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## On van der Waals friction. II: Between atom and half-space

To cite this article: Gabriel Barton 2010 New J. Phys. 12113045

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# Corrigendum: On van der Waals friction. II: Between atom and half-space 

2010 New J. Phys. 12113045

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In the second term on the right-hand side of equation (4.10), the square brackets should be divided by $\left[\omega+\omega^{\prime}-\mathbf{u} \cdot\left(\mathbf{k}+\mathbf{k}^{\prime}\right)\right]^{2}$, and the second term inside the braces on the right-hand side of (4.12) by $\left[x+x^{\prime}-\mathbf{v} \cdot\left(\kappa+\kappa^{\prime}\right)\right]^{2}$. This rectifies a copying error in the manuscript.

Consequently, to overall order $v^{2}$ and by virtue of the delta functions, the last factor on the right-hand side of (5.5) should be replaced by $\delta(\mu-x) /\left(\mu+x^{\prime}\right)^{4}$, with evident corresponding changes in (5.6), (5.7) and (6.5); and (5.8) is replaced by

$$
P_{B} \simeq \frac{9 \hbar \alpha \beta^{2} u^{2}}{64 \zeta^{5}} \cdot \frac{\omega_{S} \Omega \gamma}{\left(\Omega+\omega_{S}\right)^{3}}
$$

The discrepancy (6.4) with the result $P_{S B}$ of Scheel and Buhmann (2009) then reduces to the purely numerical ratio $P_{S B} / P_{B}=16 / 3$. This removes any indications that it might stem from differences in handling line shapes, and leaves differences in handling polarizations as the only readily visible suspects.

I am indebted to Peter Milonni for spotting the error.

# New Journal of Physics 

# On van der Waals friction. II: Between atom and half-space 

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New Journal of Physics 12 (2010) 113045 (13pp)
Received 14 June 2010
Published 26 November 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/11/113045


#### Abstract

We calculate the frictional resistance experienced by an atom, modelled as a harmonic oscillator, moving with constant velocity $\mathbf{u}$ at a fixed distance $\zeta$ outside and parallel to the surface of a Drude-modelled half-space. Our method applies in the nonrelativistic/nonretarded/electrostatic regime, where $u$ is far below $c$, and $u / \zeta$ is far below any important natural frequency of the atom or of the material. For a dissipative (e.g. for an ohmic) half-space and for the low values of $u$ least unlikely to be of practical interest, this force is dominated by a term proportional to $u / \zeta^{8}$, found perturbatively to fourth order in the interaction between atom and half-space. It appears to depend rather sensitively on how line shapes are handled.


## Contents

1. Introduction ..... 2
2. The model ..... 3
3. First-order perturbation theory ..... 5
3.1. Attraction ..... 6
3.2. Drag ..... 6
4. Second-order perturbation theory ..... 6
4.1. Generalities ..... 6
4.2. Amplitude ..... 7
4.3. Power ..... 8
5. Slow atoms ..... 9
5.1. The component $P_{A}$ ..... 9
5.2. The component $P_{B}$ ..... 10
5.3. $P_{A}$ versus $P_{B}$ ..... 11
6. Comments ..... 11
6.1. Comparisons ..... 11
6.2. Preliminary implications regarding two half-spaces ..... 12
Acknowledgment ..... 12
Appendix. Nondispersive atom ..... 12
References ..... 13

## 1. Introduction

This is the second of two papers meant as readily accessible checks on some current theories of quantum friction between finitely separated bodies in uniform relative motion. Throughout, we take retardation effects to be negligible and the temperature to be zero. The first paper (Barton 2010a), to be cited as I, introduced the exercise, and dealt with the force between two atoms: a natural preliminary, although complicated by the fact that the component of the force parallel to the motion (call it the drag force) is both position-dependent and mostly reversible, with only a small contribution from true, irreversible, frictional dissipation. The present paper (II) is free of these complications. It deals with an atom, modelled as a harmonic oscillator, moving outside a Drude-modelled half-space. Here too one finds nonzero friction. The results are very similar but not identical to the appropriate limits of those recently reported by Scheel and Buhmann (2009). References to other works will be given later, in context.

Our nonretarded approximations are restricted to separations smaller than the dominant wavelengths $\Lambda$ characterizing atom or half-space: for instance, to distances smaller than $\Lambda_{\mathrm{p}}=c / \omega_{\mathrm{p}}$ in the case of plasmas with plasma frequency $\omega_{\mathrm{p}}$. Hence they can say nothing about extensions of Casimir or of Casimir-Polder theories from perfect to imperfect but nondispersive reflectors, e.g. to insulators with finite but frequency-independent refractive index: such models lack the inner regions where, realistically, photon-dominated long-distance forces modulate to the electrostatics-dominated short-distance van der Waals (VdW) forces. On the other hand, in the small-distance nonretarded regime corrections for low but finite temperatures
remain minor ${ }^{1}$, whereas at very long distances they can become dominant even when formally free of $c$.

We stress that for our very limited purposes one requires only elementary nonrelativistic quantum mechanics with explicit Hamiltonians. No appeal is needed to QED, to stress tensors or to Green's functions. In particular, we are careful to avoid recourse to the widely invoked Lifshitz theory, because it lacks an intelligible Hamiltonian and has invited untenable inferences more than once. (A remarkable rescue operation outlined by Philbin (2010) seems capable of validating many Lifshitz-derived results by constructing an explicit and unexceptionable Hamiltonian directly from the observable imaginary parts of any dielectric response functions in question.)

Section 2 spells out our Hamiltonians for atom and half-space, and for the interaction between them, using the Huttner-Barnett theory for absorptive materials (which for our purposes is equivalent to Philbin's). One needs to distinguish sharply between (i) materials that are dispersive but not truly (i.e. not irreversibly) absorptive, e.g. plasmas with finite natural frequencies $\omega_{\mathrm{p}}$ but no ohmic resistance; and (ii) materials that are both. Contrary to widespread folklore, both types behave causally, i.e. both obey appropriate Kramers-Kronig relations ${ }^{2}$. The significant difference is that the absorption lines of the former have zero width (as e.g. in (2.8) and (2.12) below), while the lines of the latter have finite width. In particular, one must not confuse nondissipative with nondispersive limits.

Because we rely on perturbation theory, we must exclude the resonant case where the natural frequencies of atom and half-space are nearly the same.

Section 3 recalls the standard second-order expression for the attractive force and explains why, unlike most VdW effects, the friction forces are dominated by contributions not from first-order but from second-order perturbation theory, i.e. by contributions not of second but of fourth order in the interaction Hamiltonian. Section 4 derives these forces via the powerdissipation $P$ calculated by second-order time-dependent perturbation theory, and shows that $P=P_{A}+P_{B}$ splits naturally into two parts: only $P_{A}$ survives for nondispersive atoms, but it is $P_{B}$ that turns out to dominate under most physically interesting conditions. Section 5 explicates the consequences for low speeds, such as those that one is most likely to meet in practice: the main result is given by (5.7) plus (5.8). Section 6 compares our conclusions with the appropriate limit of those reached by Scheel and Buhmann (2009) and draws some tentative (because only perturbative) inferences about the notorious and until recently still contentious problem of friction between two half-spaces ${ }^{3}$.

The appendix derives the frictional force on an atom treated as nondispersive ab initio (i.e. not only in some limit), hypothetical but remarkably easy to find.

## 2. The model

The atom is constrained to move with constant velocity $\mathbf{u}=u \hat{\mathbf{x}}$ parallel to a half-space $z<0$, at a fixed distance $\zeta$. We model it as three dynamically identical simple-harmonic oscillators mutually at right angles, with frequency $\Omega$, ignoring the interactions of the three directly with

[^0]each other. The zero-point energy is dropped, and the energy eigenstates are written as $|\mathbf{n}\rangle$, with eigenvalues $n \hbar \Omega$. The ground state is $|\mathbf{0}\rangle$; we abbreviate the first-excited states as $|\boldsymbol{\eta}\rangle$, where $\eta$ is a polarization vector, used mainly as a bookkeeping device, with unit component pointing along each oscillator. For instance, for a single oscillator perpendicular to the surface, one would have $\boldsymbol{\eta}=(0,0,1)$; for our isotropic three-dimensional (3D) oscillator $\boldsymbol{\eta}=\left(\boldsymbol{\eta}_{\|}, 1\right)=(1,1,1)$. In particular,
\[

$$
\begin{equation*}
\sum_{\eta} \eta_{i} \eta_{j}=\delta_{i j} . \tag{2.1}
\end{equation*}
$$

\]

The internal coordinate is $\mathbf{s}$. Recall the oscillator range parameter $b$, the electric dipole operator $\mathbf{D}=e \mathbf{s}$ and its matrix elements, and the zero-frequency polarizability $\alpha$ :
$\left.b=\sqrt{\hbar / 2 m \Omega}, \quad\langle\boldsymbol{\eta}| D_{j}|0\rangle=\eta_{j} e b, \quad \alpha=2\left|\langle 1| D_{1}\right| 0\right\rangle\left.\right|^{2} / \hbar \Omega=e^{2} / m \Omega^{2}$.
There is an obvious transcription to an atom with a dominant excitation energy $E \sim \hbar \Omega$ and arbitrary matrix elements $\langle i| D_{j}|0\rangle$. The nondispersive limit is $\Omega \rightarrow \infty$ and $\left.\left|\langle 1| D_{1}\right| 0\right\rangle\left.\right|^{2} \rightarrow \infty$ at fixed $\alpha$, reducing the polarizability to $\alpha$ at all frequencies.

We adopt the dipole approximation, which assumes that $b \ll \zeta$.
The dielectric function of the half-space is ${ }^{4,5}$

$$
\begin{equation*}
\varepsilon=1+\frac{\omega_{\mathrm{p}}^{2}}{\omega_{0}^{2}-\omega^{2}-\mathrm{i} \omega \Gamma}, \quad \omega_{\mathrm{S}}^{2}=\omega_{0}^{2}+\omega_{\mathrm{p}}^{2} / 2, \quad \beta^{2}=\frac{\omega_{\mathrm{p}}^{2}}{2 \omega_{\mathrm{S}}^{2}}, \quad \Gamma=\frac{\omega_{\mathrm{p}}^{2}}{4 \pi \sigma} . \tag{2.3}
\end{equation*}
$$

The Drude model for metals with conductivity $\sigma$ has $\omega_{0}=0$, with squared surface-plasmon frequency $\omega_{\mathrm{S}}^{2}=\omega_{\mathrm{p}}^{2} / 2$ and $\beta=1$. Perfect reflection $(|\varepsilon| \rightarrow \infty)$ would require $\omega_{\mathrm{p}}^{2} \gg \omega_{0}^{2} \gg \omega^{2} \gg$ $\omega \Gamma$, entailing $\omega_{\mathrm{S}}^{2} \gg \omega^{2} \gg \omega \Gamma$. Imperfect but nondispersive reflection (constant real $\varepsilon=n^{2}$ ) would require $\Gamma=0$ and $\omega_{0}^{2}, \omega_{\mathrm{p}}^{2} \gg \omega^{2}$ at fixed $\left(1+\omega_{\mathrm{p}}^{2} / \omega_{0}^{2}\right)=n^{2}$. We are not committed to either approximation. But we consider only weak dissipation, $\Gamma / \omega_{\mathrm{S}} \ll 1$; in metals, this ratio is typically $10^{-2}-10^{-3}$.

The resonant scenario $\left|\omega_{\mathrm{S}}-\Omega\right| / \Omega \ll 1$ is excluded, because it requires special treatment (Barton 1978).

Because we remain in the non-retarded regime (electrostatics, as if $c \rightarrow \infty$, i.e. $\left.\left(\omega_{\mathrm{S}}, \Omega\right) \zeta / c \ll 1\right)$ there are no photons: in the absence of the half-space, excited atoms would live forever. By the same token we consider only motions slow in the sense that $u / c \ll 1$.

To make dissipation $(\Gamma \neq 0)$ and thereby (2.3) amenable to standard Hamiltonian methods, we use the classic Huttner-Barnett model (see e.g. Huttner and Barnett (1992), Matloob et al (1995), Matloob and Loudon (1996), Barnett et al (1996)). The details needed for and adapted to our regime have been spelled out elsewhere (Barton (1997), (2000), where the symbol ' $\sigma$ ' is used for the ' $\beta$ ' here).

We work in the Schrödinger picture. The quantized potential outside $(z>0)$ reads

$$
\begin{equation*}
\Phi(x, y, z)=-\omega_{\mathrm{S}} \beta \int \mathrm{~d}^{2} k \int_{0}^{\infty} \mathrm{d} \omega \frac{g_{\omega}}{F\left(\omega, \omega_{\mathrm{S}}\right)} a_{\mathbf{k} \omega} \sqrt{\frac{\hbar}{4 \pi k \omega}} \exp \left[\mathrm{i}\left(k_{1} x+k_{2} y\right)-k z\right]+\text { H.c. } \tag{2.4}
\end{equation*}
$$

4 We use unrationalized Gaussian units. The frequency of the atomic oscillator, called $\omega$ in I , has already been relabelled as $\Omega$. Here, $\omega$ is merely a variable; $\omega_{0}$ is reserved for the frequency parameterizing the (nominally nonelectric) restoring forces in the material, i.e. for the frequency at which its polarization would oscillate in the absence of electric forces between its constituents (as if in the limit of extreme dilution).
5 The symbol $\beta$ here is unrelated to the ' $\beta$ ' in I .
where H.c. stands for the Hermitian conjugate, $\mathbf{k}=\left(k_{1}, k_{2}\right)=(k \cos \phi, k \sin \phi)$ and

$$
\begin{equation*}
g_{\omega}^{2}=2 \omega^{2} \Gamma / \pi, \quad F\left(\omega, \omega_{\mathrm{S}}\right)=\omega_{\mathrm{S}}^{2}-\omega^{2}-\mathrm{i} \omega \Gamma . \tag{2.5}
\end{equation*}
$$

To realize the non-dissipative ( $n d$ ) limit $\Gamma=0$ from the outset, one replaces (2.4) by $\Gamma=0: \quad \Phi_{\mathrm{nd}}(x, y, z)=-\beta \int \mathrm{d}^{2} k a_{\mathbf{k}} \sqrt{\frac{\hbar \omega_{\mathrm{S}}}{4 \pi k}} \exp \left[\mathrm{i}\left(k_{1} x+k_{2} y\right)-k z\right]+$ H.c.
It proves convenient to introduce dimensionless variables ${ }^{6}$

$$
\begin{equation*}
x \equiv \frac{\omega}{\omega_{\mathrm{S}}}, \quad \mu \equiv \frac{\Omega}{\omega_{\mathrm{S}}}, \quad c \equiv \frac{\Gamma}{\omega_{\mathrm{S}}}, \quad \mathbf{v} \equiv \frac{\mathbf{u}}{\zeta \omega_{\mathrm{S}}}, \quad \boldsymbol{\kappa} \equiv \mathbf{k} \zeta, \quad \boldsymbol{\kappa} \cdot \mathbf{v}=\frac{\mathbf{k} \cdot \mathbf{u}}{\omega_{\mathrm{S}}}, \tag{2.7}
\end{equation*}
$$

with $\mathbf{v}$ here analogous to the differently defined $\mathbf{v}$ in I. Although our general results emerge in a form valid for any value of $v$, those of the present paper will be implemented only for small $v \ll 1$, which fortunately are those least unlikely to matter in practice. Approximations for large $v$ are surprisingly onerous.

Much use will be made of

$$
\begin{align*}
\lim _{\Gamma / \omega_{\mathrm{S}} \rightarrow 0} \frac{\omega_{\mathrm{S}} g^{2}}{|F|^{2}} & =\lim _{\Gamma / \omega_{\mathrm{S}} \rightarrow 0} \frac{2 \omega_{\mathrm{S}} \omega^{2} \Gamma}{\left[\left(\omega_{\mathrm{S}}^{2}-\omega^{2}\right)^{2}+\omega^{2} \Gamma^{2}\right]}=\lim _{c \rightarrow 0} \frac{2 x^{2} c / \pi}{\left[\left(1-x^{2}\right)^{2}+x^{2} c^{2}\right]} \\
& =\lim _{c \rightarrow 0} \frac{2 c / \pi}{\left[\left(1-x^{2}\right)^{2}+c^{2}\right]}=\delta(1-x)=\omega_{\mathrm{S}} \delta\left(\omega_{\mathrm{S}}-\omega\right) . \tag{2.8}
\end{align*}
$$

Under $\int^{\infty} \mathrm{d} \omega \ldots$, the limit applies with test functions $f(\omega)$ for which $\int^{\infty} \mathrm{d} \omega f(\omega) / \omega^{2}$ converges.

The interaction Hamiltonian is

$$
\begin{equation*}
V=-\mathbf{D} \cdot \nabla \Phi(u t, 0, \zeta), \quad V_{\mathrm{nd}}=-\mathbf{D} \cdot \nabla \Phi_{\mathrm{nd}}(u t, 0, \zeta) . \tag{2.9}
\end{equation*}
$$

The Golden Rule gives the (polarization-dependent) decay rates of the first-excited atomic states as

$$
\begin{gather*}
\gamma_{\eta}=\frac{2 \pi}{\hbar} \omega_{\mathrm{S}}^{2} \beta^{2} \int \mathrm{~d}^{2} k \int_{0}^{\infty} \mathrm{d} \omega \frac{g^{2}}{|F|^{2}} \cdot \frac{\hbar}{4 \pi k \omega}\left[(\boldsymbol{\eta} \cdot \mathbf{k})^{2}+k^{2} \eta_{3}^{2}\right] \frac{\mathrm{e}^{2} \hbar}{2 m \Omega} \mathrm{e}^{-2 k \zeta} \delta(\hbar \Omega-\hbar \omega), \\
{\left[\begin{array}{l}
\gamma_{1,2} \\
\gamma_{3}
\end{array}\right]=\left[\begin{array}{l}
1 / 2 \\
1
\end{array}\right] \gamma, \quad \gamma=\frac{\left(\mathrm{e}^{2} / m\right) \beta^{2} \Gamma \omega_{\mathrm{S}}^{2} / 4 \zeta^{3}}{\left[\left(\Omega^{2}-\omega_{\mathrm{S}}^{2}\right)^{2}+\Omega^{2} \Gamma^{2}\right]},}  \tag{2.10}\\
\gamma=\frac{\alpha \beta^{2} \Gamma \omega_{\mathrm{S}}^{2} \Omega^{2} / 4 \zeta^{3}}{\left[\left(\Omega^{2}-\omega_{\mathrm{S}}^{2}\right)^{2}+\Omega^{2} \Gamma^{2}\right]}=\frac{\alpha \beta^{2} \omega_{\mathrm{S}}}{4 \zeta^{3}} \cdot \frac{c \mu^{2}}{\left[\left(1-\mu^{2}\right)^{2}+c^{2} \mu^{2}\right]} . \tag{2.11}
\end{gather*}
$$

Written in terms of $\Omega$ this is a purely classical expression. Nondissipatively,

$$
\begin{equation*}
\lim _{\Gamma / \omega_{\mathrm{S}} \rightarrow 0} \gamma=\left(\alpha \beta^{2} \omega_{\mathrm{S}}^{2} / 4 \zeta^{3}\right)(\pi / 2) \delta\left(\Omega-\omega_{\mathrm{S}}\right) . \tag{2.12}
\end{equation*}
$$

## 3. First-order perturbation theory

Recall that, to determine energies and transition rates correct to second order in $V$, state vectors need be correct only to first order.
${ }^{6}$ Do not confuse this $x$ with the position coordinate, nor this $c$ with the speed of light.

### 3.1. Attraction

The mean potential energy between a stationary atom and a nondissipative half-space is given by the well-known expression from second-order perturbation theory using (2.6, 2.9b):
$U_{\mathrm{nd}}^{(2)}(\mathbf{u}=0)=-\frac{\mathrm{e}^{2} \beta^{2} \hbar \omega_{\mathrm{S}}}{16 m \Omega\left(\omega_{\mathrm{S}}+\Omega\right) \zeta^{3}}\left[\frac{1}{2} \boldsymbol{\eta}_{\|}^{2}+\eta_{3}^{2}\right]=-\frac{\alpha \beta^{2} \hbar \omega_{\mathrm{S}} \Omega}{16\left(\omega_{\mathrm{S}}+\Omega\right) \zeta^{3}}\left[\frac{1}{2} \boldsymbol{\eta}_{\|}^{2}+\eta_{3}^{2}\right]$.
Isotropically, $[\ldots]=2$.
For a slowly moving atom
$U_{\mathrm{nd}}^{(2)}(\mathbf{u}=u \hat{\mathbf{x}})=-\frac{\alpha \beta^{2} \hbar \omega_{\mathrm{S}} \Omega}{16\left(\omega_{\mathrm{S}}+\Omega\right) \zeta^{3}}\left\{\left[\frac{1}{2} \eta_{\|}^{2}+\eta_{3}^{2}\right]+\frac{3}{2}\left[\frac{3}{4} \eta_{1}^{2}+\frac{1}{4} \eta_{2}^{2}+\eta_{3}^{2}\right]\left(\frac{u}{\left(\omega_{\mathrm{S}}+\Omega\right) \zeta}\right)^{2}\right\}$.
Isotropically, both [...] reduce to 2, reproducing the result found by Ferrel and Ritchie (1980).

### 3.2. Drag

A steady drag force $\mathbf{F}=-\hat{\mathbf{u}} F$ on the atom dissipates power $P=u F$, supplied by the agency enforcing the motion. It operates through momentum transfer from atom to half-space. Transfer mechanisms that require the energy to rise by at least $\hbar \omega_{\min }$ in the half-space and by $\hbar \Omega_{\text {min }}$ in the atom contribute to $F$ amounts that at low speeds and/or long distances are exponentially small, in the sense of having factors $\exp \left[-2 \zeta\left(\omega_{\min }+\Omega_{\text {min }}\right) / u\right]$. In this paper, we disregard all such contributions and look only for contributions proportional to powers of $u$ and inverse powers of $\zeta$.

To second order in $V$ and with a nondissipative half-space, $\omega_{\min }=\omega_{\mathrm{S}}$. Moreover, to second order $\Omega_{\min }=\Omega$, because the atom too must be excited, while the widths of the excited states must be taken as zero, because, as (2.10) reminds one, the true widths are themselves of second order. Nondissipatively therefore $P$ is small to the tune of $\exp \left[-2 \zeta\left(\omega_{\min }+\Omega_{\min }\right) / u\right]$.

Thus, power-law contributions to $P$ enter only to fourth (or higher) order in $V$ and require state vectors correct to second order.

## 4. Second-order perturbation theory

### 4.1. Generalities

We study a dissipative half-space. Initially (as $t \rightarrow-\infty$ ) the atom is in its ground state, and there are no plasmons; we calculate the probability $|c(t)|^{2}$ that at time $t$ there are two plasmons, $\mathbf{k} \omega$ and $\mathbf{k}^{\prime} \omega^{\prime}$, with the atom still in its ground state. This becomes possible on admitting into $c$ terms of second order in $V$. It will turn out that $|c(t)|^{2}$ grows linearly with $t$ at large $t$, and we identify ${ }^{7}$

$$
\begin{equation*}
P=\lim _{t \rightarrow \infty} \frac{1}{2} \iint \mathrm{~d}^{2} k \mathrm{~d} \omega \iint \mathrm{~d}^{2} k^{\prime} \mathrm{d} \omega^{\prime} \hbar\left(\omega+\omega^{\prime}\right)|c(t)|^{2} / t \tag{4.1}
\end{equation*}
$$

where the prefactor $1 / 2$ compensates for double counting the identical states $\left|\mathbf{k} \omega, \mathbf{k}^{\prime} \omega^{\prime}\right\rangle$ and $\left|\mathbf{k}^{\prime} \omega^{\prime}, \mathbf{k} \omega\right\rangle$.

7 Alternatively one can calculate the $t \rightarrow \infty$ limit of the expectation value of the force operator, using the appropriately perturbed state vector. To $\mathcal{O}\left(V^{2}\right)$ this is an attractive option, and the best way to (3.2); but to $\mathcal{O}\left(V^{4}\right)$ the bookkeeping becomes very awkward.

In a self-explanatory notation

$$
\begin{aligned}
& |\psi(t)\rangle=\sum_{n} c_{n}(t) \exp \left(-\mathrm{i} \omega_{n} t\right)|n\rangle, \quad|\psi(-\infty)\rangle=|i\rangle, \\
& \frac{\mathrm{d} c_{j}}{\mathrm{~d} t}=-\frac{\mathrm{i}}{\hbar} \sum_{n} c_{n}(t) \exp \left(\mathrm{i} \omega_{j n} t\right)\langle j| V|n\rangle, \quad \omega_{j n} \equiv \omega_{j}-\omega_{n}, \quad c_{n}(-\infty)=\delta_{n i} .
\end{aligned}
$$

Indicating powers of $V$ by superscripts, one has (Dirac 1958)
$c_{n}(t)=\delta_{n i}+c_{n}^{(1)}(t)+c_{n}^{(2)}(t)+\cdots, \quad c_{n}^{(1)}\left(t^{\prime}\right)=-\frac{\mathrm{i}}{\hbar} \int_{-\infty}^{t^{\prime}} \mathrm{d} t^{\prime \prime} \exp \left(\mathrm{i} \omega_{n i} t^{\prime \prime}\right)\langle n| V\left(t^{\prime \prime}\right)|i\rangle$,
$c_{f}^{(2)}(t)=\left(\frac{-\mathrm{i}}{\hbar}\right)^{2} \sum_{n} \int_{-\infty}^{t} \mathrm{~d} t^{\prime} \exp \left(\mathrm{i} \omega_{f n} t^{\prime}\right)\langle f| V\left(t^{\prime}\right)|n\rangle \int_{-\infty}^{t^{\prime}} \mathrm{d} t^{\prime \prime} \exp \left(\mathrm{i} \omega_{n i} t^{\prime \prime}\right)\langle n| V\left(t^{\prime \prime}\right)|i\rangle$.
In our case, $|i\rangle=|0\rangle|\mathbf{0}\rangle$ and $|f\rangle=\left|\mathbf{k} \omega, \mathbf{k}^{\prime} \omega^{\prime}\right\rangle|\mathbf{0}\rangle$. The only intermediate states $|n\rangle$ are $\left|\mathbf{k}^{\prime} \omega^{\prime}\right\rangle|\boldsymbol{\eta}\rangle$, with $\omega_{f n}=-\Omega+\omega$ and $\omega_{n i}=\Omega+\omega^{\prime}$, plus $|\mathbf{k} \omega\rangle|\boldsymbol{\eta}\rangle$, with $\omega_{f n}=-\Omega+\omega^{\prime}$ and $\omega_{n i}=\Omega+\omega$. In each case, one must sum over the polarization $\eta$ of the virtual excited state.

### 4.2. Amplitude

To identify $P$ with confidence it helps to separate the effects of the perturbation in producing the static potential from those that depend on the motion. To this end we adopt the following scenario: for $t<0$ the atom is at rest at $(0,0, \zeta)$; the initial state vector is the direct-product ground state $|\psi(-\infty)\rangle=|0\rangle|\mathbf{0}\rangle$. For $t<0$, the interaction is taken as $V \times \exp (-\lambda t)$, where the exponential is the familiar adiabatic switching factor. As soon as it is safe we take the limit $\lambda \rightarrow 0$. For $t>0$, we use $V$ without a switching factor, but with the atom at $(u t, 0, \zeta)$.

The integrations prescribed by (4.2) are elementary but moderately tedious. Defining
$\Delta_{0} \equiv \omega-\mathbf{k} \cdot \mathbf{u}, \quad \Delta_{0}^{\prime} \equiv \omega^{\prime}-\mathbf{k}^{\prime} \cdot \mathbf{u}, \quad \Delta \equiv \Delta_{0}+\Delta_{0}^{\prime}=\omega+\omega^{\prime}-\mathbf{k} \cdot \mathbf{u}-\mathbf{k}^{\prime} \cdot \mathbf{u}$,
one eventually finds

$$
\begin{equation*}
c^{(2)}(t)=-\frac{\alpha \beta^{2} \Omega \omega_{S}^{2}}{8 \pi} \exp \left[-\left(k+k^{\prime}\right) \zeta\right] \frac{\sqrt{k k^{\prime}} g g^{\prime}}{\sqrt{\omega \omega^{\prime}} F^{*}(\omega) F^{*}\left(\omega^{\prime}\right)} Q \mathcal{M}(t), \tag{4.4}
\end{equation*}
$$

where (2.1) has been used to obtain
$Q \equiv \sum_{\eta}[\boldsymbol{\eta} \cdot(-\mathrm{i} \hat{\mathbf{k}}-\hat{\mathbf{z}})]\left[\boldsymbol{\eta} \cdot\left(-\mathrm{i} \hat{\mathbf{k}}^{\prime}-\hat{\mathbf{z}}\right)\right]=1-\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}^{\prime}=\left[1-\cos \left(\phi-\phi^{\prime}\right)\right]$,
and where

$$
\begin{align*}
& \mathcal{M}(t)=\left\{-\frac{1}{\omega+\omega^{\prime}}\left[\frac{1}{\Omega+\omega}+\frac{1}{\Omega+\omega^{\prime}}\right]-\frac{\left[\mathrm{e}^{\mathrm{i} t \Delta}-1\right]}{\Delta}\left[\frac{1}{\Omega+\Delta_{0}}+\frac{1}{\Omega+\Delta_{0}^{\prime}}\right]+\frac{\left[\mathrm{e}^{\mathrm{it}\left(-\Omega+\Delta_{0}\right)}-1\right]}{\left(-\Omega+\Delta_{0}\right)}\right. \\
&\left.\times\left[\frac{\mathbf{k}^{\prime} \cdot \mathbf{u}}{\left(\Omega+\omega^{\prime}\right)\left(\Omega+\omega^{\prime}-\mathbf{k}^{\prime} \cdot \mathbf{u}\right)}\right]+\frac{\left[\mathrm{e}^{\mathrm{it}\left(-\Omega+\Delta_{0}^{\prime}\right)}-1\right]}{\left(-\Omega+\Delta_{0}^{\prime}\right)}\left[\frac{\mathbf{k} \cdot \mathbf{u}}{(\Omega+\omega)(\Omega+\omega-\mathbf{k} \cdot \mathbf{u})}\right]\right\} . \tag{4.6}
\end{align*}
$$

The first term could of course have been found equally well by time-independent perturbation theory.

It is worth noting that $\mathcal{M}(t)$ knows nothing about $\omega_{\mathrm{s}}$.

### 4.3. Power

For insight into the behaviour of $\mathcal{M}(t)$ and of $|\mathcal{M}(t)|^{2}$ at large $t$, recall that

$$
\begin{equation*}
\frac{\left[\mathrm{e}^{\mathrm{i} t \nu}-1\right]}{\nu}=\frac{\mathrm{ie}^{\mathrm{i} t \nu / 2} \sin (t v / 2)}{(\nu / 2)} \underset{t \rightarrow \infty}{\rightarrow} \mathrm{ie}^{\mathrm{i} t \nu / 2} \pi \delta(\nu / 2)=\mathrm{i} 2 \pi \delta(\nu), \tag{4.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\frac{\left[\mathrm{e}^{\mathrm{i} t v}-1\right]}{v}\right|^{2} \underset{t \rightarrow \infty}{\rightarrow}(2 \pi)^{2} \delta^{2}(v)=t \times 2 \pi \delta(v) \tag{4.8}
\end{equation*}
$$

warranted by comparing the two representations

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\sin (t v)}{v}=\pi \delta(\nu), \quad \lim _{t \rightarrow \infty} \frac{\sin ^{2}(t v)}{t v^{2}}=\pi \delta(\nu) . \tag{4.9}
\end{equation*}
$$

Accordingly, at large $t$ contributions ${ }^{8}$ to $|\mathcal{M}(t)|^{2}$ that are nonoscillatory and $t$-proportional arise only from the individually squared moduli of the second, third and fourth terms inside the braces in (4.6). Eventually, exploiting the delta functions, one finds that

$$
\begin{align*}
\lim _{t \rightarrow \infty}\left(\frac{|\mathcal{M}(t)|^{2}}{t}\right) & =2 \pi\left\{\frac{\delta\left(\omega+\omega^{\prime}-\mathbf{u} \cdot\left(\mathbf{k}+\mathbf{k}^{\prime}\right)\right) 4 \Omega^{2}}{(\Omega+\omega-\mathbf{k} \cdot \mathbf{u})^{2}\left(\Omega+\omega^{\prime}-\mathbf{k}^{\prime} \cdot \mathbf{u}\right)^{2}}\right. \\
+ & {\left.\left[\frac{\delta(\Omega-\omega+\mathbf{k} \cdot \mathbf{u})}{\left(\Omega+\omega^{\prime}\right)^{2}}\left(\mathbf{k}^{\prime} \cdot \mathbf{u}\right)^{2}+\frac{\delta\left(\Omega-\omega^{\prime}+\mathbf{k}^{\prime} \cdot \mathbf{u}\right)}{(\Omega+\omega)^{2}}(\mathbf{k} \cdot \mathbf{u})^{2}\right]\right\} } \tag{4.10}
\end{align*}
$$

Substitution into (4.1) via (4.4) then leads to ${ }^{9}$

$$
\begin{align*}
& P=\frac{\hbar \alpha^{2} \beta^{4} \Omega^{2} \omega_{\mathrm{S}}^{4} \Gamma^{2}}{16 \pi^{3}} \iint \mathrm{~d}^{2} k \mathrm{~d}^{2} k^{\prime} \exp \left[-2\left(k+k^{\prime}\right) \zeta\right] k k^{\prime} Q^{2} \\
& \quad \times \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{d} \omega \mathrm{~d} \omega^{\prime} \frac{\omega \omega^{\prime}\left(\omega+\omega^{\prime}\right)}{\mid F\left(\omega,\left.\omega_{\mathrm{S}}\right|^{2} \mid F\left(\omega^{\prime},\left.\omega_{\mathrm{S}}\right|^{2}\right.\right.} \times\{\text { from (4.10) }\} \tag{4.11}
\end{align*}
$$

It is reassuring to note that $P$ vanishes as $\omega_{\mathrm{S}} \rightarrow \infty$ : the field cannot then penetrate into the half-space, and it cannot dissipate energy elsewhere.

[^1]To approximate sensibly one needs the scaled variables from (2.7):

$$
\begin{align*}
& P=\frac{\hbar \alpha^{2} \beta^{4} \omega_{\mathrm{S}}^{2} \mu^{2} c^{2}}{16 \pi^{3} \zeta^{6}} \iint \mathrm{~d}^{2} \kappa \mathrm{~d}^{2} \kappa^{\prime} \exp \left[-2\left(\kappa+\kappa^{\prime}\right)\right] \kappa \kappa^{\prime} Q^{2} \\
& \times \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{d} x \mathrm{~d} x^{\prime} \frac{x x^{\prime}\left(x+x^{\prime}\right)}{\left[\left(1-x^{2}\right)^{2}+c^{2} x^{2}\right]\left[\left(1-x^{\prime 2}\right)^{2}+c^{2} x^{\prime 2}\right]} \\
& \times\left\{\left[\frac{\delta\left(x+x^{\prime}-\boldsymbol{\kappa} \cdot \mathbf{v}-\boldsymbol{\kappa}^{\prime} \cdot \mathbf{v}\right) 4 \mu^{2}}{(\mu+x-\boldsymbol{\kappa} \cdot \mathbf{v})^{2}\left(\mu+x^{\prime}-\kappa^{\prime} \cdot \mathbf{v}\right)^{2}}\right]\right. \\
&\left.+\left[\frac{\delta(\mu-x+\boldsymbol{\kappa} \cdot \mathbf{v})\left(\boldsymbol{\kappa}^{\prime} \cdot \mathbf{v}\right)^{2}}{\left(\mu+x^{\prime}\right)^{2}}+\frac{\delta\left(\mu-x^{\prime}+\boldsymbol{\kappa}^{\prime} \cdot \mathbf{v}\right)(\boldsymbol{\kappa} \cdot \mathbf{v})^{2}}{(\mu+x)^{2}}\right]\right\} \equiv P_{A}+P_{B} \tag{4.12}
\end{align*}
$$

The two terms in $P_{B}$ contribute equally.
It is worth explicating some of the virtues of the scaled version (4.12). (i) The exponential allows one to treat $\kappa, \kappa^{\prime}$ as of the order of unity when approximating the integrand. (ii) It shows 'low speed' to mean $v=u / \zeta \omega_{\mathrm{S}} \ll 1$, a condition amply satisfied if, in atomic units, $\omega_{\mathrm{s}}, \Omega \sim \mathcal{O}(1)$ and $u \lesssim 1$, while $\zeta$ is macroscopic even if macroscopically small. In view of (i), small $v$ entails small $\boldsymbol{\kappa} \cdot \mathbf{v}$ and $\boldsymbol{\kappa}^{\prime} \cdot \mathbf{v}$. (iii) Nontrivially, i.e. apart from the prefactor $1 / \zeta^{6}$, it depends on $\zeta$ only through $v$. (iv) It yields the nondispersive limit by inspection:

$$
\begin{align*}
P_{n d} \equiv \lim _{\mu \rightarrow \infty} P= & \lim _{\mu \rightarrow \infty} P_{A}=\frac{\hbar \alpha^{2} \beta^{4} \omega_{\mathrm{S}}^{2} c^{2}}{4 \pi^{3} \zeta^{6}} \iint \mathrm{~d}^{2} \kappa \mathrm{~d}^{2} \kappa^{\prime} \exp \left[-2\left(\kappa+\kappa^{\prime}\right)\right] \kappa \kappa^{\prime} Q^{2} \\
& \times \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{d} x \mathrm{~d} x^{\prime} \frac{x x^{\prime}\left(x+x^{\prime}\right) \delta\left(x+x^{\prime}-\kappa \cdot \mathbf{v}-\kappa^{\prime} \cdot \mathbf{v}\right)}{\left[\left(1-x^{2}\right)^{2}+c^{2} x^{2}\right]\left[\left(1-x^{\prime 2}\right)^{2}+c^{2} x^{\prime 2}\right]} . \tag{4.13}
\end{align*}
$$

## 5. Slow atoms

We deal with $P_{A}$ and $P_{B}$ separately, approximate each to leading order in small $v$, and compare them afterwards.

### 5.1. The component $P_{A}$

The delta function in the integrand of $P_{A}$ (the first [ $\ldots$ ] inside the braces in (4.12)) shows that small $v$ entails small $x$ and $x^{\prime}$. Hence to leading order both factors [...] in its denominator reduce to unity; and the factor $(\ldots)^{2}(\ldots)^{2}$ reduces to $\mu^{4}$, cancelling the factors $\mu^{2}$ in the numerator and in the prefactor. Thus, with

$$
\begin{equation*}
a \equiv\left(\kappa+\kappa^{\prime}\right) \cdot \mathbf{v}=v\left[\kappa \cos \phi+\kappa^{\prime} \cos \phi^{\prime}\right], \tag{5.1}
\end{equation*}
$$

the inner integral reduces to

$$
\begin{equation*}
\int_{0}^{\infty} \int_{0}^{\infty} \mathrm{d} x \mathrm{~d} x^{\prime} x x^{\prime}\left(x+x^{\prime}\right) \delta\left(x+x^{\prime}-a\right)=\theta(a) a^{4} / 6 \tag{5.2}
\end{equation*}
$$

where $\theta$ is the Heaviside step function. Accordingly, with $Q$ from (4.5),

$$
\begin{equation*}
P_{A} \simeq \frac{\hbar \alpha^{2} \beta^{4} \omega_{\mathrm{S}}^{2} c^{2} v^{4}}{24 \pi^{3} \zeta^{6}} \iint \mathrm{~d}^{2} \kappa \mathrm{~d}^{2} \kappa^{\prime} \exp ^{-2\left(\kappa+\kappa^{\prime}\right)} \kappa \kappa^{\prime}\left[1-\cos \left(\phi-\phi^{\prime}\right)\right]^{2}\left[\kappa \cos \phi+\kappa^{\prime} \cos \phi^{\prime}\right]^{4} \tag{5.3}
\end{equation*}
$$

Corrections of higher order in $v^{2}$ could be found by expanding the denominator of the integrand of (4.12) in powers of $\boldsymbol{\kappa} \cdot \mathbf{v}$ and of $\boldsymbol{\kappa}^{\prime} \cdot \mathbf{v}$. The integral in (5.3) is just a number, and evaluates to $27 \pi^{2} / 8$ :

$$
\begin{equation*}
P_{A} \simeq \frac{9}{128 \pi} \cdot \frac{\hbar \alpha^{2} \beta^{4} \omega_{\mathrm{S}}^{2} c^{2} v^{4}}{\zeta^{6}}=\frac{9}{128 \pi} \cdot \frac{\hbar \alpha^{2} \beta^{4} \Gamma^{2} u^{4}}{\omega_{\mathrm{S}}^{4} \zeta^{10}} \underset{\text { Drude }}{\rightarrow} \frac{9}{512 \pi^{3}} \cdot \frac{\hbar \alpha^{2} u^{4}}{\sigma^{2} \zeta^{10}} . \tag{5.4}
\end{equation*}
$$

It is easily seen that the same approximation follows from the nondispersive limit (4.13). The overall normalization of (4.12) can be checked by comparison with the nondispersive model outlined in the appendix, which implements the limit right at the start.

### 5.2. The component $P_{B}$

To leading order in $v$ the explicit factors $(\boldsymbol{\kappa} \cdot \mathbf{v})^{2}=v^{2} \kappa^{2} \cos ^{2} \phi$ and $\left(\kappa^{\prime} \cdot \mathbf{v}\right)^{2}=v^{2} \kappa^{\prime 2} \cos ^{2} \phi^{\prime}$ in the integrand of $P_{B}$ (the second [...] inside the braces in (4.12)) show that everywhere else one may set $\mathbf{v} \rightarrow \mathbf{0}$. Since the two terms in [...] contribute equally, we may write

$$
\begin{align*}
& P_{B} \simeq \frac{\hbar \alpha^{2} \beta^{4} \omega_{\mathrm{S}}^{2} \mu^{2} c^{2} v^{2}}{8 \pi^{3} \zeta^{6}} \iint \mathrm{~d}^{2} \kappa \mathrm{~d}^{2} \kappa^{\prime} \exp \left[-2\left(\kappa+\kappa^{\prime}\right)\right] \kappa \kappa^{\prime 3}\left[1-\cos \left(\phi-\phi^{\prime}\right)\right]^{2} \cos ^{2} \phi^{\prime} \\
& \times \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{d} x \mathrm{~d} x^{\prime} \frac{x x^{\prime}\left(x+x^{\prime}\right)}{\left[\left(1-x^{2}\right)^{2}+c^{2} x^{2}\right]\left[\left(1-x^{\prime 2}\right)^{2}+c^{2} x^{\prime 2}\right]} \cdot \frac{\delta(\mu-x)}{\left(\mu+x^{\prime}\right)^{2}} \tag{5.5}
\end{align*}
$$

Corrections of higher order in $v^{2}$ could be found by expanding $\delta(\mu-x+\boldsymbol{\kappa} \cdot \mathbf{v})$ in powers of $\boldsymbol{\kappa} \cdot \mathbf{v}$.

In (5.5), the $\kappa \kappa^{\prime}$ and the $x x^{\prime}$ integrals decouple. The former is just a number, and evaluates to $9 \pi^{2} / 16$. In the latter, $\int \mathrm{d} x \ldots$ is trivial by virtue of the delta function. Redistributing the prefactor $c^{2}$ one obtains

$$
\begin{equation*}
P_{B} \simeq \frac{9 \hbar \alpha^{2} \beta^{4} \omega_{\mathrm{S}}^{2} \mu^{2} v^{2}}{128 \pi \zeta^{6}} \cdot \frac{c \mu}{\left[\left(1-\mu^{2}\right)^{2}+c^{2} \mu^{2}\right]} \int_{0}^{\infty} \mathrm{d} x^{\prime} \frac{c x^{\prime}}{\left[\left(1-x^{\prime 2}\right)^{2}+c^{2} x^{\prime 2}\right]\left(\mu+x^{\prime}\right)} . \tag{5.6}
\end{equation*}
$$

For simplicity, we now confine attention to weak damping, $c \ll 1$, and use (2.8) to evaluate $\int_{0}^{\infty} \mathrm{d} x^{\prime} \ldots$ only in the limit $c \rightarrow 0$ :

$$
\begin{align*}
\Gamma / \omega_{\mathrm{S}} \ll 1: & P_{B} \simeq \frac{9 \hbar \alpha^{2} \beta^{4} \omega_{\mathrm{S}}^{2} v^{2}}{256 \zeta^{6}(\mu+1)} \cdot \frac{c \mu^{3}}{\left[\left(1-\mu^{2}\right)^{2}+c^{2} \mu^{2}\right]}  \tag{5.7}\\
= & \frac{9 \hbar \alpha^{2} \beta^{4} u^{2} \Omega}{256 \zeta^{8}\left(\Omega+\omega_{\mathrm{S}}\right)} \cdot \frac{\Gamma \omega_{\mathrm{S}} \Omega^{2}}{\left[\left(\Omega^{2}-\omega_{\mathrm{S}}^{2}\right)^{2}+\Gamma^{2} \Omega^{2}\right]} .
\end{align*}
$$

Even though this expression is well defined for all $\mu=\Omega / \omega_{\mathrm{s}}$, one should bear in mind that near $\mu=1$ our perturbative approach is unwarranted.

Finally, it is entertaining to rewrite (5.7) in terms of the parameter $\gamma$ (equation (2.11)) governing the decay rates of the excited states:

$$
\begin{equation*}
P_{B} \simeq \frac{9 \hbar \alpha \beta^{2} u^{2}}{64 \zeta^{5}} \cdot \frac{\Omega \gamma}{\omega_{\mathrm{S}}\left(\Omega+\omega_{\mathrm{S}}\right)} \tag{5.8}
\end{equation*}
$$

## 5.3. $P_{A}$ versus $P_{B}$

By (5.4) and (5.7)

$$
\begin{equation*}
\frac{P_{A}}{P_{B}}=\left\{\frac{2 \Gamma v^{2}}{\pi \omega_{\mathrm{S}}}\right\}\left\{\frac{\left(\Omega+\omega_{\mathrm{S}}\right)\left[\left(\Omega^{2}-\omega_{\mathrm{S}}^{2}\right)^{2}+\Gamma^{2} \Omega^{2}\right]}{\Omega^{3} \omega_{\mathrm{S}}^{2}}\right\}, \quad v^{2} \equiv\left(\frac{u}{\zeta \omega_{\mathrm{S}}}\right)^{2} . \tag{5.9}
\end{equation*}
$$

In the regime we are considering, $v^{2} \ll 1$ because the motion is slow, and $\Gamma / \omega_{\mathrm{S}} \ll 1$ because damping is weak. Hence the factor in the first pair of braces is small. Moreover, as explained in section 1 , we are excluding the near-resonant case where $\left|\Omega-\omega_{S}\right| \ll \Omega$, whence the factor in the second pair of braces is of order unity. Accordingly, in our regime $P_{B}$ dominates:

$$
\begin{equation*}
P_{A} / P_{B} \sim \Gamma v^{2} / \omega_{\mathrm{S}}=\Gamma u^{2} / \zeta^{2} \omega_{\mathrm{S}}^{3} \ll 1 \quad \Rightarrow \quad P=u F \simeq P_{B} \tag{5.10}
\end{equation*}
$$

## 6. Comments

### 6.1. Comparisons

Scheel and Buhmann (2009) use Green's function techniques to look for the properly retarded $\mathcal{O}\left(V^{4}\right)$ force on an isotropic ground-state atom with a single excitation frequency $\omega_{A}$ and squared dipole matrix element $d^{2}$ (our $|\mathbf{D}|^{2}$ ). Our results and the non-retarded limits of theirs should tally, but in fact, although they vary in the same way with $u$ and $\zeta$, they vary differently with $\Omega$ and $\omega_{\mathrm{s}}$.

Their expressions translate into ours according to the following dictionary:

$$
\begin{array}{lclllllll}
\text { SB : } & \left(4 \pi \varepsilon_{0}\right) & v & z_{A} & \omega_{A} & d^{2} & \gamma & \omega_{\mathrm{p}}^{2} & |\mathbf{F}|  \tag{6.1}\\
\text { here : } & 1 & u & \zeta & \Omega & \left|\mathbf{D}^{2}\right|=3 \alpha \hbar \Omega / 2 & \Gamma / 2 & \left(2 \omega_{\mathrm{S}}^{2} \beta^{2}\right) & (P / u) .
\end{array}
$$

Write the power loss corresponding to their result as $P_{\mathrm{SB}}$, found by substituting their (83) into their (82) and multiplying by the velocity. Then, re-expressed in our notation and in terms of $\alpha$ instead of $|\mathbf{D}|^{2}$, one finds that

$$
\begin{equation*}
P_{\mathrm{S} B}=\frac{3 \hbar \alpha^{2} \beta^{4} u^{2} \Gamma \Omega^{3} \omega_{\mathrm{S}}^{3}}{16 \zeta^{8}\left(\Omega^{2}-\omega_{\mathrm{S}}^{2}\right)^{2}\left(\Omega+\omega_{\mathrm{S}}\right)^{3}} . \tag{6.2}
\end{equation*}
$$

By contrast, for weak damping our own result (5.7) may be written as

$$
\begin{equation*}
P_{B}=\frac{9 \hbar \alpha^{2} \beta^{4} u^{2} \Gamma \Omega^{3} \omega_{\mathrm{S}}}{256 \zeta^{8}\left(\Omega^{2}-\omega_{\mathrm{S}}^{2}\right)^{2}\left(\Omega+\omega_{\mathrm{S}}\right)} \tag{6.3}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\frac{P_{\mathrm{S} B}}{P_{B}}=\frac{4}{3} \times\left(\frac{2 \omega_{\mathrm{S}}}{\Omega+\omega_{\mathrm{S}}}\right)^{2} \tag{6.4}
\end{equation*}
$$

a ratio that looks as if it might stem from or at least be related to tacit differences regarding the line shape. The writer understands that some of their expressions might need revision in order to take full account of the difference between the decay rates of $\left|\eta_{1,2}\right\rangle$ and of $\left|\eta_{3}\right\rangle$.

Essentially the same nonretarded problem was considered in at least two much earlier papers. For low speeds, Mahanty (1980; equation (23)) gave $F \sim-\left(u \hbar \alpha / 32 \zeta^{5}\right)\left(n^{2}-1\right) /\left(n^{2}+1\right)$ as re-expressed in our notation, with $n$ being the zero-frequency dielectric constant. Schaich and

Harris ${ }^{10}$ (1981; equation (43)) gave $F \sim-u \alpha^{2} e^{4} / \hbar \omega_{\mathrm{S}}^{2} \zeta^{10}$. The writer believes that neither can be right, if only because, even without dissipation in the material, each makes $F$ vary like some power of $u$ and inverse power of $\zeta$ rather than exponentially.

In fact the $u^{2} / \zeta^{8}$-proportionality seems to be remarkably robust. For instance, ZuritaSánchez et al (2004) predict and cite experimental evidence for it, at finite temperature, in the quite different problem where instead of the atom one has a macroscopic sphere, with radius well below the pertinent $\Lambda$, but large enough for its response to be characterized by a Drude response function of its own.

### 6.2. Preliminary implications regarding two half-spaces

Consider a second half-space, optically dilute, made of atoms of the kind we have been considering, $\bar{n}$ per unit volume. Hence, it is dispersive but nondissipative, with dielectric response $\bar{\varepsilon}(\omega) \simeq 1+4 \pi \bar{n} \alpha \Omega^{2} /\left[\Omega^{2}-\left(\omega^{2}+\mathrm{i} 0\right)\right]$. Let it be parallel to the original half-space and move laterally, with a gap of width $Z$ in between. It experiences a drag force $-R \hat{\mathbf{u}}$ per unit surface area; then the power dissipation (localized in the original half-space) is $u R$. By simple addition ${ }^{11}$ of the forces $P / u$ per atom, and in virtue of (5.7),

$$
\begin{equation*}
R=\bar{n} \int_{Z}^{\infty} \mathrm{d} \zeta \frac{P_{B}}{u} \simeq \frac{9}{(7 \times 256)} \cdot \frac{u \hbar \bar{n} \alpha^{2} \beta^{4} \Omega}{Z^{7}\left(\Omega+\omega_{S}\right)} \cdot \frac{\Gamma \omega_{\mathrm{S}} \Omega^{2}}{\left[\left(\Omega^{2}-\omega_{\mathrm{S}}^{2}\right)^{2}+\Gamma^{2} \Omega^{2}\right]} \tag{6.5}
\end{equation*}
$$

To forestall confusion, we remark that it makes no sense to try and compare (6.5) in any detail with the force $\mathbb{F}$ in appendix B of I, even though both feature dilute material. The latter applies to two nondissipative half-spaces, both dilute, with identical natural frequencies, and applies to second order in the coupling between them. By contrast, (6.5) applies when one halfspace is dissipative and need not be dilute, while the other must be both nondissipative and dilute; it excludes the case where the natural frequencies are the same and applies to fourth order in the coupling between them. We study elsewhere how the methods of the present paper extend to two half-spaces that need not be dilute and may be conducting (Barton 2010b).

## Acknowledgment

It is a pleasure to thank Stefan Buhmann for very helpful comments.

## Appendix. Nondispersive atom

Call the interaction $W$, to forestall confusion with $V_{\mathrm{nd}}$ from (2.9):

$$
\begin{equation*}
W=-\alpha \mathbf{E}^{2}(u t, 0, \zeta) / 2=-\alpha[\nabla \Phi(u t, 0, \zeta)]^{2} / 2 . \tag{A.1}
\end{equation*}
$$

[^2]The state vector now acquires two-plasmon admixtures to first order in $W$. Following the same scenario as in section 3.2, one finds that the power dissipation to second order in $W$ reads

$$
\begin{align*}
& P_{W}=\frac{\hbar \alpha^{2} \beta^{4} \omega_{S}^{4} \Gamma^{2}}{4 \pi^{3}} \iint \mathrm{~d}^{2} k \mathrm{~d}^{2} k^{\prime}\left\{\frac{\left[k k^{\prime}-\mathbf{k} \cdot \mathbf{k}^{\prime}\right]^{2}}{k k^{\prime}}\right\} \exp \left[-2\left(k+k^{\prime}\right) \zeta\right] \\
& \times \int_{0}^{\infty} \mathrm{d} \omega \int_{0}^{\infty} \mathrm{d} \omega^{\prime} \frac{\omega \omega^{\prime}\left(\omega+\omega^{\prime}\right) \delta\left(\omega+\omega^{\prime}-\mathbf{u} \cdot\left(\mathbf{k}+\mathbf{k}^{\prime}\right)\right)}{\left[\left(\omega_{\mathrm{S}}^{2}-\omega^{2}\right)^{2}+\omega^{2} \Gamma^{2}\right]\left[\left(\omega_{\mathrm{S}}^{2}-\omega^{\prime 2}\right)^{2}+\omega^{\prime 2} \Gamma^{2}\right]} \tag{A.2}
\end{align*}
$$

The factor in braces is just $k k^{\prime} Q^{2}$, with $Q$ from (4.5), whence on scaling (A.2) tallies as it should with (4.13).

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[^0]:    1 This contrasts sharply with VdW friction between two half-spaces, which at any fixed finite temperature $T$ and small enough velocity is dominated by a term proportional to $T^{2}$ (Barton 2010b).
    ${ }^{2}$ For a general introduction to these relations see e.g. Goldberger and Watson (1964) and Jackson (1999). Their application to Casimir problems is discussed e.g. by Klimchitskaya et al (2009).
    ${ }^{3}$ See e.g. Leonhardt (2010) and Pendry (2010).

[^1]:    ${ }^{8}$ How large $t$ should be emerges only from (4.11) below: the $\delta(\nu)$ delivered by (4.7)-(4.9) approximate peaks that must be narrower than the factors multiplying $\{\ldots\}$ in the integrand of (4.11), whence we require $t \Gamma \gg 1$. (Recall that here we are concerned specifically with nonzero $\Gamma$.) Finite- $t$ contributions are much harder to determine. For instance, even in second order there must be transients just after $t=0$, if only to supply the $u^{2}$-proportional part of (3.2).
    ${ }^{9}$ The first term inside the braces in (4.10) and (4.11) has poles where $\Omega+\omega-\mathbf{k} \cdot \mathbf{u}$ or $\Omega+\omega^{\prime}-\mathbf{k}^{\prime} \cdot \mathbf{u}$ vanishes: they signal that fast enough atoms can create physical rather than merely virtual plasmons, the mechanism responsible for the $\mathcal{O}\left(V^{2}\right)$ contribution to $P$. But to the $\mathcal{O}\left(V^{4}\right)$ contributions, dominant at the low speeds we are considering, section 5 will demonstrate that the poles are irrelevant.

[^2]:    ${ }^{10}$ Much of their paper deals with two half-spaces responding like a Fermi gas, whose excitation spectrum differs significantly from that of the Drude model. But they do use the Drude model for the half-space interacting with an atom, so that comparison makes sense.
    ${ }^{11}$ Unfortunately no such simple argument applies if the second half-space contains mobile charge carriers however dilute, nor in the presence of appreciable dissipation.

