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A random matrix approach to VARMA processes

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Abstract. We apply random matrix theory to derive the spectral density of large sample covariance matrices generated by multivariate VMA\((q)\), VAR\((q)\) and VARMA\((q_1, q_2)\) processes. In particular, we consider a limit where the number of random variables \(N\) and the number of consecutive time measurements \(T\) are large but the ratio \(N/T\) is fixed. In this regime, the underlying random matrices are asymptotically equivalent to free random variables (FRV). We apply the FRV calculus to calculate the eigenvalue density of the sample covariance for several VARMA-type processes. We explicitly solve the VARMA\((1, 1)\) case and demonstrate perfect agreement between the analytical result and the spectra obtained by Monte Carlo simulations. The proposed method is purely algebraic and can be easily generalized to \(q_1 > 1\) and \(q_2 > 1\).

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1. Introduction

Vector auto-regressive (VAR) models play an important role in contemporary macro-economics, being an example of an approach called ‘dynamic stochastic general equilibrium’ (DSGE), which is superseding traditional large-scale macro-econometric forecasting methodologies [1]. The motivation behind them is based on the assertion that more recent values of a variable are more likely to contain useful information about its future movements than the older ones. On the other hand, a standard tool in multivariate time series analysis is vector moving average (VMA) models, which is really a linear regression of the present value of the time series with respect to the past values of a white noise. A broader class of stochastic processes used in macro-economics comprises both these types in the form of vector auto-regressive moving average (VARMA) models. These methodologies can capture certain spatial and temporal structures of multidimensional variables, which are often neglected in practice; including them not only results in more accurate estimation, but also leads to models that are more interpretable. They are widely used by academia and central banks (cf the European Central Bank’s Smets–Wouters model for the Euro zone [2]), as they constitute quite a simple version of the DSGE equations.

VARMA models are constructed from a number of univariate ARMA (Box–Jenkins; see, for example, [3]) processes, typically coupled with each other. In this paper, we investigate only a significantly simplified circumstance when there is no coupling between the many ARMA components. One may argue that this is too far fetched and will be of no use in describing an economic reality. However, one may also treat it as a ‘zeroth-order hypothesis’, analogously to the idea of [4, 5] in finance, namely that the case with no cross-covariances is considered theoretically and subsequently compared with some real-world data modeled by a VARMA process; any discrepancy between the two will reflect nontrivial cross-covariances present in the system, thus permitting their investigation. This latter route is taken in this paper.

A challenging and yet increasingly important problem is the estimation of large covariance matrices generated by these stationary VARMA\( (q_1, q_2) \) processes, since high dimensionality of the data as compared with the sample size is quite common in many statistical problems (the ‘dimensionality curse’). Therefore, an appropriate ‘noise cleaning’ procedure has to be implemented, and random matrix theory (RMT) provides a natural and efficient means of doing
that. In particular, the mean spectral densities (a.k.a. ‘limiting spectral distributions’, LSD) of the Pearson estimators of the cross-covariances for the VMA(1) and VAR(1) models, in the relevant high-dimensionality sector and under the full decoupling, have been derived in [6] by applying the framework proposed in [7].

In this paper, we suggest that such calculations can be considerably simplified by resorting to a mathematical concept of the free random variables (FRV) calculus [8, 9], succinctly introduced in section 2. Our general FRV formula [10] allows us to not only rediscover, with much less strain, the two fourth-order polynomial equations obtained in [6] in the VMA(1) and VAR(1) cases, but also to derive a sixth-order equation (A.6) that produces the mean spectral density for a more involved VARMA(1, 1) model. The results are verified by numerical simulations, which show perfect agreement. Also, practical relevance of VARMA(1, 1) is reinforced by fitting the density retrieved from (A.6) to real macroeconomic data. All this is done in section 3.

2. Doubly correlated Wishart ensembles and free random variables (FRV)

2.1. Doubly correlated Wishart ensembles

2.1.1. Correlated Gaussian random variables. VARMA($q_1$, $q_2$) stochastic processes, as we will see below, fall under quite a general setup encountered in many areas of science where a probabilistic nature of multiple degrees of freedom evolving in time is relevant, for example, multivariate time series analysis in finance, applied macro-econometrics and engineering. To describe this framework, consider a situation of $N$ time-dependent random variables that are measured at $T$ consecutive time moments (separated by some time interval $\delta t$). Let $Y_{ia}$ be the value of the $i$th ($i = 1, \ldots, N$) random number at the $a$th time moment ($a = 1, \ldots, T$).

Together, they make up a rectangular $N \times T$ matrix $Y$. In what would usually be the first approximation, each $Y_{ia}$ is supposed to be drawn from a Gaussian probability distribution. We will also assume that they have mean value zero, $\langle Y_{ia} \rangle = 0$. These degrees of freedom may in principle display mutual correlations. A set of correlated zero-mean Gaussian numbers is fully characterized by the two-point covariance function, $C_{ia,jb} \equiv \langle Y_{ia}Y_{jb} \rangle$, if the underlying stochastic process generating these numbers is stationary. Linear stochastic processes, including VARMA($q_1$, $q_2$), belong to this category. We will restrict our attention to an even narrower class where the cross-correlations between different variables and the auto-correlations between different time moments are factorized, i.e.

$$\langle Y_{ia}Y_{jb} \rangle = C_{ij}A_{ab}.$$  

In this setting, the inter-variable covariances do not change in time (and are described by an $N \times N$ cross-covariance matrix $C$), and also the temporal covariances are identical for all the numbers (and are included in a $T \times T$ auto-covariance matrix $A$; both these matrices are symmetric and positive-definite). The Gaussian probability measure with this structure of covariances is known from textbooks,

$$P_{c.G.}(Y)DY = \frac{1}{\mathcal{N}_{c.G.}} \exp \left( -\frac{1}{2} \sum_{i,j=1}^{N} \sum_{a,b=1}^{T} Y_{ia}[C^{-1}]_{ij}Y_{jb}[A^{-1}]_{ba} \right) DY$$

$$= \frac{1}{\mathcal{N}_{c.G.}} \exp \left( -\frac{1}{2} \text{Tr}Y^{T}C^{-1}YA^{-1} \right) DY.$$  

where the normalization constant $N_{c.G.} = (2\pi)^{NT/2}(\text{Det}C)^{T/2}(\text{Det}A)^{N/2}$, and the integration measure $D\tilde{Y} \equiv \prod_{i=1}^{N} \prod_{a=1}^{T} dY_{ia}$, while the letters ‘c.G.’ stand for ‘correlated Gaussian’.

Now, a standard way to approach correlated Gaussian random numbers is to recall that they can always be decomposed as linear combinations of uncorrelated Gaussian degrees of freedom. Indeed, this is achieved through the transformation

$$Y = \sqrt{C}\tilde{Y}\sqrt{A},$$

which yields

$$P_{G}(\tilde{Y})D\tilde{Y} = \frac{1}{N_{G.}}\exp\left(-\frac{1}{2}\text{Tr}\tilde{Y}^{T}\tilde{Y}\right)D\tilde{Y}.$$  \hfill (3)

where the square roots of the covariance matrices, necessary to facilitate the transition, exist due to the positive definiteness of $C$ and $A$. The new normalization reads $N_{G.} = (2\pi)^{NT/2}$.

### 2.1.2. Estimating equal-time cross-covariances.

An essential problem in multivariate analysis is to determine (estimate) the covariance matrices $C$ and $A$ from a given $N$ time series of length $T$ of the realizations of our random variables $Y_{ia}$. For simplicity, we do not distinguish in notation between random numbers, i.e. the population, and their realizations in actual experiments, i.e. the sample. Since the realized cross-covariance between degrees $i$ and $j$ at the same time $a$ is $Y_{ia}Y_{ja}$, the simplest method to estimate today’s cross-covariance $c_{ij}$ is to compute the time average,

$$c_{ij} \equiv \frac{1}{T} \sum_{a=1}^{T} Y_{ia}Y_{ja}, \quad \text{i.e.} \quad c = \frac{1}{T} YY^{T} = \frac{1}{T} \sqrt{C}\tilde{Y}\tilde{A}\tilde{Y}^{T}\sqrt{C}. \hfill (4)$$

This is usually named the ‘Pearson estimator’, up to the prefactor, which, depending on the context, is $1/(T-1)$ or $1/T$. Other estimators might be introduced, such as between distinct degrees of freedom at separate time moments (‘time-delayed estimators’), or with certain decreasing weights given to older measurements to reflect their growing obsolescence (‘weighted estimators’), but we will not investigate them in this paper. Furthermore, in the last equality in (4), we cast $c$ through the uncorrelated Gaussian numbers contained in $\tilde{Y}$, the price to pay for this being that the covariance matrices now enter into the expression for $c$, making it more complicated. This will be the form used hereafter. The random matrix $c$ is called a ‘doubly correlated Wishart ensemble’ [11].

Let us also mention that the auto-covariance matrix $A$ can be estimated through $a \equiv (1/N)YY^{T}$. However, it is verified that this object carries identical information to the one contained in $c$ (it is ‘dual’ to $c$) and therefore may safely be discarded. Indeed, these two estimators have the same nonzero eigenvalues (modulo an overall rescaling by $r$), and the larger one has $|T-N|$ additional zero modes.

Any historical estimator is inevitably marred by the measurement noise; it will reflect the true covariances only to a certain degree, with a superimposed broadening due to the finiteness of the time series. More precisely, there are $N(N+1)/2$ independent elements in $C$, to be estimated from $NT$ measured quantities $Y$. Hence the estimation accuracy will depend on the ‘rectangularity ratio’,

$$r \equiv \frac{N}{T}, \hfill (5)$$

the closer $r$ is to zero, the more truthful is the estimate. This is a cornerstone of classical multivariate analysis. Unfortunately, a practical situation will typically feature a large number...
of variables sampled over a comparably big number of time snapshots, so that we may approximately talk about the ‘thermodynamical limit’,

$$N \to \infty, \quad T \to \infty, \quad \text{such that} \quad r = \text{fixed}. \quad (6)$$

On the other hand, it is exactly this limit in which the FRV calculus (see the subsection below for its brief elucidation) can be applied. Hence, the challenge of de-noising is somewhat counterbalanced by the computationally powerful FRV techniques.

2.2. A short introduction to the FRV calculus: the multiplication algorithm

2.2.1. The M-transform and the spectral density. Any study of a (real symmetric $K \times K$) random matrix $H$ will most surely include a fundamental question about the average values of its (real) eigenvalues $\lambda_1, \ldots, \lambda_K$. They are concisely encoded in the ‘mean spectral density’,

$$\rho_H(\lambda) \equiv \frac{1}{K} \sum_{i=1}^{K} \langle \delta(\lambda - \lambda_i) \rangle = \frac{1}{K} \langle \text{Tr} (\lambda 1_K - H) \rangle. \quad (7)$$

Here, the expectation map $\langle \ldots \rangle$ is understood to be taken with respect to the probability measure $P(H) dH$ of the random matrix. We will always have this distribution rotationally (i.e. $H \to O^T H O$, with $O$ orthogonal) invariant, and hence the full information about $H$ resides in its eigenvalues, distributed on average according to (7).

On the practical side, it is more convenient to work with either of the two equivalent objects,

$$G_H(z) \equiv \frac{1}{K} \left\langle \text{Tr} \frac{1}{z 1_K - H} \right\rangle \quad \text{or} \quad M_H(z) \equiv z G_H(z) - 1, \quad (8)$$

referred to as the ‘Green’s function’ (or the ‘resolvent’) and the ‘M-transform’ of $H$. The latter is also called the ‘moments’ generating function’ because, if the ‘moments’ $M_{H,n} \equiv (1/K) \langle \text{Tr} H^n \rangle$ of $H$ exist, it can be expanded into a power series around $z \to \infty$ as $M_H(z) = \sum_{n \geq 1} M_{H,n} / z^n$. It should, however, be underlined that, even for probability measures disallowing such an expansion (heavy-tailed distributions, preeminent in finance, being an example), the quantities (8) still manage to entirely capture the spectral properties of $H$. Hence the name ‘M-transform’ is more appropriate, in addition to being more compact.

We will show that, for our purposes (multiplication of random matrices; see section 2.2.2), the $M$-transform serves better than the Green’s function. However, it is customary to write the relationship between (7) and (8) in terms of this latter,

$$\rho_H(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \text{Im} G_H(\lambda + i \epsilon) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left( G_H(\lambda + i \epsilon) - G_H(\lambda - i \epsilon) \right), \quad (9)$$

resulting from a well-known formula for generalized functions, $\lim_{\epsilon \to 0^+} 1/(x \pm i \epsilon) = \text{pv}(1/x) \mp i\pi \delta(x)$. 

2.2.2. The $N$-transform and FRV. The doubly correlated Wishart ensemble $c$ (4) may be viewed as a product of several random and non-random matrices. The general problem of multiplying random matrices seems formidable. In classical probability theory, it can be effectively handled in the special situation when the random terms are independent: then, the exponential map reduces it to the addition problem of independent random numbers, solved by considering the logarithm of the characteristic functions of the respective probability density functions (PDFs), which proves to be additive. In matrix probability theory, a crucial insight came from Voiculescu et al and Speicher [8, 9], who showed how to parallel the commutative construction in the noncommutative world. It starts with the notion of ‘freeness’, which basically comprises probabilistic independence together with a lack of any directional correlation between two random matrices. This nontrivial new property happens to be the right extension of classical independence, as it allows for an efficient algorithm of multiplying FRV, which we state below:

**Step 1:** Suppose that we have two random matrices, $H_1$ and $H_2$, mutually free. Their spectral properties are best wrought into the $M$-transforms (8), $M_{H_1}(z)$ and $M_{H_2}(z)$.

**Step 2:** The critical maneuver is to turn attention to the functional inverses of these $M$-transforms, the so-called ‘$N$-transforms’,

$$M_H(N_H(z)) = N_H(M_H(z)) = z.$$  

(10)

**Step 3:** The $N$-transforms submit to a very straightforward rule upon multiplying free random matrices (the ‘FRV multiplication law’),

$$N_{H_1,H_2}(z) = \frac{z}{1+z}N_{H_1}(z)N_{H_2}(z), \quad \text{for free } H_1, H_2.$$  

(11)

**Step 4:** Finally, it remains to functionally invert the resulting $N$-transform $N_{H_1,H_2}(z)$ to obtain the $M$-transform of the product, $M_{H_1,H_2}(z)$, and consequently, all the spectral properties via formula (9).

It is stunning that such a simple prescription (relying on the choice of the $M$-transform as the carrier of the mean spectral information, and the construction of its functional inverse, the $N$-transform, which is essentially multiplicative for the free product) resolves the multiplication problem for free random noncommutative objects.

Let us just mention that addition may be tackled along similar lines. In this case, the Green’s function should be exploited, its functional inverse considered $(G_H(B_H(z)) = B_H(G_H(z)) = z$. It is sometimes called the ‘Blue’s function’ [12, 13], which obeys the ‘FRV addition law’,

$$B_{H_1+H_2}(z) = B_{H_1}(z) + B_{H_2}(z) - 1/z,$$  

for two free random matrices. In this paper, we do not resort to using this addition formula, even though our problem could be approached through it as well.

Let us also remark that, in the original mathematical formulations [8, 9] of these frames, a slightly different language is employed: instead of the $N$-transform, the ‘$S$-transform’ is found convenient, $S_{H_1}(z) = (1+z)/(zN_{H_1}(z))$, while in place of the Blue’s function, one engages the ‘$R$-transform’, $R_H(z) = B_H(z) - 1/z$. They fulfill simpler laws, $S_{H_1,H_2}(z) = S_{H_1}(z)S_{H_2}(z)$ and $R_{H_1+R_2}(z) = R_{H_1}(z) + Y_{H_2}(z)$, respectively.

2.2.3. Doubly correlated Wishart ensembles from FRV. The innate potential of the FRV multiplication algorithm (11) is surely revealed when inspecting the doubly correlated Wishart random matrix $c = (1/T)^{1/2}(\mathbf{Y}^\mathsf{T}\mathbf{Y})^{1/2}$ (4). This has been done in detail in [10], so we will
only accentuate the main results here, referring the reader to the original paper for a thorough explanation.

The idea is that one uses twice the cyclic property of the trace (which permits cyclic shifts in the order of the terms), and twice the FRV multiplication law (11) (to break the $N$-transforms of products of matrices down to their constituents), in order to reduce the problem to solving the uncorrelated Wishart ensemble $(1/T)\tilde{Y}^T \tilde{Y}$. This last model is further simplified, again by the cyclic property and the FRV multiplication rule applied once, to the standard GOE random matrix squared (and the projector $P \equiv \text{diag}(1_N, 0_{T-N})$, designed to chip the rectangle $\tilde{Y}$ off the square GOE), whose properties are firmly established. Let us sketch the derivation.

$$N_c(z) \overset{\text{cyclic}}{=} N_1 r \tilde{Y} \tilde{A} r \tilde{Y} C(z) \overset{\text{FRV}}{=} \frac{z}{1+z} N_1 r \tilde{Y} \tilde{A} r \tilde{Y} (z) N_c(z)$$

$$\overset{\text{cyclic}}{=} \frac{z}{1+z} N_1 r \tilde{Y} \tilde{A} (r z) N_c(z) \overset{\text{FRV}}{=} \frac{z}{1+z} \frac{r z}{1+r z} N_1 r \tilde{Y} \tilde{Y} (r z) N_A(r z) N_c(z)$$

$$= r z N_A(r z) N_c(z). \quad (12)$$

This is the basic formula. Since the spectral properties of $c$ are given by its $M$-transform, $M \equiv M_c(z)$, it is more pedagogical to recast (12) as an equation for the unknown $M$,

$$z = r M N_A(r M) N_c(M). \quad (13)$$

It provides a means of computing the mean spectral density of a doubly correlated Wishart random matrix once the ‘true’ covariance matrices $C$ and $A$ are given.

In this paper, only a particular instance of this fundamental formula is applied, namely with an arbitrary auto-covariance matrix $A$, but with trivial cross-covariances, $C = 1_N$. Using that $N_1(z) = 1 + 1/z$, equation (13) thins out to

$$r M = M_A \left( \frac{z}{r (1 + M)} \right), \quad (14)$$

which will be strongly exploited below. Let us mention that these equalities (13) and (14) have been derived through other, more tedious, techniques (the planar Feynman-diagrammatic expansion, the replica trick) in [14]–[18].

3. VARMA from FRV

In what follows, we will assume that the VMA($q$), VAR($q$) or VARMA($q_1, q_2$) stochastic processes are covariance (weak) stationary. For details, we refer to [19]. It implies certain restrictions on their parameters, but we will not bother about this issue in the current work. Another consequence is that the processes display some interesting features, such as invertibility.

For all this, we must in particular take both $N$ and $T$ large from the start, with their ratio $r \equiv N/T$ fixed (6). More precisely, we stretch the range of the $a$-index from minus to plus infinity. This means that all the finite-size effects (appearing at the ends of the time series) are readily disregarded. In particular, there is no need to care about initial conditions for the processes, and all the recurrence relations are assumed to continue to the infinite past.
3.1. The VMA(q) process

3.1.1. The definition of VMA(q). We consider a situation when \( N \) stochastic variables evolve according to identical independent VMA(q) processes, which we sample over \( T \) consecutive moments of time. This is a simple generalization of the standard univariate weak-stationary moving average MA(q). In such a setting, the value \( Y_{ia} \) of the \( i \)th \((i = 1, \ldots, N)\) random variable at time moment \( a \)(\(a = 1, \ldots, T\)) can be expressed as

\[
Y_{ia} = \sum_{a=0}^{q} a_a \epsilon_{i,a-a}. \tag{15}
\]

Here, all the \( \epsilon_{ia} \)'s are IID standard (mean zero, variance one) Gaussian random numbers (white noise), \( \langle \epsilon_{ia} \epsilon_{jb} \rangle = \delta_{ij} \delta_{ab} \). The \( a_a \)'s are some \((q + 1)\) real constants; importantly, they do not depend on the index \( i \), which reflects the fact that the processes are identical and independent (no ‘spatial’ covariances among the variables). The rank \( q \) of the process is a positive integer.

3.1.2. The auto-covariance matrix. In order to handle such a process (15), note that the \( Y_{ia} \)'s, being linear combinations of uncorrelated Gaussian numbers, must also be Gaussian random variables, albeit displaying some correlations. Therefore, to fully characterize these variables, it is sufficient to calculate their two-point covariance function. This is straightforwardly done (see appendix A.1 for details),

\[
\langle Y_{ia} Y_{jb} \rangle = \delta_{ij} A_{ab}^{(1)}, \tag{16}
\]

where

\[
A_{ab}^{(1)} = \kappa_{0}^{(1)} \delta_{ab} + \sum_{d=1}^{q} \kappa_{d}^{(1)} (\delta_{a,b-d} + \delta_{a,b+d}), \quad \text{with} \quad \kappa_{d}^{(1)} \equiv \sum_{a=0}^{q-d} a_a a_{a+d}, \quad d = 0, 1, \ldots, q. \tag{17}
\]

In other words, the cross-covariance matrix is trivial, \( C = I_N \) (no correlations between different variables), while the auto-covariance matrix \( A^{(1)} \), responsible for temporal correlations, can be called ‘\((2q + 1)\)-diagonal’. In the course of this paper, we will use several different auto-covariance matrices, and, for brevity, we decide to label them with superscripts. Their definitions are all collected in appendix A.2.

For example, in the simplest case of VMA(1), it is tri-diagonal,

\[
A_{ab}^{(1)} = (a_0^2 + a_1^2) \delta_{ab} + a_0 a_1 (\delta_{a,b-1} + \delta_{a,b+1}). \tag{18}
\]

3.1.3. The Fourier transform and the M-transform of the auto-covariance matrix. Such an infinite matrix (17) is translationally invariant (as stated, it is one of the implications of the weak stationarity), i.e. the value of any of its entries depends only on the distance between its indices, \( A_{ab}^{(1)} = A^{(1)}(a - b) \); specifically, \( A^{(1)}(\pm d) = \kappa_{d}^{(1)} \), for \( d = 0, 1, \ldots, q \), and \( A^{(1)}(|d| > q) = 0 \). Hence, it is convenient to rewrite this matrix in the Fourier space,

\[
\hat{A}^{(1)}(p) \equiv \sum_{d \in \mathbb{Z}} e^{ipd} A^{(1)}(d) = \kappa_{0}^{(1)} + 2 \sum_{d=1}^{q} \kappa_{d}^{(1)} \cos(dp). \tag{19}
\]

In this representation, the \( M \)-transform of \( A^{(1)} \) is readily obtained [10],

\[
M_{A^{(1)}}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp \frac{\hat{A}^{(1)}(p)}{z - A^{(1)}(p)}. \tag{20}
\]
This integral can be evaluated by the method of residues for any value of $q$, which we do in appendix A.3, where we also give the general result (A.3). In particular, for $q = 1$,

$$M_A(z) = \frac{z}{\sqrt{z - (a_0 + a_1)^2} \sqrt{z - (a_0 - a_1)^2}} - 1,$$

where the square roots are principal.

### 3.1.4. The Pearson estimator of the covariances from FRV

We will be interested in investigating the spectral properties of the Pearson estimator $\epsilon = (1/T)YY^T = (1/T)\tilde{Y}A^{(1)}\tilde{Y}^T$ (4). The $M$-transform of this correlated Wishart random matrix, $M \equiv M_c(z)$, can be retrieved from equation (14). We could write it for any $q$ using (A.3), but we will restrict ourselves only to $q = 1$, in which case the substitution of (21) leads to a fourth-order polynomial (Ferrari) equation for the unknown $M$,

$$r^4(a_0^2 - a_1^2)^2M^4 + 2r^3(-a_0^2 + a_1^2)z + (a_0^2 - a_1^2)^2(r + 1)M^3$$

$$+ 2r^2(z^2 - 2(a_0^2 + a_1^2)(r + 2)z + (a_0^2 - a_1^2)^2(2r^2 + 4r + 1))M^2$$

$$+ 2r(z^2 - (a_0^2 + a_1^2)(2r + 1)z + (a_0^2 - a_1^2)^2r(r + 1))M$$

$$+ r(-2(a_0^2 + a_1^2)z + (a_0^2 - a_1^2)^2r) = 0. \quad (22)$$

The FRV technique allowed us, therefore, to find this equation in a matter of a few lines of a simple algebraic computation. It has already been derived in [6], and (22) may be verified to coincide with the version given in that paper. In [6], the pertinent equation is printed before (A.6), and to compare the two, one needs to change their variables into ours according to $y \rightarrow 1/r$, $x \rightarrow z/r$ and $m \rightarrow -r(1 + M)/z$. The last equality means that $m$ and $m$ of [6] correspond in our language to the Green’s functions $-rG_c(z)$ and $-G_a(z/r)$, respectively, where $a = (1/N)Y^TY$ is the Pearson estimator dual to $c$. As mentioned, a quick extension to the case of arbitrary $q$ is possible. However, the resulting equations for $M$ will be significantly more complicated. For instance, for $q = 2$, a lengthy ninth-order polynomial equation is discovered.

### 3.2. The VAR($q$) process

#### 3.2.1. The definition of VAR($q$)

A setup of $N$ identical and independent VAR($q$) (vector autoregressive) processes is somewhat akin to (15), i.e. we consider $N$ decoupled copies of a standard univariate AR($q$) process,

$$Y_{ia} - \sum_{\beta=1}^q b_\beta Y_{i,a-\beta} = a_0 \epsilon_{ia}. \quad (23)$$

It is again described by the demeaned and standardized Gaussian white noise $\epsilon_{ia}$ (which triggers the stochastic evolution), as well as $(q + 1)$ real constants $a_0$, $b_\beta$, with $\beta = 1, \ldots, q$. As stated earlier, the time stretches to the past infinity, so no initial condition is necessary. Although at first sight equation (23) may appear to be a more involved recurrence relation for the $Y_{ia}$s, it is actually easily reduced to the VMA($q$) case: it remains to be remarked that if one exchanges the $Y_{ia}$s with the $\epsilon_{ia}$s, one precisely arrives at the VMA($q$) process with the constants $a_0^{(2)} \equiv 1/a_0$, $a_\beta^{(2)} \equiv -b_\beta/a_0$, $\beta = 1, \ldots, q$. In other words, the auto-covariance matrix $A^{(3)}$ of the VAR($q$)
process (23) is simply the inverse of the auto-covariance matrix \( A^{(2)} \) of the corresponding VMA\((q)\) process with the described modification of the parameters,

\[
A^{(3)} = (A^{(2)})^{-1}.
\]

(24)

This inverse exists thanks to the weak stationarity supposition.

3.2.2. The Fourier transform and the M-transform of the auto-covariance matrix. The Fourier transform of the auto-covariance matrix \( A^{(3)} \) of VAR\((q)\) is therefore a (number) inverse of its counterpart for VMA\((q)\) with its parameters appropriately changed,

\[
\widehat{A}^{(3)}(p) = \frac{1}{A^{(2)}(p)} = \frac{1}{\kappa_0^{(2)} + 2 \sum_{d=1}^{q} \kappa_d^{(2)} \cos(dp)},
\]

(25)

where

\[
\kappa_d^{(2)} = \frac{1}{a_0^2} \sum_{a=0}^{q-d} b_a b_{a+d}, \quad d = 0, 1, \ldots, q,
\]

(26)

and where we define \( b_0 \equiv -1 \).

In order to find the M-transform of the inverse matrix, \( A^{(3)} = (A^{(2)})^{-1} \), one employs a general result, true for any (real symmetric) random matrix \( H \), and obtainable through an easy algebra,

\[
M^{-1}(z) = -M\left(\frac{1}{z}\right) - 1.
\]

(27)

Since the quantity \( M_{A^{(2)}}(z) \) is known for any \( q \) (A.3), so is \( M_{A^{(3)}}(z) \) via (27), but we will not present it explicitly. Let us just give it for \( q = 1 \), in which case equations (27) and (21) yield

\[
M_{A^{(3)}}(z) = -\frac{1}{\sqrt{1 - \frac{(1-b_1)^2}{a_0^2} z} \sqrt{1 - \frac{(1+b_1)^2}{a_0^2} z}}.
\]

(28)

3.2.3. The auto-covariance matrix. Despite being somewhat outside the main line of thought of this article, an interesting question would be to search for an explicit expression for the auto-covariance matrix \( A^{(3)} \) from its Fourier transform (25),

\[
A^{(3)}(d) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp \frac{1}{e^{-idp} \kappa_0^{(2)} + 2 \sum_{l=1}^{q} \kappa_l^{(2)} \cos(lp)},
\]

(29)

where we exploited the fact that \( A^{(3)} \) must be translationally invariant, \( A^{(3)}_{ab} = A^{(3)}(a-b) \). This computation would shed light on the structure of temporal correlations present in a VAR setting.

This integral is evaluated by the method of residues in a very similar manner to the one shown in appendix A.3, and we do this in appendix A.4. We discover that the auto-covariance matrix is a sum of \( q \) exponential decays,

\[
A^{(3)}(d) = \sum_{\gamma=1}^{q} C_{\gamma} e^{-|d|/T_{\gamma}},
\]

(30)

where \( C_{\gamma} \) are constants, and \( T_{\gamma} \) are the characteristic times (A.5), \( \gamma = 1, \ldots, q \). These constituents are given explicitly in (A.4). This is a well-known fact. Nevertheless, we wanted to establish it again within our approach.
For example, for \( q = 1 \), the auto-covariance matrix of VAR(1) is one exponential decay,

\[
A^{(3)}(d) = \frac{\alpha_0^2}{1 - b_1^2} b_1^{[d]},
\]

where we assumed, for simplicity, \( 0 < b_1 < 1 \) (the formula can easily be extended to all values of \( b_1 \)).

### 3.2.4. The Pearson estimator of the covariances from FRV

Having found an expression for the \( M \)-transform of the auto-covariance matrix \( A^{(3)} \) of a VAR(\( q \)) (27), (A.3), we may proceed to investigate equation (14) for the \( M \)-transform \( M \equiv M_c(z) \) of the correlated Wishart random matrix \( c = (1/T)YY^T = (1/T)\tilde{Y}A^{(3)}\tilde{Y}^T \) (4). We will do this explicitly only for \( q = 1 \), when (28) leads to a fourth-order (Ferrari) polynomial equation for the unknown \( M \),

\[
a_0^4 r^2 M^4 + 2a_0^2 r(-1 + b_1^2)z + a_0^2 r M^3 + ((1 - b_1^2)^2 z^2 - 2a_0^2 r(1 + b_1^2)z^4 + (r^2 - 1)a_0^4 M^2 - 2a_0^4 M - a_0^4 = 0.
\]

This equation has been derived by another method in [6], and our result confirms their equation (A.8), with the change in notation, \( y \rightarrow 1/r, x \rightarrow z/r, \tilde{z} \rightarrow rM \).

### 3.3. The VARMA(\( q_1, q_2 \)) process

#### 3.3.1. The definition of VARMA(\( q_1, q_2 \))

The two types of processes that we elaborated on above, VAR(\( q_1 \)) and VMA(\( q_2 \)), can be combined into one stochastic process called VARMA(\( q_1, q_2 \)),

\[
Y_{\alpha} - \sum_{\beta=1}^{q_1} b_\beta Y_{\alpha-\beta} = \sum_{\alpha=0}^{q_2} a_\alpha \epsilon_{\alpha-\alpha}.
\]

Now, it is a straightforward and well-known observation (which can be verified by a direct calculation) that the auto-covariance matrix \( A^{(5)} \) of this process is simply the product (in any order) of the auto-covariance matrices of the VAR and VMA pieces. More precisely,

\[
A^{(5)} = (A^{(4)})^{-1} A^{(1)},
\]

where \( A^{(1)} \) corresponds to the generic VMA(\( q_2 \)) model (17), while \( A^{(4)} \) denotes the auto-covariance matrix of VMA(\( q_1 \)) with a slightly different modification of the parameters compared with the previously used, namely \( a_0^{(4)} \equiv 1, a_\beta^{(4)} \equiv -b_\beta \), for \( \beta = 1, \ldots, q_1 \). We have thus already made use here of the fact that the auto-covariance matrix of a VAR process is the inverse of the auto-covariance matrix of a certain corresponding VMA process (24), but the new change in parameters necessary in moving from VAR to VMA has effectively \( a_0 = 1 \) with respect to what we had before (24). It is understandable: this ‘missing’ \( a_0 \) is now included in the matrix of the other VMA(\( q_2 \)) process.

#### 3.3.2. The Fourier transform and the \( M \)-transform of the auto-covariance matrix

The Fourier transform of the auto-covariance matrix \( A^{(5)} \) of VARMA(\( q_1, q_2 \)) (34) is simply the product of the respective Fourier transforms (19) and (25),

\[
\tilde{A}^{(5)}(p) = \frac{\kappa_0^{(1)} + 2 \sum_{d_1=1}^{q_1} \kappa_{d_1}^{(1)} \cos (d_1 p)}{\kappa_0^{(4)} + 2 \sum_{d_1=1}^{q_1} \kappa_{d_1}^{(4)} \cos (d_1 p)} .
\]
where
\[
\kappa_{a_1}^{(4)}(q_1) = \sum_{a_2=0}^{q_1-d_1} b_{a_1} b_{a_1+d_1}, \quad \kappa_{a_2}^{(1)}(q_2) = \sum_{a_2=0}^{q_2-d_2} a_{a_2} a_{a_2+d_2}, \quad d_1 = 0, 1, \ldots, q_1, \quad d_2 = 0, 1, \ldots, q_2,
\]
(36)
where we recall \(b_0 = -1\). For instance, for VARMA(1, 1) (it is described by three constants, \(a_0\), \(a_1\) and \(b_1\)), one explicitly has
\[
\widehat{A}^{(5)}(p) = \frac{a_0^2 + a_1^2 + 2a_0a_1 \cos p}{1 + b_1^2 - 2b_1 \cos p}
\]
(37)
The \(M\)-transform of \(A^{(5)}\) can consequently be derived from the general formula (20). We will evaluate here the pertinent integral only for the simplest VARMA(1, 1) process, even though an arbitrary case may be handled by the technique of residues,
\[
M_{A^{(5)}}(z) = \frac{1}{a_0a_1 + b_1z} \left( -a_0a_1 + \frac{z(a_0a_1 + (a_0^2 + a_1^2)b_1 + a_0a_1b_1^2)}{\sqrt{(1 - b_1)^2z - (a_0 + a_1)^2(1 + b_1)^2z - (a_0 - a_1)^2}} \right)
\]
(38)
3.3.3. The auto-covariance matrix. One might again attempt to track the structure of temporal covariances in a VARMA process. This can be done either by the inverse Fourier transform of (35) or through a direct computation based on the recurrence relation (33) (importantly, adhering to the assumption that it stretches to the past infinity). Let us give the result just for VARMA(1, 1),
\[
A^{(5)}(d) = -\frac{a_0a_1}{b_1} \delta_{d,0} + \frac{(a_1 + a_0b_1)(a_0 + a_1b_1)}{b_1(1 - b_1^2)} b_1^{d-1},
\]
(39)
where, for simplicity, \(0 < b_1 < 1\). This is an exponential decay, with the characteristic time of the VAR piece, with an additional term on the diagonal.

3.3.4. The Pearson estimator of the covariances from FRV. Expression (38), along with the fundamental FRV formula (14), allows us to write the equation satisfied by the \(M\)-transform \(M \equiv M_c(z)\) of the Pearson estimator \(c = (1/T) YY^T = (1/T) \tilde{Y} A^{(5)} \tilde{Y}^T\) (4) of the cross-covariances in the VARMA(1, 1) process. It happens to be polynomial of order six, and we present it as (A.6) in appendix A.5. It may be solved numerically, a proper solution chosen (the one which leads to a sensible density: real, positive-definite, normalized to unity) and, finally, the mean spectral density \(\rho_c(\lambda)\) derived from (9). We show the shapes of this density for a variety of the values of the parameters \(r\), \(a_0\), \(a_1\) and \(b_1\) in figure 1. Moreover, in order to test the result (A.6) and, more broadly, to further establish our FRV framework in the guise of formula (14), the theoretical form of the density is compared with Monte Carlo simulations in figure 2. They remain in excellent concord. These are the main findings of this paper.

3.3.5. An application to macroeconomic data. In this paragraph, we further pursue the above analysis of the VARMA(1, 1) model on a concrete example of real data. Namely, we investigate \(N = 52\) various macroeconomic time series for Poland of length \(T = 118\). They have been selected on a monthly basis in such a manner so as to cover most of the main sectors of the Polish economy, i.e. the money market, domestic and foreign trade, labor market, balance of payments, inflation in different sectors, etc. The full list can be gained from the authors upon
Figure 1. The mean spectral density $\rho_c(\lambda)$ of the Pearson estimator $c$ of the cross-covariances in the VARMA($1, 1$) process computed numerically from the sixth-order polynomial equation (A.6), for various values of the process’ parameters. The scale of these parameters is determined by choosing $a_0 = 1$ everywhere. Recall that the theoretical formula (A.6) is valid in the thermodynamical limit ($N, T \to \infty$, with $r = N/T$ kept finite).

Top left: we set the remaining VARMA parameters to $a_1 = 0.3$, $b_1 = 0.2$, while the rectangularity ratio takes the values $r = 0.5$ (the purple line), 0.1 (red), 0.02 (magenta) and 0.004 (pink); each one is 5 times smaller than the preceding one. We observe how the graphs become increasingly peaked (narrower and taller) around $\lambda = 1$ as $r$ decreases, which reflects the movement of the estimator $c$ toward its underlying value $C = I_N$.

Top right: we set $r = 0.25$ and consider the two VARMA parameters as equal to each other, with the values $a_1 = b_1 = 0.6$ (purple), 0.4 (red), 0.2 (magenta) and 0.01 (pink).

Bottom left: we set $r = 0.25$ and $b_1 = 0.2$, and modify $a_1 = 0.6$ (purple), 0.4 (red), 0.2 (magenta) and 0.0 (pink); for this last value, the VARMA($1, 1$) model reduces to VAR(1).

Bottom right: similarly to the above, but this time we assign $r = 0.25$ and $a_1 = 0.2$, while changing $b_1 = 0.6$ (purple), 0.4 (red), 0.2 (magenta) and 0.0 (pink); this last value corresponds to VMA(1).
Figure 2. Monte Carlo simulations of the mean spectral density $\rho_c(\lambda)$ (the green plots) compared to the theoretical result obtained numerically from the sixth-order equation (A.6) (the dashed red lines). The conformity is nearly perfect. We generate the matrices $Y$ of sizes $N = 50$, $T = 200$ (i.e. $r = 0.25$) from the VARMA$(1, 1)$ process with the parameters $a_0 = 1$, $a_1 = 0.3$, $b_1 = 0.2$. The Monte Carlo simulation is repeated 1000 (left) or 10,000 (right) times; in this latter case, a significant improvement in the quality of the agreement is seen. One notes the finite-size effects at the edges of the spectrum (‘leaking out’ of eigenvalues); in the numerical simulations, $N$ and $T$ are obviously finite, while equation (A.6) is legitimate in the thermodynamical limit (6) only, hence the small discrepancies; by enlarging the chosen dimensions $50 \times 200$ one would diminish this fallout.

request). Although longer time series for Poland are accessible, we restrict ourselves to the last 10 years in order to avoid the effects of structural change. We assume that each economic variable is affected by the same shock (i.e. the ‘global market shock’) of an ARMA$(1, 1)$ type with unknown parameters, which we are to estimate. The AR part implies that the shock dies quite quickly, while the MA part is responsible for the persistency of the shock. To preserve the proper VARMA representation, the original time series have been normalized and the seasonal component has been removed. In figure 3, we plot these time series (left), and we make (right) a histogram of the mean spectral density $\rho_c(\lambda)$, which we compare to a theoretical prediction from our FRV equation (A.6) with the estimated values of the parameters $a_0$, $a_1$ and $b_1$. Perfect agreement is found.

4. Conclusions

In this paper, we attempted to advertise the power and flexibility of the FRV calculus for multivariate stochastic processes of the VARMA type. The FRV calculus is ideally suited to multidimensional time series problems, provided that the dimensions of the underlying matrices are large. The operational procedures are simple, algebraic and transparent. The structure of the final formula, which relates the moments’ generating function of the population covariance and the sample covariance, allows one to easily derive eigenvalue density of the sample covariance. We illustrated in detail how this procedure works for VARMA$(1, 1)$, confronted the theoretical prediction with numerical data obtained by Monte Carlo simulations of the VARMA process and observed perfect agreement.

The FRV calculus is not restricted to Gaussian variables. It also works for non-Gaussian processes, including those with heavy-tailed increments belonging to the Lévy basin of attraction, where the moments do not exist. Since the majority of data collected nowadays are naturally stored in the form of huge matrices, we believe that the FRV technique is the most natural candidate for the ‘matrix-valued probability calculus’ that can provide efficient algorithms for cleaning (de-noising) large sets of data and unraveling essential but hidden correlations.

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Appendix

A.1. The auto-covariance matrix for VMA(q)

In this appendix, we sketch a proof of formula (17) for the auto-covariance matrix of the VMA(q) process. As mentioned, since the random variables are centered Gaussian, this matrix alone suffices to completely capture all their properties. We set $i = j$. The dependence on this index may be dropped as there are no correlations here. We use definition (15) of VMA(q), as
well as the auto-covariance structure of the white noise, \(<\epsilon_{i_a}\epsilon_{j_b}) = \delta_{i_j}\delta_{a_b}\). This leads to

\[
A_{ab}^{(1)} = (Y_{i_a}Y_{j_b}) = \sum_{\alpha=0}^{q} \sum_{\beta=0}^{q} a_\alpha a_\beta \langle \epsilon_{i_{\alpha-a}} \epsilon_{i_{1-b-\beta}} \rangle = \sum_{\alpha=0}^{q} \sum_{\beta=0}^{q} a_\alpha a_\beta \delta_{\alpha-a, \beta-b} = \cdots.
\]

The double sum is symmetrized, the index \(\beta\) replaced by \(d \equiv \beta - \alpha\),

\[
\cdots = \frac{1}{2} \sum_{\alpha=0}^{q} \sum_{\beta=-\alpha}^{\alpha} a_\alpha a_\beta (\delta_{b,a+d} + \delta_{b,a-d}) = \cdots,
\]

and the order of the sums interchanged (an elegant method for which is explained in [20]),

\[
\cdots = \frac{1}{2} \sum_{d=-q}^{q} q \min(0,d) a_\alpha a_{\alpha+d} (\delta_{b,a+d} + \delta_{b,a-d}),
\]

which, upon splitting the sum over \(d\) into three pieces (from \(-q\) to \(-1\), \(d = 0\), and from \(1\) to \(q\)), is quickly seen to coincide with (17).

### A.2. A list of the various auto-covariance matrices used

For the reader’s convenience, let us collect in this appendix the five auto-covariance matrices that are defined throughout this paper:

- **By A\(^{(1)}\)** we denote the auto-covariance matrix of the VMA\((q)\) process with the generic constants \(a_\alpha\), with \(\alpha = 0, 1, \ldots, q\), as defined in (15).

- **By A\(^{(2)}\)** we denote the auto-covariance matrix of the VMA\((q)\) process with the constants \(a_0^{(2)} = 1/a_0\), \(a_\beta^{(2)} = -b_\beta/a_0\), where \(\beta = 1, \ldots, q\).

- **By A\(^{(3)}\)** we denote the auto-covariance matrix of the VAR\((q)\) process with the generic constants \(a_0, b_\beta\), with \(\beta = 1, \ldots, q\), as defined in (23). There holds \(A^{(3)} = (A^{(2)})^{-1}\) (24).

- **By A\(^{(4)}\)** we denote the auto-covariance matrix of the VMA\((q_1)\) process with the constants \(a_0^{(4)} = 1\), \(a_\beta^{(4)} = -b_\beta\), where \(\beta = 1, \ldots, q_1\).

- **By A\(^{(5)}\)** we denote the auto-covariance matrix of the VARMA\((q_1, q_2)\) process with the generic constants \(b_\beta, a_\alpha\), with \(\beta = 1, \ldots, q_1\) and \(\alpha = 0, 1, \ldots, q_2\), according to the definition (33). There is \(A^{(5)} = (A^{(4)})^{-1}A^{(1)}\) (34), where in the latter piece \(q = q_2\).

### A.3. The M-transform of the auto-covariance matrix for VMA\((q)\)

We will derive here the \(M\)-transform (20) of the auto-covariance matrix \(A^{(1)}\) of an arbitrary VMA\((q)\) process, using the expression for its Fourier transform (19). It is a little simpler to consider the Green’s function,

\[
G_{A^{(1)}}(z) = \frac{1 + MA^{(1)}(z)}{z} = \frac{1}{\pi} \int_{0}^{\pi} dp \frac{1}{z - A^{(1)}(p)}, \quad (A.1)
\]

where the integration range has been halved due to the evenness of the integrand.
This integral is performed with help of the change of variables \( y \equiv 2 \cos p \). The measure, when \( p \in [0, \pi] \), reads \( dp = -dy / \sqrt{4 - y^2} \). A basic observation is that the denominator of the integrand is a linear combination of \( \cos(dp) \), for \( d = 1, \ldots, q \), and each such cosine can be cast as a polynomial of order \( d \) in \( y \) through the de Moivre formula. Hence, the denominator is a polynomial of order \( q \) in \( y \),

\[
A^{(1)}(p) - z = \kappa_0^{(1)} - z + 2 \sum_{d=1}^{q} \kappa_d^{(1)} \cos(dp) = \psi \prod_{\beta=1}^{q} (y - y_{\beta}),
\]  

(A.2)

where the \( y_{\beta} \)s are the \( q \) roots (which we assume to be single), and \( \psi \) is the coefficient at \( y^q \). Using the method of residues, one readily finds

\[
G_{A^{(1)}}(z) = -\frac{1}{\pi} \psi \int_{-2}^{2} \frac{dy}{\sqrt{4 - q^2 - 4y^2}} \prod_{\beta=1}^{q} (y - y_{\beta}) = \frac{1}{\psi} \sum_{\gamma=1}^{q} \prod_{\beta \neq \gamma}^{q} \frac{1}{y_{\gamma} - y_{\beta}} \sqrt{y_{\gamma} - 2\sqrt{y_{\gamma} + 2}},
\]

(A.3)

where the two square roots on the rhs are principal. This is an explicit formula for the Green’s function of \( A^{(1)} \), provided that one has factorized the order-\( q \) polynomial (A.2).

A.4. The auto-covariance matrix for \( \text{VAR}(q) \)

Let us argue now that the Fourier transform (25) leads to the auto-covariance matrix of \( \text{VAR}(q) \) (29) of the form of a sum of exponential decays (30), and let us give precise expressions for the constants \( C_{\gamma} \) and the characteristic times \( T_{\gamma}, \gamma = 1, \ldots, q \).

We proceed by the technique of residues, analogously to appendix A.3, but this time with the aid of another variable, \( x = e^{-ip} \), related to the one previously used through \( y = 2 \cos p = x + 1/x \). The integration measure is \( dp = idx / x \), and the integration path is counterclockwise around the centered unit circle. The denominator of the integrand is a polynomial of order \( q \) in the variable \( y \), having thus some \( q \) roots \( \tilde{y}_{\beta}, \beta = 1, \ldots, q \). Therefore, there are \( 2q \) corresponding solutions for the variable \( x \), with half of them inside the integration path and half outside. Let \( \tilde{x}_{\beta} \) be the solutions to \( x + 1/x = \tilde{y}_{\beta} \) with the absolute values less than 1. Only they contribute to the integral, and their residues straightforwardly give

\[
A^{(3)}(d) = \frac{1}{\psi} \sum_{\gamma=1}^{q} \prod_{\beta \neq \gamma}^{q} \frac{(\tilde{x}_{\gamma})^{d|\gamma| - 1}}{(\tilde{x}_{\gamma} - \tilde{x}_{\beta}) \prod_{\beta=1}^{q} (\tilde{x}_{\gamma} - \frac{1}{\tilde{x}_{\beta}})}.
\]

(A.4)

This is indeed \( q \) exponents \( (\tilde{x}_{\gamma})^{d|\gamma|}, \gamma = 1, \ldots, q \). Note that the solutions may be complex. Hence, this is really \( q \) different exponential decays \( \exp(-|d|/T_{\gamma}) \), with the characteristic times

\[
T_{\gamma} \equiv -\frac{1}{\log |\tilde{x}_{\gamma}|}
\]

(A.5)

(these times are positive as the roots have absolute values of less than 1), possibly modulated by sinusoidal oscillations when a root has an imaginary part.

For example, for \( q = 1 \) there is one exponential decay (31), while for \( q = 2 \) one obtains either two exponential decays (the two roots are real and different) or one exponential decay modulated by oscillations (the two roots are complex and mutually conjugate), etc.
A.5. The equation for the M-transform of the Pearson estimator of the covariances for VARMA(1, 1)

The sixth-order polynomial equation obeyed by $M \equiv M_6(z)$ in the case of VARMA(1, 1) reads

$$
M = a_0 a_1^2 (a_0^2 - a_1^2)^2 M^4 + 2 r^3 a_0 a_1 (a_0^4 - 6 a_0^2 a_1^2 + a_1^4) b_1 + a_0 a_1 (a_0^2 + a_1^2) (b_1^2 + 1) z
$$

$$
+ (1 + 2 r) a_0 a_1 (a_0^2 - a_1^2)^2 M^5 + r^2 ((a_0^4 - 20 a_0^2 a_1^2 + a_1^4) b_1 - 4 a_0 a_1 (a_0^2 + a_1^2) \times b_1 (b_1^2 + 1) + a_0^2 a_1^2 (b_1^4 + 1)) z^2 + 2 a_0 a_1 ((1 + 3 r) (a_0^4 + a_1^4) - 2 (5 + 9 r) a_0^2 a_1^2) b_1
$$

$$
- (2 + 3 r) a_0 a_1 (a_0^2 + a_1^2) (b_1^2 + 1) z + (1 + 8 r + 6 r^2) a_0^2 a_1^2 (a_0^2 - a_1^2)^2 M^4
$$

$$
+ 2 r (b_1 (-6 a_0 a_1 b_1 - (a_0^2 + a_1^2) b_1 (b_1^2 + 1) + a_0 a_1 (b_1^4 + 1)) z^3
$$

$$
+ ((-10 (1 + 2 r) a_0^2 a_1^2 + r (a_0^4 + a_1^4)) b_1^2 - 2 (1 + 2 r) a_0 a_1 (a_0^2 + a_1^2) b_1 (b_1^2 + 1)
$$

$$
+ (1 + r) a_0^2 a_1^2 (b_1^4 + 1)) z^2 + a_0 a_1 ((3 r (1 + r) (a_0^4 + a_1^4) - 2 (2 + 15 r + 9 r^2) a_0^2 a_1^2) b_1
$$

$$
- (1 + 6 r + 3 r^3) a_0 a_1 (a_0^2 + a_1^2) (b_1^2 + 1)) z + 2 r (1 + 3 r + r^2) a_0^2 a_1^2 (a_0^2 - a_1^2)^2 M^3
$$

$$
+ (b_1^2 (1 - b_1^2)^2 z^4 + 2 b_1 (-2 (1 + 3 r) a_0 a_1 b_1^2 - r (a_0^2 + a_1^2) b_1 (b_1^2 + 1) + (1 + r)
$$

$$
\times a_0 a_1 (b_1^4 + 1) z^3 + (-((-1 - r^2) (a_0^4 + a_1^4) + 2 (3 + 20 r + 10 r^2) a_0^2 a_1^2) b_1^2
$$

$$
- 2 (1 + 4 r + 2 r^2) a_0 a_1 (a_0^2 + a_1^2) b_1 (b_1^2 + 1) + r (4 + r) a_0^2 a_1^2 (b_1^2 + 1)) z^2
$$

$$
+ 2 r a_0 a_1 ((r (3 + r) (a_0^4 + a_1^4) - 6 (2 + 5 r + r^2) a_0^2 a_1^2) b_1 - (3 + 6 r + r^2)
$$

$$
\times a_0 a_1 (a_0^2 + a_1^2) (b_1^2 + 1)) z + r^2 (6 + 8 r + r^2) a_0^2 a_1^2 (a_0^2 - a_1^2)^2) M^2
$$

$$
+ 2 (a_0 a_1 b_1 (1 - b_1^2)^2 z^3 + ((a_0^4 + a_1^4) + 2 (3 + 5 r) a_0^2 a_1^2) b_1^2 - 2 (1 + r)
$$

$$
\times a_0 a_1 (a_0^2 + a_1^2) b_1 (b_1^2 + 1) + r a_0 a_1 (b_1^4 + 1)) z^2 + r a_0 a_1 ((r (a_0^4 + a_1^4) - 2 (6 + 5 r)
$$

$$
\times a_0 a_1 (a_0^2 + a_1^2) b_1 - (3 + 2 r) a_0 a_1 (a_0^2 + a_1^2) (b_1^2 + 1)) z + r^2 (2 + r) a_0^2 a_1^2 (a_0^2 - a_1^2) M
$$

$$
- b_1 ((a_0^4 + 6 a_0^2 a_1^2 + a_1^4) b_1 + 2 a_0 a_1 (a_0^2 + a_1^2) (b_1^2 + 1)) z^2
$$

$$
+ 2 r a_0^2 a_1^2 (4 a_0 a_1 b_1 + (a_0^2 + a_1^2) (b_1^2 + 1)) z + r^2 a_0^2 a_1^2 (a_0^2 - a_1^2)^2 = 0.
$$

(A.6)

This equation in the form of a Mathematica file can be obtained from the authors upon request.

References


