New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG IOP Institute of Physics



# Area–angle variables for general relativity

To cite this article: Bianca Dittrich and Simone Speziale 2008 *New J. Phys.* **10** 083006

New Journal of Physics
The open-access journal for physics

# Area–angle variables for general relativity

**Bianca Dittrich and Simone Speziale**
Perimeter Institute, 31 Caroline St. N, Waterloo, ON, N2L 2Y5, Canada
E-mail: sspeziale@perimeterinstitute.ca

*New Journal of Physics* **10** (2008) 083006 (12pp)
Received 21 May 2008
Published 5 August 2008
Online at http://www.njp.org/
doi:10.1088/1367-2630/10/8/083006

**Abstract.** We introduce a modified Regge calculus for general relativity on a triangulated four-dimensional Riemannian manifold where the fundamental variables are areas and a certain class of angles. These variables satisfy constraints which are local in the triangulation. We expect the formulation to have applications to classical discrete gravity and non-perturbative approaches to quantum gravity.

**Contents**



## 1. Introduction

Discrete approaches have proved useful in many areas of physics. Regge calculus [1] is a discrete formulation of general relativity (GR) where space-time is approximated by a triangulated manifold, and the fundamental variables used to describe the metric are the lengths of the edges of the triangulation. This approach has been applied with some success to classical gravity [2, 3], and used as a starting point for a lattice quantization of GR [2], [4]–[6]. Like other non-perturbative approaches to quantum gravity, quantum Regge calculus suffers from the problem of defining a unique gauge-invariant measure in the path integral.

 

The background-independent spinfoam approach [7] suggests an original route based on the well-defined quantum measure of BF theory. The latter is a topological theory where area variables appear naturally, and whose action can be reduced to GR by means of the so-called simplicity constraints, as discovered by Plebanski [8]. This and other motivations have led Rovelli [9] to suggest that 4D quantum gravity should be related to a modification of Regge calculus where the fundamental variables are the areas of triangles rather than the edge lengths. Some effort was put into this line of research by Makela and Williams among others [2], [10]–[12], but the problem has been open for more than 10 years. The main difficulty lies in the fact that a generic triangulation has many more triangles than edges, thus area variables should be constrained. An explicit expression of these constraints is obscured by their non-local nature in the triangulation.

In this paper, we introduce a description of discrete gravity that overcomes this difficulty. The key idea is to enlarge the set of variables from areas only, to areas and angles. In this way the constraints become local, are easy to write explicitly, and further they are related to the simplicity constraints of Plebanski's formulation of GR[1].

We approximate the space-time manifold by a simplicial triangulation, where each 4-simplex[2] is flat and the curvature is described by deficit angles associated with the triangles. Regge calculus uses the fact that on each 4-simplex $\sigma$ the ten components of the (constant) metric tensor $g_{\mu\nu}(\sigma)$ can be straightforwardly expressed in terms of the ten edge lengths $\ell_{\mathrm{e}}$. A further advantage of using the edge lengths as variables is that they endow each tetrahedron with six quantities which are sufficient to completely characterize the tetrahedron's geometry. Therefore the gluing of 4-simplices, obtained by identifying a shared tetrahedron, is trivial and causes no complications.

On a single 4-simplex, there are also ten triangles, suggesting that areas can be equivalently taken as the metric variables. There are two difficulties with this idea. First, it is less straightforward to express $g_{\mu\nu}(\sigma)$ in terms of areas. For instance the change of variables from edge lengths to areas on a 4-simplex is singular for orthogonal configurations [11], which is where right angles among the edges are present. Even the equal area configuration has a two-fold ambiguity where the same set of areas corresponds to two different sets of edge lengths. This is a more significant difficulty than it might seem at first glance, as such configurations are relevant in the case of a regular lattice, the simplest flat solution to Regge calculus. The second issue is even more serious. Ten areas might be enough to describe the 4-geometry of the simplex, but how about its boundary 3-geometry? Consider any of the five tetrahedra in the 4-simplex; its geometry is not uniquely defined by the areas of its four triangles (two more quantities are needed, corresponding for instance to (non-opposite) dihedral angles). So we need the geometry of the full 4-simplex to determine the individual geometry of any of its boundary tetrahedra. As a consequence, two adjacent 4-simplices in a triangulation will typically induce different geometries on the common tetrahedron, leading to discontinuities in the metric [12], or to non-local constraints involving the two 4-simplices [10].

A solution to the problem can be achieved by adding enough variables to the areas so that the geometry of each of the five tetrahedra can be independently and completely determined. A natural choice is to add the tetrahedral dihedral angles. Of course, this pleonastic set of

[1] Similar ideas have been investigated by Reisenberger [13] and Rovelli [14]. Some ideas were then developed into the classical action used in [15].

[2] The 4-simplex, also known as pentachoron in the mathematical literature, is the convex hull of five points. A 4-simplex contains five tetrahedra, ten triangles and ten edges.

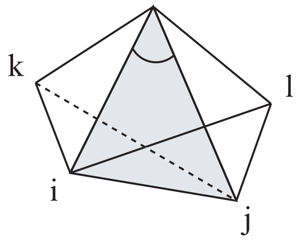


**Figure 1.** The geometric meaning of equation (2): the 2D angle $\alpha_{ij,kl}$ belonging to the shaded triangle can be expressed in terms of 3D angles associated with the thick edges of the tetrahedron $k$, or equivalently of the tetrahedron $l$.

variables needs to be constrained in order to successfully reproduce the dynamics of GR. We now turn to the study of these constraints.

## 2. *De natura* pentachori

Let us study how to characterize the geometry of a 4-simplex and its five boundary tetrahedra, using areas and 3D dihedral angles. We use a notation which might seem counterintuitive at first, but that pays off well in terms of efficiency and extends to any dimension. We denote by $V$ the 4-volume of the simplex $\sigma$ (or the $n$-volume in general), by $V(i)$ the 3-volume of the tetrahedron $\sigma(i)$ obtained by removing the vertex $i$ from the 4-simplex, by $V(ij)$ the area of the triangle $\sigma(ij)$ obtained by removing the vertices $i$ and $j$, and so on. For the dihedral angles, we use the following notation: $\theta_{ij}$ is the 4D dihedral angle between the tetrahedra $\sigma(i)$ and $\sigma(j)$, hinged at the triangle $\sigma(ij)$; $\phi_{ij,k}$ is the 3D dihedral angle between the two triangles $\sigma(ik)$ and $\sigma(jk)$, hinged at $\sigma(ijk)$ within the tetrahedron $\sigma(k)$; finally, $\alpha_{ij,kl}$ is the 2D dihedral angle between the edges $\sigma(ijk)$ and $\sigma(ijl)$ belonging to the triangle $\sigma(kl)$. All dihedral angles are *internal*; thus, for instance, an equilateral 4-simplex has $\cos\theta = 1/4$.

These various types of dihedral angles satisfy a number of relations in a closed 4-simplex, which we present together with their proofs in appendix A. An important role in our construction is played by the following expression of the 2D $\alpha$s in terms of the 3D $\phi$s:

$$\cos\,\alpha_{ij,kl} = \frac{\cos\,\phi_{ij,k} + \cos\,\phi_{il,k}\cos\,\phi_{jl,k}}{\sin\,\phi_{il,k}\sin\,\phi_{jl,k}}. \tag{1}$$

In this formula the 2D angle, belonging to the triangle $kl$, is described in terms of three 3D angles all belonging to the *same* tetrahedron $k$. In a closed 4-simplex, a triangle is shared by two tetrahedra; thus there are two possible choices. Consistency of the two choices, i.e. $\alpha_{ij,kl} = \alpha_{ij,lk}$ (see figure 1), gives

$$\mathcal{C}_{kl,ij}(\phi) \equiv \frac{\cos\,\phi_{ij,k} + \cos\,\phi_{il,k}\cos\,\phi_{jl,k}}{\sin\,\phi_{il,k}\sin\,\phi_{jl,k}} - \frac{\cos\,\phi_{ij,l} + \cos\,\phi_{ik,l}\cos\,\phi_{jk,l}}{\sin\,\phi_{ik,l}\sin\,\phi_{jk,l}} = 0. \tag{2}$$

Thus a consistent gluing of the tetrahedra in a 4-simplex gives relations among the $\phi$s. There are three relations per triangle, hence 30 in total, of which only 20 are independent. To see this, we linearized equation (2) around generic non-degenerate configurations, including the potentially harmful orthogonal one, and used an algebraic manipulator to study the rank.

The good behaviour of the orthogonal configuration can also be anticipated by the absence of cosines in the denominator of (1). Of course, our construction would fail for degenerate configurations where one or more angles equal 0 or $\pi$.

These relations are important to characterize the geometry of a 4-simplex. Consider a generic 4-simplex. Its ten 4D angles $\theta_{ij}$ define the Gram matrix $G_{ij}(\theta) \equiv \cos\theta_{ij}$ (with the convention $\cos\theta_{ii} \equiv -1$). If the simplex is closed and flat, these ten angles cannot be all independent, but have to satisfy the condition of vanishing of the Gram determinant, $\det G = 0$ (e.g [16, 17])[3]. The nine independent quantities parametrize the space of shapes of the 4-simplex (a scale factor being the tenth and last metric variable).

We then expect that to characterize the geometry in terms of the thirty 3D angles $\phi_{ij,k}$, there must exist 21 relations among them. These can be found as follows. First, we consider the Gram matrices $G^k_{ij}(\phi)$ associated with the five tetrahedra; imposing the vanishing of their determinant guarantees that the tetrahedra are closed. These are five independent conditions. Next, we use the 2D angle consistency relations (2) to ensure a consistent gluing of the tetrahedra into a 4-simplex. The complete set

$$\det\, G^k(\phi) = 0, \qquad \mathcal{C}_{kl,ij}(\phi) = 0 \tag{3}$$

can be shown, again by linearization, to have rank 21. Notice that the first constraint is local on each tetrahedron, unlike the second that involves two adjacent tetrahedra. Hence we found a necessary and sufficient set of relations among the $\phi$ angles to be the 3D dihedral angles of a 4-simplex. Other sets are possible (see appendix B); the advantage of this one is the transparency of its geometric meaning.

## 3. Area–angle Regge calculus

With the understanding of the geometry of a 4-simplex gained above, we now come to the main point of this paper: describing the dynamics of GR on a discrete manifold, using areas and 3D angles as variables. For simplicity, we consider here the case of a Riemannian manifold with no boundaries. The extension to Lorentzian signature and to boundary terms will be discussed elsewhere.

Using the standard notation ($t$, a triangle; $e = tt'$, an edge), the variables on the full triangulation are $A_t$ and $\phi^\tau_e$. The gluing conditions (2) refer to each pair of edges in a triangle shared by two tetrahedra. In a triangulation there will be in general many tetrahedra around the same triangle, and (2) has to hold for any choice of two. However, transitivity ensures that it is enough to impose (2) to the pairs of tetrahedra belonging to the same 4-simplex. We can then write these constraints as

$$\mathcal{C}^\sigma_{ee'}(\phi^\tau_e) = 0, \tag{4}$$

where $\mathcal{C}^\sigma_{ee'}$ is given by (2) for $e = \sigma(kli)$ and $e' = \sigma(klj)$ sharing a vertex in a 4-simplex $\sigma$, and zero otherwise.

On a single 4-simplex we have ten areas and thirty 3D angles; thus we need 30 independent constraints to reduce the total number of variables to ten. The situation parallels the analysis we performed in the previous section. We can still take the triangle gluing conditions (4) involving only $\phi$ angles, and include the areas in the closure conditions for the five tetrahedra. Denoting $n_t$ the normal to a triangle, we have by definition $|n_t|^2 = A_t^2$ and $n_t \cdot n_{t'} = -A_t A_{t'} \cos\phi^\tau_{tt'}$.

[3] This condition is the origin of the well-known Schläfli identity.

The closure condition on a tetrahedron $\tau$ reads $N_\tau \equiv \sum_{t\in\tau} n_t = 0$. By sequentially taking the scalar product of $N_\tau$ with the four $n_t$ we obtain four constraints,

$$\mathcal{N}_t^\tau(A,\phi) = A_t - \sum_{t'\neq t} A_{t'} \cos\phi_{tt'}^\tau = 0. \tag{5}$$

Considering the five tetrahedra on the whole 4-simplex (5) gives 20 constraints, to be added to the 30 constraints (4). Again we studied the number of independent constraints by linearization, and found that the resulting system has rank 30 for a generic configuration and also for the orthogonal one. Consequently, only ten of the 40 variables used are truly independent. This is consistent with the kinematical degrees of freedom of discrete GR. As shown explicitly in appendix B, the 40 variables $(A_t, \phi_e^\tau)$ satisfying these 30 independent relations determine completely the geometry of the 4-simplex *and* of its five tetrahedra; thus each tetrahedron has a well-defined geometry, and gluing 4-simplices causes no problems. In other words, satisfying constraints (4) and (5) allows us to reconstruct uniquely a set of edge lengths from the variables $(A_t, \phi_e^\tau)$.

We then consider the following action for GR:

$$S[A_t, \phi_e^\tau, \lambda_t^\tau, \mu_{ee'}^\sigma] = \sum_t A_t \epsilon_t(\phi) + \sum_\tau \sum_{t\in\tau} \lambda_t^\tau \mathcal{N}_t^\tau(A,\phi) + \sum_\sigma \sum_{ee'\in\sigma} \mu_{ee'}^\sigma \mathcal{C}_{ee'}^\sigma(\phi). \tag{6}$$

The first term is just the Regge action with independent area–angle variables[4], and the other terms are the constraints (5) and (4) imposed by the Lagrange multipliers $\lambda_t^\tau$ and $\mu_{ee'}^\sigma$. As discussed above, they effectively reduce the set of variables $(A_t, \phi_e^\tau)$ to the edge lengths $\ell_e$; therefore (6) is equivalent to the conventional Regge action, $S_R[\ell_e] = \sum_t A_t(\ell_e)\epsilon_t(\ell_e)$.

Notice that our approach should not be seen as a first-order formulation of Regge calculus (see for instance [16, 18]), because we are adding 3D dihedral angles (the $\phi$s), not 4D dihedral angles (the $\theta$s): only the latter encode the extrinsic curvature of a 3D slice and are thus conjugate to the areas.

The reader might wonder at this point whether (6) is a discretization of a continuum action for GR, like Regge's is a discretization of the Einstein–Hilbert action $\int \sqrt{g}R$. We argue that this is the case, the continuum avatar of (6) being Plebanski's action [8]. The latter is a modified BF action, which schematically reads $S = \int B\wedge F + \mu\mathcal{C}(B)$ (we refer the reader to the literature for more details [19]). The term $\mathcal{C}(B)$ is a set of constraints reducing topological BF theory to GR. We are naturally led towards the interpretation of the first two terms of (6) as a discretization of the BF theory, with the closure constraint (5) implementing the Gauss constraint of the continuum BF action, and the third term as the simplicity constraints. Recall that Plebanski's constraints state that the bi-normal $B$ to any triangle $\sigma(ij)$ must be simple, i.e. it must be the wedge product of (any) two edge vectors. In our notation, $B_{ij} = \pm e_{ijk}\wedge e_{ijl}$ with $k$ and $l$ different from $i$ and $j$. Then if the closure and simplicity constraints are satisfied, they imply

$$\begin{aligned} B_{ij}\cdot B_{ij} &= {e_{ijk}}^2 {e_{ijl}}^2 - (e_{ijk}\cdot e_{ijl})^2 \\ &= {V_{ijk}}^2 {V_{ijl}}^2 \sin\alpha_{ijkl}^2 \\ &= {V_{ij}}^2, \end{aligned} \tag{7}$$

[4] The deficit angles $\epsilon_t$ are given by the sum over 4D angles on the 4-simplices sharing the triangle $t$, $\epsilon_t = 2\pi - \sum_\sigma \theta_t^\sigma$. We describe in appendix B how to express them in terms of the $\phi$s, or in terms of edge lengths as it is done in Regge calculus.

$$\begin{aligned} B_{ij} \cdot B_{ik} &= {e_{ijk}}^2 (e_{ijl} \cdot e_{ikl}) - (e_{ijk} \cdot e_{ijl})(e_{ijk} \cdot e_{ikl}) \\ &= {V_{ijk}}^2 V_{ijl} V_{ikl} \left( \cos\,\alpha_{il,jk} - \cos\,\alpha_{ij,kl} \cos\,\alpha_{ik,jl} \right) \\ &= V_{ij} V_{ik} \cos\,\phi_{jk,i}. \end{aligned} \tag{8}$$

Thus using (7) in (8) gives

$$\cos\,\phi_{jk,i} = \frac{\cos\,\alpha_{il,jk} - \cos\,\alpha_{ij,kl} \cos\,\alpha_{ik,jl}}{\sin\,\alpha_{ij,kl} \sin\,\alpha_{ik,jl}}. \tag{9}$$

This relation can be inverted to give (1) with (2) holding (see appendix B). Conversely if (2) and (5) are satisfied, we can proceed backwards and define bi-normals satisfying the simplicity constraints[5].

To further study this correspondence, a canonical analysis of the action (6) is in progress, and will appear elsewhere [22].

## 4. Conclusions

We introduced a modified Regge calculus where the fundamental variables are areas and 3D dihedral angles between triangles. The action, given in (6), is the conventional Regge term with independent area–angle variables, plus two additional constraints. The first imposes the closure of each tetrahedron in the triangulation, and the second guarantees a consistent gluing between adjacent tetrahedra, by imposing the (conformal) geometry of the common triangle to be the same. All the constraints are local in the triangulation. We expect our action to be related to a discretization of Plebanski's action.

Our main result is to show that the full set of constraints guarantees that local variables determine completely the geometry of each 4-simplex *and of each tetrahedron* in the triangulation. As 4-simplices are glued together identifying a tetrahedron in their boundary, being able to determine the tetrahedra's geometry is crucial to have a consistent propagation of the degrees of freedom in the triangulation. The crucial counting of the independent constraints was performed by linearizing the constraints around generic non-degenerate configurations (see appendix B). In particular, we checked that the potentially harmful orthogonal configuration is well behaved. The ambiguity of two sets of lengths giving the same areas [11] is removed in our formalism simply because the two sets give different 3D angles. On the basis of our analysis, we cannot exclude completely the presence of pathological configurations. However, a well-behaved orthogonal configuration reassures us that at least for the regular lattice our approach solves both difficulties with area Regge calculus described in section 1. This opens the way, for instance, to perturbation theory on a flat background.

While studying the constraints we found a number of relations between the dihedral angles of various dimensions of a 4-simplex. We present them together with their derivation in appendix A. We provide an explicit algorithm to compute the edge lengths from area–angles in a tetrahedron and in a 4-simplex in appendix B.

We expect our result to have a number of applications, and before concluding, we would like to briefly point out a few potentially promising ones.

[5] Notice that the simplicity constraints have typically solutions in two sectors. Here we are imposing directly the solution in the geometric sector, so we do not comment about this ambiguity, which however plays an important role in quantum models [15, 20, 21].

At the classical level, the canonical analysis of (6) could shed light on the description of the Hamiltonian algebra of deformation of discrete manifolds [22]. The applicability of this approach to numerical studies of lattice gravity has to be explored. It would for instance be interesting to study whether our approach keeps the good convergence properties of conventional Regge calculus in the continuum limit [23].

At the quantum level, there are possible links to the spinfoam formalism that are worth exploring. The formalism is expected to provide a well-defined measure for a regularized path integral for non-perturbative quantum gravity (however, see also [24]). Recently, a spinfoam model has been proposed [15] (see also [25]), whose dynamical variables can be expressed as normals to triangles [20] (see also [21]). The scalar reduction of these quantities produces exactly the variables $(A_t, \phi_e^\tau)$ considered here. The matching of variables suggests that the discrete calculus introduced here is a candidate for the semiclassical limit of this new spinfoam model, mimicking what happens in the 3D case with Regge calculus [26]. Indeed, the recent advances [27] in calculating the graviton propagator from spinfoams [28] are based precisely on such a link. In the context of pure area Regge calculus on a single 4-simplex, this idea has been investigated in [29].

From this viewpoint, it would be useful to study the quantum theory defined on a regular lattice in perturbative expansion around flat space-time, as done by Rocek and Williams [4].

## Acknowledgments

We thank Ruth M Williams for her inspirational work and Carlo Rovelli for useful discussions. This research was supported by the Perimeter Institute for Theoretical Physics. Research at the Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

## Appendix A. Relations between dihedral angles

To study the geometry of simplices, we use the affine (or barycentric) coordinates for Regge calculus introduced in [30] and further developed in [31] (see also [32]). With the help of these we derive relations between dihedral angles, and show how constrained areas and angles suffice to reconstruct edge lengths in 4-simplices[6].

Consider an $n$-dimensional simplex $\sigma$ with $n \geqslant 3$ and choose an $(n-3)$-dimensional subsimplex $\sigma(ijk)$. (Here, $\sigma(ijk\ldots)$ denotes the subsimplex spanned by all the vertices of $\sigma$ except for $(ijk\ldots)$.) We have three $(n-2)$-dimensional subsimplices $\sigma(ij), \sigma(ik)$ and $\sigma(jk)$ meeting at $\sigma(ijk)$. These carry the three $n$-dimensional dihedral angles $\theta_{ij}$, $\theta_{ik}$ and $\theta_{jk}$, respectively.

In addition, the subsimplex $\sigma(ijk)$ is shared by three $(n-1)$-dimensional subsimplices $\sigma(i), \sigma(j)$ and $\sigma(k)$. In the intrinsic geometry of these subsimplices we can define the $(n-1)$-dimensional angles $\phi_{lm,i}$, where $\phi_{lm,p}$ denotes the dihedral angle in the subsimplex $\sigma(p)$ between the simplices $\sigma(lp)$ and $\sigma(mp)$. There are again three $(n-1)$-dimensional angles $\phi_{ij,k}$, $\phi_{jk,i}$ and $\phi_{ik,j}$ meeting at the subsimplex $\sigma(ijk)$.

[6] Given our interest in a Regge-like formulation of discrete geometry, we focus here only on the case of flat simplices, with curvature concentrated on the $(n-2)$-subsimplices. The same technique can be used to find similar relations in the case of spherical or hyperbolical $n$-simplices.

It turns out that the set of angles $\theta_{ij}$, $\theta_{ik}$ and $\theta_{jk}$ can be computed from the set of $(n-1)$-dimensional angles $\phi_{ij,k}$, $\phi_{jk,i}$ and $\phi_{ik,j}$ and vice versa. There are different methods to derive these relations. Here we use the affine metric, referring to the appendix of [31] for an introduction to affine coordinates and more details.

The affine metric $\tilde{g}^{ij}$ associated with an $n$-simplex satisfies the following properties:

$$\tilde{g}^{ij} = -\frac{1}{V^2}\frac{\partial V^2}{\partial \ell_{ij}{}^2}, \quad i \neq j, \qquad \tilde{g}^{ii} = \frac{1}{n^2}\frac{V(i)^2}{V^2}. \tag{A.1}$$

Given the normal $n(i)$ to a subsimplex $i$, we can use the affine metric to compute the scalar products

$$n(i)\cdot n(i) = \tilde{g}^{ii} \equiv |n(i)|^2, \tag{A.2}$$

$$n(i)\cdot n(j) = \tilde{g}^{ij} \equiv -|n(i)||n(j)| \cos\theta_{ij}. \tag{A.3}$$

Notice that with these conventions the closure condition reads

$$V(j) = \sum_{i\neq j} V(i) \cos\theta_{ij}.$$

In analogy with the continuum, the normal vectors allow us to introduce the induced metric on a subsimplex $k$,

$$\tilde{g}^{ij}(k) = \tilde{g}^{ij} - \frac{\tilde{g}^{ik}\tilde{g}^{jk}}{\tilde{g}^{kk}}. \tag{A.4}$$

From (A.3) and (A.1(b)) we obtain

$$\cos\theta_{ij} = -n^2 \frac{V^2}{V(i)V(j)}\tilde{g}^{ij}. \tag{A.5}$$

This relation can be straighforwardly pushed one dimension down using the induced metric (A.4), to give

$$\cos\phi_{ij,k} = -(n-1)^2 \frac{V(k)^2}{V(ik)V(jk)}\tilde{g}^{ij}(k). \tag{A.6}$$

Using (A.5) and the well-known generalized law of sines

$$\sin\theta_{ij} = \frac{n}{(n-1)}\frac{V(ij)V}{V(i)V(j)} \tag{A.7}$$

to eliminate the volume factors in (A.6), we finally get

$$\cos\phi_{ij,k} = \frac{\cos\theta_{ij} + \cos\theta_{ik}\cos\theta_{jk}}{\sin\theta_{ik}\sin\theta_{jk}}. \tag{A.8}$$

These are relations between the three $\phi$ angles and the three $\theta$ angles at the subsimplex $\sigma(ikl)$. The inverse, giving the $\theta$ angles as a function of the three $\phi$ angles, has the following remarkable form:

$$\cos\theta_{ij} = \frac{\cos\phi_{ij,k} - \cos\phi_{ik,j}\cos\phi_{jk,i}}{\sin\phi_{ik,j}\sin\phi_{jk,i}}. \tag{A.9}$$

The formulae (A.8) and (A.9) can be adapted to one dimension down, giving the relations (1) and (9) in the main text between $(n-1)$- and $(n-2)$-dimensional angles[7].

For the special case of 3D space, these formulae had already appeared in the mathematical (e.g. [33]) and physical (e.g. [17, 31, 34]) literature.

From the formulae above, we can derive a number of other useful relations between areas and angles or between angles alone. Introducing the shorthand notation $\mathrm{c}(ij,k) \equiv \cos\phi_{ij,k}$, $\mathrm{s}(ij,k) \equiv \sin\phi_{ij,k}$, we have

$$\begin{aligned}
\frac{\mathrm{s}(jl,i)\mathrm{s}(ip,j)}{\mathrm{s}(jp,i)\mathrm{s}(il,j)} &= \frac{V(ip)V(lj)}{V(il)V(pj)},\\
\frac{\mathrm{c}(kl|i)-\mathrm{c}(ki|l)\mathrm{c}(li|k)}{\mathrm{s}(ki|l)\mathrm{s}(li|k)} &= \frac{\mathrm{c}(kl|j)-\mathrm{c}(kj|l)\mathrm{c}(lj|k)}{\mathrm{s}(kj|l)\mathrm{s}(lj|k)},\\
\mathrm{s}(jl|i)\mathrm{s}(kl|j)\mathrm{s}(il|k) &= \mathrm{s}(kl|i)\mathrm{s}(il|j)\mathrm{s}(jl|k),
\end{aligned}$$

as well as

$$\mathrm{s}(ij|k)\mathrm{s}(ik|l)\mathrm{s}(jl|i)\mathrm{s}(kl|j) = \mathrm{s}(ij|l)\mathrm{s}(ik|j)\mathrm{s}(kl|i)\mathrm{s}(jl|k). \tag{A.10}$$

Any of these relations can be taken as the starting point to constrain the area–angle variables. Their geometric interpretation is less immediate than the relations (2), and we do not discuss them here. The interested reader can work them out easily.

## Appendix B. Lengths from area–angles

In this appendix, we show how to explicitly construct the edge lengths from area–angles in a tetrahedron and in the full 4-simplex. Let us begin by considering the tetrahedron $i$, in the notation used so far. The closure reads

$$V(ik) = \sum_{j\neq i} V(jk)\cos\phi_{ij,k}. \tag{B.1}$$

Heron's formula for the area gives

$$16V(ij)^2 = 2\sum_{k,l\neq i,j} V(ijk)^2V(ijl)^2 - \sum_{k\neq i,j} V(ijk)^4.$$

Using the law of sines (A.7) adapted to 3D, the rhs gives $f_{ij}(V(ik), \sin\phi_{kl,i})/V^4$ where $f_{ij}$ is a simple polynomial in the areas and the sines. From this we read an (asymmetric) expression for

[7] An interesting alternative derivation of (A.8) and (A.9) uses a vanishing curvature condition. Consider the three $(n-1)$-dimensional subsimplices joined at $\sigma(ijk)$ as a piecewise linear $(n-1)$-dimensional geometry. Its intrinsic curvature is given by the deficit angle $\epsilon_{ij} = 2\pi - \phi_{ij,k} - \phi_{jk,i} - \phi_{ki,j}$, i.e. $2\pi$ minus the sum of the three $(n-1)$-dimensional angles meeting at $\sigma(ijk)$. In the flat $n$-dimensional embedding provided by the $n$-simplex, its extrinsic curvature is given by the three $n$-dimensional dihedral angles. That is, if we parallel-transport an $n$-dimensional vector according to the $n$-dimensional geometry around $\sigma(ijk)$ we should obtain the identity. The latter is a rotation in the 3D subspace orthogonal to $\sigma(ijk)$ that can be expressed using the $\theta$s and $\phi$s. Hence requiring this rotation to be equal to the identity leads to three conditions which can be used to express one set of angles as functions of the other set.

the tetrahedron's volume in terms of areas and angles,

$$V(i)^4 = \frac{1}{16V(ij)^2} f_{ij}\left(V(ik), \sin\phi_{kl,i}\right). \tag{B.2}$$

Finally, we can use (A.7) to express the six edge lengths in terms of areas and angles,

$$V(ijk) = \tfrac{4}{3} f_{ij}^{-1/4} V(ij)^{3/2} V(ik) \sin\phi_{jk,i}. \tag{B.3}$$

Notice that the procedure becomes ill-defined for degenerate configurations where $V(i) = 0$, but it is perfectly well defined otherwise, including orthogonal configurations with right angles.

For the 4-simplex we proceed exactly as above, using the relations (B.3) in each tetrahedron. All that remains to prove is that conditions (2) are enough to ensure that (B.3) applied to the different tetrahedra sharing the same edge gives the same value, namely that

$$\begin{aligned}
V(ijk) &= \frac{2}{3}\frac{V(ij)V(ik)}{V(i)}\sin\phi_{jk,i} \\
&= \frac{2}{3}\frac{V(ij)V(jk)}{V(j)}\sin\phi_{ik,j} \\
&= \frac{2}{3}\frac{V(ik)V(jk)}{V(k)}\sin\phi_{ij,k}.
\end{aligned}$$

Thus, for instance, we need to show that

$$\frac{V(i)}{V(ik)\sin\phi_{jk,i}} = \frac{V(j)}{V(jk)\sin\phi_{ik,j}}.$$

This is a tedious but straighforward check that can be done using (B.2) for the two volumes, and then twice (A.10).

This construction shows explicitly how constraints (2) and (5), which imply (A.10) and (B.2), allow us to reconstruct unambiguously the edge lengths of a tetrahedron and of a 4-simplex starting from areas and angles.

To make sure that the set of constraints (2) and (5) are sufficient, we studied the matrix of coefficients of their linearization in a 4-simplex. The entries in this matrix depend on the configuration around which we choose to linearize. We considered two different configurations: the equilateral and the orthogonal ones. The equilateral one has no $\phi$ angles equal to $\pi/2$ (which would lead to zeros in the matrix of coefficients), and in this sense is representative of a generic configuration. The orthogonal one has the maximal number of $\phi$ angles equal to $\pi/2$ and thus the maximal number of zeros in the matrix of coefficients. It is the dangerous configuration where area Regge calculus is ill-defined, and it corresponds to a 4-simplex fitting into a regular hypercubical lattice. In both cases the rank of the matrix turned out to be exactly 30, meaning that the initial 40 variables can be reduced to ten, which can then be chosen to be the edge lengths using the explicit procedure described above. The same procedure was used to check that (3) has rank 21.

This shows that there cannot be singular configurations where two different sets of lengths (and thus two different metrics) correspond to the same set of area–angle variables, as is the case for area Regge calculus.

## References


[1] Regge T 1961 General relativity without coordinates *Nuovo Cimento* **19** 558

[2] Regge T and Williams R M 2000 Discrete structures in gravity *J. Math. Phys.* **41** 3964 (*Preprint* gr-qc/0012035)

[3] Loll R 1998 Discrete approaches to quantum gravity in four dimensions *Living Rev. Relativity* **1** 13 (*Preprint* gr-qc/9805049)

Gentle A P and Miller W A 2001 A brief review of Regge calculus in classical numerical relativity *Preprint* gr-qc/0101028

Hamber H W 2007 Discrete and continuum quantum gravity *Preprint* 0704.2895 [hep-th]

[4] Rocek M and Williams R M 1981 Quantum Regge calculus *Phys. Lett.* B **104** 31

Rocek M and Williams R M 1984 The quantization of Regge calculus *Z. Phys.* C **21** 371

Hamber H W and Williams R M 1993 Simplicial quantum gravity in three-dimensions: analytical and numerical results *Phys. Rev.* D **47** 510

[5] Hamber H W and Williams R M 2004 Non-perturbative gravity and the spin of the lattice graviton *Phys. Rev.* D **70** 124007 (*Preprint* hep-th/0407039)

[6] Immirzi G 1997 Quantum gravity and Regge calculus *Nucl. Phys. Proc. Suppl.* **57** 65 (*Preprint* gr-qc/9701052)

[7] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)

[8] Plebanski J F 1977 On the separation of Einsteinian substructures *J. Math. Phys.* **18** 2511

[9] Rovelli C 1993 The basis of the Ponzano–Regge–Turaev–Viro–Ooguri quantum gravity model in the loop representation basis *Phys. Rev.* D **48** 2702

[10] Makela J 1994 On the phase space coordinates and the Hamiltonian constraint of Regge calculus *Phys. Rev.* D **49** 2882

Makela J 2000 Variation of area variables in Regge calculus *Class. Quantum Grav.* **17** 4991 (*Preprint* gr-qc/9801022)

Makela J and Williams R M 2001 Constraints on area variables in Regge calculus *Class. Quantum Grav.* **18** L43 (*Preprint* gr-qc/0011006)

[11] Barrett J W, Rocek M and Williams R M 1999 A note on area variables in Regge calculus *Class. Quantum Grav.* **16** 1373 (*Preprint* gr-qc/9710056)

[12] Wainwright C and Williams R M 2004 Area Regge calculus and discontinuous metrics *Class. Quantum Grav.* **21** 4865 (*Preprint* gr-qc/0405031)

[13] Reisenberger M P 1997 A left-handed simplicial action for Euclidean general relativity *Class. Quantum Grav.* **14** 1753 (*Preprint* gr-qc/9609002)

Reisenberger M P 1998 Classical Euclidean general relativity from 'left-handed area=right-handed area' *Preprint* gr-qc/9804061

[14] Rovelli C 2007 unpublished

[15] Engle J, Pereira R and Rovelli C 2007 The loop-quantum-gravity vertex-amplitude *Phys. Rev. Lett.* **99** 161301 (*Preprint* 0705.2388) [gr-qc]

Engle J, Pereira R and Rovelli C 2008 Flipped spinfoam vertex and loop gravity *Nucl. Phys.* B **798** 251 (*Preprint* 0708.1236) [gr-qc]

[16] Barrett J W 1994 First order Regge calculus *Class. Quantum Grav.* **11** 2723 (*Preprint* hep-th/9404124)

[17] Freidel L and Louapre D 2003 Asymptotics of 6j and 10j symbols *Class. Quantum Grav.* **20** 1267 (*Preprint* hep-th/0209134)

[18] Caselle M, D'Adda A and Magnea L 1989 Regge calculus as a local theory of the Poincare group *Phys. Lett.* B **232** 457

[19] Capovilla R, Jacobson T and Dell J 1989 General relativity without the metric *Phys. Rev. Lett.* **63** 2325

De Pietri R and Freidel L 1999 SO(4) Plebanski action and relativistic spin foam model *Class. Quantum Grav.* **16** 2187 (*Preprint* gr-qc/9804071)

Perez A 2002 Spin foam quantization of SO(4) Plebanski's action *Adv. Theor. Math. Phys.* **5** 947

Perez A 2003 *Adv. Theor. Math. Phys.* **6** 593 (*Preprint* gr-qc/0203058)

Smolin L 2007 The Plebanski action extended to a unification of gravity and Yang–Mills theory *Preprint* 0712.0977 [hep-th]

[20] Livine E R and Speziale S 2007 New spinfoam vertex for quantum gravity *Phys. Rev.* D **76** 084028 (*Preprint* 0705.0674) [gr-qc]

Livine E R and Speziale S 2008 Solving the simplicity constraints for spinfoam quantum gravity *Europhys. Lett.* **81** 50004 (*Preprint* 0708.1915) [gr-qc]

[21] Freidel L and Krasnov K 2007 A new spin foam model for 4d gravity *Preprint* 0708.1595 [gr-qc]

[22] Dittrich B and Ryan J P 2008 Phase space descriptions for 4d simplicial geometries *Preprint* 0807.2806 [gr-qc]

[23] Brewin L C and Gentle A P 2001 On the convergence of Regge calculus to general relativity *Class. Quantum Grav.* **18** 517 (*Preprint* gr-qc/0006017)

[24] Bojowald M and Perez A 2003 Spin foam quantization and anomalies *Preprint* gr-qc/0303026

[25] Engle J, Livine E, Pereira R and Rovelli C 2007 LQG vertex with finite Immirzi parameter *Preprint* 0711.0146 [gr-qc]

[26] Ponzano G and Regge T 1968 Semiclassical limit of Racah coefficients *Spectroscopy and Group Theoretical Methods in Physics* ed F Bloch (Amsterdam: North-Holland)

[27] Ashtekar A 2007 Loop quantum gravity: four recent advances and a dozen frequently asked questions *Preprint* 0705.2222 [gr-qc]

[28] Rovelli C 2006 Graviton propagator from background-independent quantum gravity *Phys. Rev. Lett.* **97** 151301 (*Preprint* gr-qc/0508124)

Speziale S 2006 Towards the graviton from spinfoams: the 3d toy model *J. High Energy Phys.* JHEP05(2006)039 (*Preprint* gr-qc/0512102)

Bianchi E, Modesto L, Rovelli C and Speziale S 2006 Graviton propagator in loop quantum gravity *Class. Quantum Grav.* **23** 6989 (*Preprint* gr-qc/0604044)

Livine E R, Speziale S and Willis J L 2007 Towards the graviton from spinfoams: higher order corrections in the 3d toy model *Phys. Rev.* D **75** 024038 (*Preprint* gr-qc/0605123)

Livine E R and Speziale S 2006 Group integral techniques for the spinfoam graviton propagator *J. High Energy Phys.* JHEP11(2006)092 (*Preprint* gr-qc/0608131)

Christensen J D, Livine E R and Speziale S 2007 Numerical evidence of regularized correlations in spin foam gravity *Preprint* 0710.0617 [gr-qc]

Alesci E and Rovelli C 2007 The complete LQG propagator: I. Difficulties with the Barrett-Crane vertex *Phys. Rev.* D **76** 104012 (*Preprint* 0708.0883) [gr-qc]

Alesci E and Rovelli C 2008 The complete LQG propagator: II. Asymptotic behavior of the vertex *Phys. Rev.* D **77** 044024 (*Preprint* 0711.1284) [gr-qc]

[29] Bianchi E and Modesto L 2007 The perturbative Regge-calculus regime of loop quantum gravity *Preprint* 0709.2051 [gr-qc]

[30] Sorkin R 1975 The electromagnetic field on a simplicial net *J. Math. Phys.* **16** 2432

Sorkin R 1978 *J. Math. Phys.* **19** 1800

[31] Dittrich B, Freidel L and Speziale S Linearized dynamics from the 4-simplex Regge action *Phys. Rev.* D **76** 104020 (*Preprint* 0707.4513) [gr-qc]

[32] Dittrich B and Loll R 2006 Counting a black hole in Lorentzian product triangulations *Class. Quantum Grav.* **23** 3849 (*Preprint* gr-qc/0506035)

[33] Luo F 2005 Volume and angle structures on 3-manifolds *Preprint* math/0504049

[34] 't Hooft G 1993 Canonical quantization of gravitating point particles in (2 + 1)-dimensions *Class. Quantum Grav.* **10** 1653 (*Preprint* gr-qc/9305008)