \hfill $\Box$

If GM of $\rho_N$ is additive, proposition 23 implies that $G(\rho_N) \leq \frac{1}{2} \log d_T - \frac{1}{2} \log \text{tr}(\rho_N^2)$. In other words, GM cannot be additive if the states are too entangled with respect to GM. This intuition will be made more rigorous in theorem 24.

If $d_j = d$, $\forall j$ and $d \geq N$, the universal upper bound to $G(\rho_N^{\otimes 2})$ given in proposition 23 is saturated for the antisymmetric projector states (see section 4.3). If $d_j = d$, $\forall j$ and $N$ is even, there is a simple scheme to construct a pure state whose GM saturates the upper bound: divide the parties into $N/2$ pairs and choose a maximally entangled state for each pair of parties, then the tensor product of the $N/2$ maximally entangled states is such a candidate. Moreover, GM of the state so constructed is additive, so are REE and LGR. A more attractive example that saturates the upper bound is the cluster state with even number of qubits, whose GM, REE and LGR are all equal to $N/2$ and are additive [31]. Hence, proposition 23 implies that, in any multipartite Hilbert space with even number of parties and equal local dimension, any pure state cannot be more entangled with respect to AGM than the tensor product of bipartite maximally entangled states, or the cluster state, for a multiqubit system.

If $N$ is odd, however, there may exist no pure state that can saturate the upper bound in proposition 23. For example, W state has been shown to be the maximally entangled state with respect to GM among three-qubit states [62, 63], while its AGM, which equals its GM $\log \frac{9}{4}$, is strictly smaller than the upper bound $\frac{3}{7} \log 2$ given by proposition 23.

It will be interesting to know whether there is a similar universal upper bound to REE and LGR, in particular, whether AREE or ALGR are also upper bounded by $\frac{1}{7} \log d_T$. It is also not clear whether REE and LGR are non-additive for generic multipartite states. We have shown in section 4.3 that AREE is upper bounded by $\frac{1}{7} \log d_T$ for antisymmetric basis states. The same is true for all symmetric basis states, according to (36). However, a complete picture is still missing. We hope that our results can stimulate more progress in this direction.

5.2. Non-additivity theorem for GM of generic multipartite states: a statistical approach

In this section, we prove the following theorem.
Theorem 24. Suppose that pure states are drawn according to the Haar measure from the Hilbert space $\bigotimes_{j=1}^{N} \mathcal{H}_j$ with $N \geq 3$ and $\text{Dim} \mathcal{H}_j = d_j$ ($d_j \geq 2$, $\forall j$); define $d_T = \prod_{j=1}^{N} d_j$ and $d_S = \sum_{j=1}^{N} d_j$. The fraction of pure states whose GM is additive is smaller than $\exp\left[-\frac{2}{3}\sqrt{d_T} + d_S \ln(59Nd_T)\right]$. The fraction of pure states $|\psi\rangle$ such that $[\log d_T - \log(d_S \ln d_T) - \log \frac{9}{2}] \leq G(|\psi\rangle) < G(|\psi\rangle)^{\otimes 2}$ is less than $d_T^{-d_S}$.

Theorem 24 implies that GM is non-additive for almost all pure multipartite states if the number of parties is sufficiently large, and the dimensions of the local Hilbert spaces are comparable. Moreover, GM for one copy and two copies, respectively, is almost equal. If the dimensions of the local Hilbert spaces are equal, the probability that GM is additive decreases doubly exponentially with the number of parties $N$. The generalization to mixed states is immediate, since GM of any mixed state is equal to GM of its purification [61] (see also (12)).

Theorem 24 is an immediate consequence of proposition 23 in section 5.1 and proposition 25 presented below. The later proposition, which is inspired by a similar result on multiqubit pure states of [35], states that almost all multipartite pure states are nearly maximally entangled with respect to GM.

Proposition 25. Suppose pure states are drawn according to the Haar measure from the Hilbert space $\bigotimes_{j=1}^{N} \mathcal{H}_j$ with $N \geq 3$ and $\text{Dim} \mathcal{H}_j = d_j$ ($d_j \geq 2$, $\forall j$); define $d_T = \prod_{j=1}^{N} d_j$ and $d_S = \sum_{j=1}^{N} d_j$. The fraction of pure states whose GM is smaller than $\log d_T - \log(d_S \ln d_T) - \log \frac{9}{2}$ is less than $d_T^{-d_S}$; the fraction of pure states whose GM is smaller than $\frac{1}{2} \log d_T$ is less than $\exp[-\frac{2}{3}\sqrt{d_T} + d_S \ln(59Nd_T)]$.

By means of the relation among the three measures GM, REE and LGR (see (7)), we obtain

Corollary 26. Suppose that pure states are drawn according to the Haar measure from the Hilbert space $\bigotimes_{j=1}^{N} \mathcal{H}_j$ with $N \geq 3$ and $\text{Dim} \mathcal{H}_j = d_j$ ($d_j \geq 2$, $\forall j$); define $d_T = \prod_{j=1}^{N} d_j$ and $d_S = \sum_{j=1}^{N} d_j$. The fraction of pure states whose REE or LGR is smaller than $\log d_T - \log(d_S \ln d_T) - \log \frac{9}{2}$ is less than $d_T^{-d_S}$; the fraction of pure states whose REE or LGR is smaller than $\frac{1}{2} \log d_T$ is less than $\exp[-\frac{2}{3}\sqrt{d_T} + d_S \ln(59Nd_T)]$.

Note that $G(\rho) \leq E_R(\rho) \leq \log d_T$ for any pure state $\rho$, since $S(\rho\|I/d_T) = \log d_T$. Proposition 25 and corollary 26 imply that almost all multipartite pure states are nearly maximally entangled with respect to GM and REE if the number of parties is sufficiently large, and the dimensions of the local Hilbert spaces are comparable. In particular, if the dimensions of the local Hilbert spaces are equal, then the probability that GM (REE, LGR) is smaller than $\log d_T - \log(d_S \ln d_T) - \log \frac{9}{2}$ decreases exponentially with the number of parties $N$, and the probability that GM (REE, LGR) is smaller than $\frac{1}{2} \log d_T$ decreases doubly exponentially.

Proof. To prove the proposition, we need the concept of $\varepsilon$-net. An $\varepsilon$-net $\mathcal{N}_{\varepsilon,N}$ on the set of product states is a set of states that satisfy

$$\max_{|\varphi\rangle \in \text{PRO}} \min_{|\tilde{\varphi}\rangle \in \mathcal{N}_{\varepsilon,N}} \| |\varphi\rangle \langle \varphi| - |\tilde{\varphi}\rangle \langle \tilde{\varphi}| \|_1 \leq \varepsilon,$$

or, equivalently,

$$\min_{|\varphi\rangle \in \text{PRO}} \max_{|\tilde{\varphi}\rangle \in \mathcal{N}_{\varepsilon,N}} |\langle \varphi|\tilde{\varphi}\rangle|^2 \geq 1 - \frac{\varepsilon^2}{4}.$$
We shall show that there exists an \( \epsilon \)-net with \( |\mathcal{N}_{\epsilon,N}| \leq (5\sqrt{N}/\epsilon)^{2d_3} \), where \( |\mathcal{N}_{\epsilon,N}| \) denotes the number of elements in the \( \epsilon \)-net. From \cite{83}, we know that there is an \( \epsilon \)-net \( \mathcal{M} \) on the space of single qudit with \( |\mathcal{M}| \leq (5/\epsilon)^{2d_3} \). Let \( \mathcal{M}_j \) be an \( (\epsilon/\sqrt{N}) \)-net on \( \mathcal{H}_j \) with \( |\mathcal{M}_j| \leq (5\sqrt{N}/\epsilon)^{2d_j} \) for \( j = 1, \ldots, N \) and \( \mathcal{N}_{\epsilon,N} := \{ \bigotimes_{j=1}^N \{ \tilde{a}_j \} : \tilde{a}_j \in \mathcal{M}_j \} \). Suppose that \( |\varphi_4| = \bigotimes_{j=1}^4 (c_j |0\rangle + s_j |1\rangle) \) is an arbitrary product state, by definition of the \( (\epsilon/\sqrt{N}) \)-net, for each \( j \), there exists \( \tilde{a}_j \in \mathcal{M}_j \) such that \( \langle \tilde{a}_j | \tilde{a}_j \rangle \geq 1 - \epsilon^2/4N \). It follows that for \( |\tilde{\varphi}\rangle = \bigotimes_{j=1}^N |\tilde{a}_j\rangle \in \mathcal{N}_{\epsilon,N} \), the following relation holds:

\[
|\langle \varphi | \tilde{\varphi} \rangle |^2 = \left( 1 - \frac{\epsilon^2}{4N} \right)^N \geq 1 - \frac{\epsilon^2}{4}. \tag{83}
\]

Hence, \( \mathcal{N}_{\epsilon,N} \) is an \( \epsilon \)-net on the set of product states with \( |\mathcal{N}_{\epsilon,N}| \leq (5\sqrt{N}/\epsilon)^{2d_3} \).

Another ingredient in our proof is the following result for the concentration of measure: let \( |\Phi\rangle \) be any given normalized pure state, and \( |\Psi\rangle \) be chosen according to the Haar measure; then we have (assuming \( 0 < \epsilon < 1 \))

\[
\text{Prob}[|\langle \Phi | \Psi \rangle|^2 \geq \epsilon] = (1 - \epsilon)^{d_1 - 1} < \exp[-(d_T - 1)\epsilon]. \tag{84}
\]

Since

\[
||\langle \Phi | \Psi \rangle|^2 - |\langle \tilde{\Phi} | \Psi \rangle|^2|| = |\text{tr}(|\langle \Phi | \Psi \rangle - |\tilde{\Phi} | \Psi \rangle)|| = \frac{1}{2} ||\langle \Phi | \Psi \rangle - |\tilde{\Phi} | \Psi \rangle||_1 \leq \frac{\epsilon}{2}, \tag{85}
\]

the probability that \( G(|\Psi\rangle) \leq -\log(\frac{1}{2}\epsilon) \) is not larger than

\[
\text{Prob}\left\{ \max_{|\tilde{\psi}\rangle \in \mathcal{N}_{\epsilon,N}} |\langle \tilde{\psi} | \Psi \rangle|^2 \geq \epsilon \right\} < \exp[-(d_T - 1)\epsilon] |\mathcal{N}_{\epsilon,N}| \leq \exp[-(d_T - 1)\epsilon] \left( \frac{5\sqrt{N}}{\epsilon} \right)^{2d_3}. \tag{86}
\]

Now let \( \epsilon = 3d_3 \ln d_T/d_T \), the probability that \( G(|\Psi\rangle) \leq \log d_T - \log(d_s \ln d_T) - \log \frac{9}{2} \) is no larger than

\[
\text{Prob}\left\{ \max_{|\tilde{\psi}\rangle \in \mathcal{N}_{\epsilon,N}} |\langle \tilde{\psi} | \Psi \rangle|^2 \geq 3d_3 \ln d_T \right\} < \exp[-3d_3 \ln d_T + d_s \left( 3 \ln d_T + 3 \ln 3d_3 \ln d_T \right)]
\]

\[
< \exp(-3d_3 \ln d_T) = d_T^{-d_3}. \tag{87}
\]

Here, the last inequality can be derived as follows. Under our assumption \( N \geq 3, d_j \geq 2, \forall j \) \( (d_3 \geq 6, d_T \geq 8) \), the maximum of \( 3 \ln d_T/d_T + 2 \ln[5\sqrt{N}/(3d_3 \ln d_T)] \) for each \( N \), which is obtained when \( d_j = 2, \forall j \), decreases monotonically with \( N \). Hence, the global maximum of \( 3 \ln d_T/d_T + 2 \ln[5\sqrt{N}/(3d_3 \ln d_T)] \) is obtained when \( N = 3, d_1 = d_2 = d_3 = 2 \). Since this maximum is negative, the inequality follows. This completes the proof of the first part of the proposition.

Next, let \( \epsilon = \frac{2}{3}d_T^{-1/2} \), the probability that \( G(|\Psi\rangle) \leq \frac{1}{2} \log d_T \) is not larger than

\[
\text{Prob}\left\{ \max_{|\tilde{\psi}\rangle \in \mathcal{N}_{\epsilon,N}} |\langle \tilde{\psi} | \Psi \rangle|^2 \geq \frac{2}{3 \sqrt{d_T}} \right\} \leq \exp\left[ -\frac{2}{3} \sqrt{d_T} + \frac{2}{3 \sqrt{d_T}} + d_s \ln\left( \frac{225}{4} N d_T \right) \right]
\]

\[
= \exp\left[ -\frac{2}{3} \sqrt{d_T} + d_s \ln(59N d_T) \right] \exp\left[ \frac{2}{3} d_T^{-1/2} + d_s \ln\left( \frac{225}{236} \right) \right]
\]

\[
< \exp\left[ -\frac{2}{3} \sqrt{d_T} + d_s \ln(59N d_T) \right]. \tag{88}
\]
Here, the last inequality can be seen as follows. Under our assumption \( N \geq 3, d_j \geq 2, \forall j \) (\( d_s \geq 6, d_T \geq 8 \)), the maximum of \( \frac{2}{3}d_T^{-1/2} + d_S \ln(\frac{225}{236}) \), which is obtained when \( N = 3, d_1 = d_2 = d_3 = 2 (d_s = 6, d_T = 8) \), is negative. \( \square \)

5.3. Impact of additivity property on one-way quantum computation and asymptotic state transformation

Recently, Gross et al [35] (see also [84]) showed that most quantum states are too entangled to be useful as computational resources. One of the key ingredients in their proof is the observation that almost all multiqudit pure states are nearly maximally entangled with respect to GM. However, their arguments would break down if measurements were allowed on two copies of multiqudit states, since two copies of generic multiqudit states are just moderate entangled (GM is nearly one half of the maximal possible value), according to theorem 32. Hence, it is conceivable that we may realize universal quantum computation on a certain family of multiqudit states if they come in pairs, even if this is impossible on a single copy. It would be very desirable to construct an explicit example of such a family of multiqudit states or disprove this possibility. However, a detailed investigation in this direction would well go beyond the scope of this paper.

Corollary 26 has a significant implication for asymptotic state transformation. In particular, it implies that almost all multiqudit pure states cannot be prepared reversibly with multipartite GHZ states (of different number of parties) under asymptotic LOCC, unless REE is non-additive for generic multiqudit states. This can be seen as follows. According to the result of Linden et al [23], reversible transformation between two pure states under asymptotic LOCC would mean that the ratio of the bipartite AREE \( E^\infty_R(A_j : \tilde{A}_j) \) to the \( N \)-partite AREE \( E^\infty_R \) is conserved, for \( j = 1, 2, \ldots, N \), where \( \tilde{A}_j \) denotes all the parties except \( A_j \). As a result, the ratio \( \left[ \sum_{j=1}^N E^\infty_R(A_j : \tilde{A}_j) \right] / E^\infty_R \) is conserved. If a state \( |\psi\rangle \langle \psi| \) can be prepared reversibly with \( n_k \) copies of \( k \)-partite GHZ states for \( k = 2, \ldots, N \), then

\[
\frac{\sum_{j=1}^N E^\infty_R(|\psi\rangle_{A_j:\tilde{A}_j})}{E^\infty_R(|\psi\rangle)} = \frac{\sum_{k=2}^N n_k}{\sum_{k=2}^N n_k} \geq 2, \tag{89}
\]

where we have used the fact that REE of the tensor product of GHZ-type states is additive. On the other hand, \( E^\infty_R(|\psi\rangle_{A_j:\tilde{A}_j}) = E_R(|\psi\rangle_{A_j:\tilde{A}_j}) \leq \log d \); in addition, \( E_R(|\psi\rangle) > \frac{1}{2} \log d_T = \frac{N}{2} \log d \) for almost all multiqudit pure states, according to corollary 26. If REE of \(|\psi\rangle\) is additive, then we have

\[
\frac{\sum_{j=1}^N E^\infty_R(|\psi\rangle_{A_j:\tilde{A}_j})}{E^\infty_R(|\psi\rangle)} = \frac{\sum_{j=1}^N E_R(|\psi\rangle_{A_j:\tilde{A}_j})}{E_R(|\psi\rangle)} < \frac{N \log d}{\frac{N}{2} \log d} = 2, \tag{90}
\]

which contradicts (89). Hence, almost all multiqudit pure states cannot be prepared reversibly under asymptotic LOCC, unless REE is non-additive for generic multiqudit pure states. Our observation adds to the evidence that a reversible entanglement generating set \([24, 25]\) with a finite cardinality may not exist.

As a concrete example, a similar reasoning has been employed by Ishizaka and Plenio [85] to show that \(|\psi_{3-}\rangle\) cannot be generated reversibly from the GHZ state and EPR pairs under asymptotic LOCC if its REE is additive. The same is true for \(|\psi_{N-}\rangle\) with \( N \geq 3 \), since \( E_R(|\psi_{N-}\rangle) = \log(N!) > \frac{N}{2} \log N \) [30, 55, 56] (see also (63) in section 4.3).

In this paper, we have studied the additivity property of three main multipartite entanglement measures, namely GM, REE and LGR.

Firstly, we proved the strong additivity of GM of any non-negative states, thus simplifying the computation of GM and AGM of a large family of states of either experimental or theoretical interest. Thanks to the connection among the three measures, GM of non-negative states provides a lower bound to AREE and ALGR and a new approach to proving the additivity of REE and LGR for states with a certain group symmetry. In particular, we proved the strong additivity of GM and the additivity of REE of Bell diagonal states, maximally correlated generalized Bell diagonal states, generalized Dicke states and their reduced states after tracing out one party, the Smolin state, Dür’s multipartite entangled states, etc. The additivity of LGR of generalized Dicke states and the Smolin state was also shown. These results can be applied to studying state discrimination under LOCC \([29, 30]\), the classical capacity of quantum multi-terminal channels. The result for AREE is also useful in studying state transformation either under asymptotic LOCC or under asymptotic non-entangling operations. For non-negative bipartite states, the result for AREE also leads to a new lower bound to entanglement of formation and entanglement cost. The result for GM and AGM may also find applications in the study of quantum channels due to the connection between pure tripartite states and quantum channels \([32]\).

Secondly, we established a simple connection between the permutation symmetry and the additivity property of multipartite entanglement measures. In particular, we showed that GM is non-additive for antisymmetric states shared over three or more parties. Also, we gave a unified explanation of the non-additivity of the three measures GM, REE and LGR of the antisymmetric projector states and derive analytical formulae for the three measures for one copy and two copies of such states. Our results for antisymmetric states are expected to be useful in the study of fermion systems, which are described by antisymmetric wave functions due to the super-selection rule.

Thirdly, we showed that almost all multipartite pure states are maximally entangled with respect to GM and REE. However, their GM is non-additive, and GM of two copies and one copy, respectively, of these states is almost equal. Based on these results, we showed that more states may be suitable for universal quantum computation, if measurements can be performed on two copies of the resource states. We also showed that, for almost all multipartite pure states, the additivity of their REE implies the irreversibility in generating them from GHZ-type states under LOCC, even in the asymptotic sense.

There are also quite a few open problems that can provide new directions in the future study of multipartite entanglement:

1. Are GM and REE of arbitrary two-qubit states and pure three-qubit states additive?
2. Are GM and REE of arbitrary symmetric states additive? We cannot find any counter-examples at the moment; however, the possibility has not been excluded.
3. When are GM and REE of bipartite mixed antisymmetric states additive or non-additive?
4. How can we compute AGM, AREE and ALGR of the antisymmetric projector states? It is enough to compute any one of the three measures, since they are related to each other by the simple equalities in proposition 19.
5. Are REE and LGR non-additive for generic multipartite states?
6. Does there exist a family of quantum states such that two copies are universal for quantum computation while one copy is not?

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Note added. Upon completion of this work, we found that Plastino et al [86] had derived a similar criterion as proposition 15 for determining when an antisymmetric state is a Slater determinant state.

Appendix

Suppose that \( \rho_N \) and \( \rho'_N \) are two \( N \)-partite states on the Hilbert space \( \bigotimes_{j=1}^{N} \mathcal{H}_j \) with \( \text{dim} \mathcal{H}_j = d_j \); define \( d_T = \prod_{j=1}^{N} d_j \) and define

\[
|\varphi_N \rangle = \bigotimes_{j=1}^{N} |a_{V_j} \rangle, \quad |a_{V_j} \rangle = \frac{1}{\sqrt{d_j}} \sum_{k,l=0}^{d_j-1} V_{j,kl} |k \rangle \langle l |
\]

\( V_j = V_{j,kl} |k \rangle \langle l | \), \( \text{tr}(V_j V_j^\dagger) = d_j \), \( \mathcal{V} = \bigotimes_{j=1}^{N} V_j \).

In this appendix, we prove the following formula,

\[
\langle \varphi_N | \rho_N \otimes \rho'_N | \varphi_N \rangle = \frac{1}{d_T} \text{tr}(\rho_N^{1/2} \mathcal{V} \rho'_N \mathcal{V}^\dagger \rho_N^{1/2}),
\]

which reduces to

\[
\langle \varphi_N | \rho_N \otimes \rho'_N | \varphi_N \rangle = \frac{1}{d_N} \text{tr}(\rho_N^{1/2} \mathcal{V} \otimes \rho'_N \mathcal{V}^\dagger \otimes \rho_N^{1/2}),
\]

in the special case \( d_j = d \), \( V_j = V, \forall j \).

Proof.

\[
\langle \varphi_N | \rho_N \otimes \rho'_N | \varphi_N \rangle = \| \rho_N^{1/2} \otimes \rho'_N^{1/2} | \varphi_N \rangle \|^2
\]

\[
= \frac{1}{d_T} \left\| \sum_{k_1,l_1,\ldots,k_N,l_N} V_{1,k_1 l_1} \cdots V_{N,k_N l_N} \rho_N^{1/2} |k_1,\ldots,k_N \rangle \otimes \rho'_N^{1/2} |l_1,\ldots,l_N \rangle \right\|^2
\]
\[
= \frac{1}{d_T} \left\| \sum_{k_1,l_1,\ldots,k_N,l_N} V_{k_1,l_1} \cdots V_{k_N,l_N} \rho_{N}^{1/2} |k_1,\ldots,k_N\rangle |l_1,\ldots,l_N\rangle \rho_{N}^{1/2} \right\|_{\text{HS}}^2
\]
\[
= \frac{1}{d_T} \left\| \rho_{N}^{1/2} \left( \bigotimes_{j=1}^{N} V_j \right) \rho_{N}^{1/2} \right\|_{\text{HS}}^2 = \frac{1}{d_T} \left\| \rho_{N}^{1/2} \mathcal{V} \rho_{N}^{1/2} \right\|_{\text{HS}}^2
\]
\[
= \frac{1}{d_T} \text{tr} \left( \rho_{N}^{1/2} \mathcal{V} \rho_{N}^{1/2} \mathcal{V}^\dagger \rho_{N}^{1/2} \right).
\]

(A.4)
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