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Non-resonant instability of coupled Alfvén and drift compressional modes in magnetospheric plasma

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Abstract
A new mechanism of generation of the high-$m$ compressional ULF waves in the magnetosphere is considered. It is suggested that the wave can be generated by the non-resonant instability of coupled Alfvén and drift compressional modes in the energetic component of the magnetospheric plasma. A stability analysis of the of the coupled modes in the inhomogeneous finite-$\beta$ plasma in the dipole-like field in gyrokinetics is performed. A quadratic equation was obtained that determines mode frequency and the growth rate. The frequencies of both modes depend on the azimuthal wave number, $m$. The branches are merged at some critical $m$ value, forming a mode with both real and imaginary parts of the wave frequency. This mode is amplified due to the instability called the drift coupling instability. The instability criterion was found. Its growth rate is determined by the mode coupling.

Keywords: space plasma, plasma waves, ULF waves, magnetosphere, plasma instabilities, gyrokinetics

(Some figures may appear in colour only in the online journal)

1. Introduction

The Alfvén waves are often observed in terrestrial magnetosphere [1]. The azimuthal large-scale Alfvén waves (azimuthal wave numbers $m \sim 1$) are probably generated due to the direct interaction with the solar wind, while the azimuthal small-scale Alfvén waves ($m \gg 1$) generated due to instabilities in hot space plasma component. In the latter case, the wave energy can be derived from unstable particle populations via the drift of drift-bounce resonance [2, 3],

$$\omega - m \bar{\omega}_d - K \omega_b = 0,$$

where $\omega$ is the wave’s frequency, $m$ is the azimuthal wave number, $\bar{\omega}_d$ is the bounce averaged drift frequency, $\omega_b$ is the bounce frequency, and $K$ is integer ($K = 0$ and $K = 0$ cases correspond to the drift-bounce and drift resonances, respectively). The conditions for those instabilities were considered in a number of publications [4–6]. According to [7, pp 18–19], these instabilities belong to the dissipative-type instabilities. Recent observational cases for drift of drift-bounce instabilities can be found in [8, 9]. The high-$m$ Alfvén waves can be generated also by means of alternating currents of substorm-injected energetic particles drifting in the azimuthal direction [10–12]. In this case, the wave’s frequency is formally determined by equation (1) with $K = 0$.

Another ULF wave’s branch that can exist in the space plasmas is the drift compressional mode [13, 14]. This mode can be responsible for the so-called ‘compressional storm time Pc5 pulsations’ [15, 16] and the flapping waves in the magnetotail [17]. Observational evidences for these modes were obtained in recent radar studies [18–20]. The generation mechanisms of drift compressional modes have been considered in [21–23]. The instability responsible for the generation of this mode has the dissipative character. The necessary condition for the wave generation considered in those papers is the same drift bounce resonance condition, equation (1).

The Alfvén and drift compressional modes are coupled with each other due to the plasma and magnetic field inhomogeneity [24–26]. However, the effects of the coupling are...
not clearly understood. The stability of these coupled modes in simple model with circular field lines was investigated by [27]. The instability develops when the Alfvén and drift compressional oscillation branches are merged at some critical values of the westward electric current. The drift bounce resonance condition (1) is not satisfied for this instability, and the wave’s frequency is determined in different way, by the mode coupling characteristics. Thus, that instability has non-resonant character. This instability can be classified as the reactive-type plasma instabilities [7, pp 19–20]. The instability threshold depends on the diamagnetic drift frequency, which, in turn, is determined by the azimuthal wave number m. The unstable mode has a real part of the oscillation frequency of the order of the drift frequency. The observational signatures for this instability were presented in [18].

However, the model with constant field line curvature used in [27] is far from realistic. Also, that paper did not consider dependence of the instability threshold on the azimuthal wave number m, an important issue because this value determines the drift compression mode and Alfvén modes in the dipole-like model by taking into account not only the inhomogeneity across the magnetic shells but also in field-aligned direction, including the magnetic traps and associated bounce motion.

2. Model and initial equations

As an approximation of the magnetosphere, we consider an axisymmetric model, in which plasma is inhomogeneous both across the magnetic shells and along the magnetic field lines. We use orthogonal curvilinear coordinate system \( \{ x^1, x^2, x^3 \} \) where \( x^1 \) is the radial coordinate (across the magnetic shells), \( x^3 \) is the field-aligned coordinate, and \( x^2 \) is the azimuthal coordinate (Figure 1). The increment of physical length along a coordinate line is \( dl = \sqrt{g_{ij}}dx^i \), where \( g_{ij} \) is the component of the metric tensor, and \( \sqrt{g} \) is the Lamé coefficient.

We assume that the plasma is isotropic, and composed of core cold particles and an admixture of hot protons with the Maxwell distribution

\[
F(\varepsilon, x^i) = \frac{n(x^i)}{(2\pi T(x^i)))^{3/2}} e^{-\varepsilon/T(x^i)},
\]

where \( \varepsilon = (v^2 + v^2)/2 \) is the particle energy per unit mass, \( v_1 \) and \( v_2 \) are the particle velocities along and across the ambient magnetic field, \( T \) is the particle temperature in units of e and \( n \) is the particle number density. For such plasma the equilibrium condition is written as

\[
(\beta/2) \kappa_p + \kappa_B + \kappa_R = 0
\]

and

\[
\frac{\partial P}{\partial x^3} = 0,
\]

where

\[
\kappa_p = \frac{1}{\sqrt{g_1 g_2}} \frac{\partial P}{\partial x^1}, \quad \kappa_B = \frac{1}{\sqrt{g_1 g_2}} \frac{\partial B}{\partial x^1}, \quad \kappa_R = \frac{1}{\sqrt{g_2}} R.
\]

\( R \) is the local field line curvature radius, \( P \) is the plasma pressure, \( B \) is the equilibrium magnetic field, and \( \beta = 8\pi P/B^2 \). Note that the \( \kappa_p, \kappa_B, \) and \( \kappa_R \) are functions of the radial \( x^1 \) and parallel \( x^3 \) coordinates.

The oscillations with high azimuthal wave numbers (\( m \gg 1 \)) are considered. Then the transverse WKB approximation can be used, where the wave variables depend on time and coordinates as

\[
A(x^1, x^3) \exp[-i\omega t + i \int k_1(x^3)dx^1 + ik_2x^2].
\]

Here \( \omega \) is wave frequency, \( A(x^1, x^3) \) denotes amplitudes of all the perturbations, \( k_1(x^3) \) and \( k_2 \) are the radial and azimuthal components of the wave vector, respectively. If the azimuthal angle is used as the \( x^2 \) coordinate, the \( k_2 \) is the azimuthal wave number. Note that the transverse WKB approximation assumes that [28]

\[
\left| \frac{1}{\sqrt{g_1}} \frac{\partial A}{\partial x^2} \right| \ll \frac{k_1}{\sqrt{g_1}}.
\]

The wave phenomena with the wave frequency \( \omega \) much less than the gyrofrequency \( \omega_g \), are conventionally studied within the gyrokinetic formalism [25, 29–31]. The gyrokinetic system of wave equations obtained in [25] can be used for the variables \( \psi \) and \( b \) describing respectively the transverse electric \( \vec{E}_b = -\nabla_\psi \psi \) and parallel magnetic \( \vec{b}_b = \frac{\omega_p}{\omega} \psi \) field of the wave. Moreover, the wave frequencies are assumed to be much smaller than proton bounce frequency \( \omega_p \), and wave lengths are much larger than the proton Larmor radius. In this case the gyrokinetic wave equations from [25] can be reduced to a system of two integral-differential equations (equations (2) and (3) from [23]):

\[
B_0 \frac{\partial}{\partial t} \left[ k_2 \frac{\partial}{\partial x^2} \psi + \frac{\omega p^2}{V_A^2} \psi + \frac{1}{2} \beta k_p (\kappa_B - \kappa_R) \psi \right] + k_2 \frac{4\pi q^2}{m_e c^2} \left( \frac{\hat{\Omega} F}{\omega - \omega_d} \right) + k_2 \frac{1}{2} \beta k_p b + \frac{4\pi q}{e} \left( \frac{\hat{\Omega} F}{\omega - \omega_d} \right) = 0,
\]
Here \( l \) is the physical length along a field line, \( V_A \) is the Alfvén velocity, \( m_p \) is the proton mass, \( k_p^2 = k_l^2/g_0 + k_2^2/g_2 \) is the wave vector’s transverse component squared, 
\[
(\ldots) = 4\pi \int (\ldots) \frac{B}{|v|} \, d\mu_{\omega} \varepsilon
\]
is the integral over velocity space, 
\[
(\ldots) = \frac{2}{\tau_B} \int_{-l_0}^{l_0} (\ldots) |v|^{-1} \, dl
\]
is an average for a bounce period \( \tau_B \),
\[
\tau_B = 2 \int_{-l_0}^{l_0} |v|^{-1} \, dl,
\]
where \( \pm l_0 \) are reflection points of a particle with energy \( \varepsilon \) and magnetic moment \( \mu = v_l^2/(2B) \), bounce frequency \( \omega_b = 2\pi/\tau_B \),
\[
\omega_d = \frac{k_l \cdot V_d}{\omega_l} = \frac{k_2}{\omega_l} \left( \frac{\kappa_R}{2} - \kappa_l v_l^2 \right)
\]
is the drift frequency, which corresponds to the frequency of a wave with azimuthal phase velocity as the magnetic (gradient and curvature) drift velocity \( V_d \), \( \omega_l \) is the proton gyrofrequency, and \( Q \) is the operator defined as
\[
\hat{Q} = \frac{\omega}{\omega_l} \frac{\partial}{\partial \varepsilon} + \frac{k_2}{\omega_l} \frac{1}{\sqrt{g_2}} \frac{1}{\partial \lambda^1}.
\]

The equation (6) is the Alfvén mode equation determining the transverse electric and magnetic perturbations, and equation (7) is the compressional mode equation determining the parallel magnetic perturbations. If \( \beta = 0 \), then equation (6) becomes the Alfvén equation for a cold plasma in a curved magnetic field,
\[
B \frac{\partial}{\partial l} \frac{k_l^2}{B} \frac{\partial}{\partial l} \frac{\omega_d^2}{V_A^2} \psi + 0 = 0,
\]
which coincides with the Alfvén mode equation in cold plasma in MHD theory (for example, see equation (15) in [28]). The equation (7) in zero-pressure plasma is reduced to the trivial equality \( 0 = 0 \). If \( \beta > 0 \), but the mode coupling is neglected, then (7) describes the drift compressional mode. The coupling can be neglected in the case \( |k_l| \rightarrow \infty \) (the compressional resonance) [23].

Changing variables \( \varepsilon, \mu \) to \( \xi, \lambda \) in equations (6) and (7), where \( \lambda = \sin^2 \alpha = \mu_0 B_0 \varepsilon, \alpha \) is the pitch-angle, \( B_0 \) is the magnetic field value at the geomagnetic equator, and \( \xi = \sqrt{\varepsilon}/T \), after integration over \( \xi \) and \( \lambda \), assuming that \( \omega_b \) and \( \omega_d \) weakly depend on \( \lambda \), we reduce the system of two integral-differential equations (6) and (7) to the canonical form:
\[
\begin{align*}
B_0 \frac{\partial}{\partial l} \frac{k_l^2}{B} \frac{\partial}{\partial l} \frac{\omega_d^2}{V_A^2} \psi + k_2^2 \Lambda(\omega) & \int_0^{l_B} \frac{B_0}{B} K_A \psi' \, dl' + k_2 \frac{1}{2} \beta \kappa_l \psi \\
& + k_2 \Lambda(\omega) \int_0^{l_B} \frac{B_0}{B} K_M b' \, dl' - b + k_2 \frac{1}{2} \beta \kappa_l \psi \\
& + k_2 \Lambda(\omega) \int_0^{l_B} \frac{B_0}{B} K_C \psi' \, dl' = 0,
\end{align*}
\]
(a detailed description of the transformations can be found in [23]). Here prime denotes that a variable depends on \( l' \) instead \( l \) and the integration is performed from the geomagnetic equator, \( l = 0 \), to the ionosphere, \( l = l_I \).

\[
K_A = (\kappa_R + 2\kappa_k)(\kappa_R b' + 2\kappa_k) B_0^2 B^2 / B_0^2
\]
\[
\int \left[ 2\kappa_R (\kappa_R b' + 2\kappa_k) B_0^2 B^2 + 2\kappa_k (\kappa_R + 2\kappa_k) B_0^2 B^2 \right] \, dl + 4\kappa_R \kappa_k l_0,
\]
\[
K_C(l, l') = (\kappa_R + 2\kappa_k) B_0^2 B^2 / B_0^2,
\]
\[
K_M(l, l') = K_C(l, l'), \quad K_M = B_0^2 B^2 / B_0^2
\]
\[
I_0 = \Theta(0 - l') \int_0^{B_0/B} \frac{\lambda d\lambda}{u(\lambda)} u'(\lambda) \, \Theta(l' - l)
\]
\[
x \int_0^{B_0/B} \frac{\lambda d\lambda}{u(\lambda)} u'(\lambda), \quad (i = 0, 1, 2),
\]
\[
u(\lambda) = \sqrt{1 - \lambda B/B_0}
\]
\[
I_0 = \frac{B_0}{\sqrt{B B^2}} \ln \left( \frac{\sqrt{B} + \sqrt{B^2}}{\sqrt{B} - \sqrt{B^2}} \right), \quad I_1 = \frac{B_0^2}{2} - \frac{1}{2} \left( \frac{B_0}{B} + \frac{B_0}{B^2} \right) l_0,
\]
\[
I_2 = - \frac{3 B_0^2}{4 B} \left( \frac{B_0}{B} + \frac{B_0}{B^2} \right) + \frac{3}{8} \left( \frac{B_0}{B} + \frac{B_0^2}{B^2} \right) l_0,
\]
\[
\Lambda(\omega) = \frac{2\beta_0}{L_0} \left[ - \frac{15}{8} \frac{\omega_d}{\Omega_d} \right] + \left[ \frac{\omega - \omega_b}{\omega} - \frac{\omega_b}{\Omega_d} \right]
\]
\[
\times \left[ \frac{3}{4} \frac{\omega}{\Omega_d} + \frac{1}{2} \left( \frac{\omega}{\Omega_d} \right)^2 + \frac{3}{2} \frac{\omega}{\Omega_d} \right] Z \left( \frac{\omega}{\Omega_d} \right),
\]
\[
Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt
\]
is the plasma dispersion function, \( \Omega_d = \omega_d T/\varepsilon \) is the bounce averaged magnetic drift frequency of particles with energy \( \varepsilon = T \).
\[
\omega_b = \omega_{\eta \mu} - \frac{3}{2} \omega_{e\varepsilon}, \quad \omega_{\eta \mu} = \frac{k_2 T}{\omega_c} \frac{\partial n}{\omega_c \sqrt{g_2} n \partial \lambda^1},
\]
\[
\omega_{e\varepsilon} = \frac{k_2 T}{\omega_c \sqrt{g_2} T} \frac{\partial T}{\partial \lambda^1}
\]
are the diamagnetic frequencies,

\[ L_b = v_p \beta_0 = 4 \int_0^{\omega_b(\lambda)} u(l, \lambda)^{-1} dl \]

is the particle path length over the bounce period, and \( \beta_0 \) is the equatorial value of the plasma to magnetic pressure ratio.

In this paper we consider only waves propagating in the direction of the energetic proton drift, so that drift frequency \( \Omega_d > 0 \) (proton drift velocity \( V_d < 0 \) and wave vector azimuthal component \( k_2 < 0 \)). Wave with \( k_2 > 0 \) cannot interact with energetic protons, \( \Lambda(\omega) \) is real value and determined the other expression than (11).

In two limiting cases the expression for the \( \Lambda(\omega) \) function can be sufficiently simplified: for large frequencies, \( \omega \gg \Omega_d \):

\[ \Lambda(\omega) = \frac{15}{4} \beta_0 \left( \frac{\omega_b - \omega}{L_b} + \frac{7}{2} \frac{\omega_b}{\omega} \right) \]  

and for small frequencies, \( \omega \ll \Omega_d \):

\[ \Lambda(\omega) = \left( \frac{3}{2} \frac{\beta_0^2}{L_b} \right) \left( 1 - \frac{2 \omega_b}{3 \Omega_d} \right) - \frac{\omega_b}{\Omega_d} \frac{\omega_b}{\Omega_d} \]  

These two expressions are obtained under an assumption that imaginary part of the frequency is negligibly small. The imaginary part appear due to wave-particle resonance. The effects of the wave-particle interactions can result in additional plasma instabilities [16, 23]. However, we deal with the instability of another kind, the one caused by the mode coupling. Thus, the wave-particle resonances will not concern us here.

3. Quadratic form

Let us consider the auxiliary equation

\[ b = \Lambda(\omega) \int_0^{l_b} B_0 B' K_M b' d l' \]  

representing equation (10) with \( \psi = 0 \). As [23] showed, this equation governs the compressional mode structure for the case \( k_1 \rightarrow \infty \) (the compressional resonance). Let us designate the eigenvalues of this inhomogeneous Fredholm integral equation as \( \Lambda_N \), and its as eigenfunctions \( b_N \). These eigenfunctions can be conveniently normalized so that

\[ \int_0^{l_b} B_0 B' b_N d l' = 1. \]

It should be noted that the functions \( b_N \) are localized near the geomagnetic equator, and \( \Lambda_N \) are positive real values increasing with increasing harmonic number \( N \), i.e. with decreasing field-aligned wave length [23]. The calculated in [23] first three eigenvalues are \( \Lambda_1 \approx 0.5/R_0 \), \( \Lambda_2 \approx 1.5/R_0 \), \( \Lambda_3 \approx 2.5/R_0 \), where \( R_0 \) is the equatorial field line curvature radius.

The \( \Lambda_N \) values lead to the eigenfrequencies of the compressional resonances trough equation (11). In what following, the frequency of the principal harmonic of the compressional resonance (\( N = 1 \)) will be designated as \( \Omega_M \).

That is, \( \Lambda(\Omega_M) = \Lambda_1 \). The explicit expressions for \( \Omega_M \) can be found in the limiting cases of large and small frequencies. For the large frequencies, \( \omega \gg \Omega_d \), it follows from equation (13):

\[ \Omega_M = \left( \frac{-7 \omega_b}{\omega} + \frac{4}{15} \frac{L_b}{\beta_0} \right)^{-1}. \]

For the small frequencies, \( \omega \ll \Omega_d \), it follows from equation (14):

\[ \Omega_M = \Omega_d \left( \frac{\omega_b}{\Omega_d} + \frac{2L_b}{3 \beta_0} \right) \left( 1 - \frac{2 \omega_b}{3 \Omega_d} \right)^{-1}. \]

Remind that according to equations (8) and (12) all the values \( \Omega_d \), \( \omega_b \), \( \omega_b \), \( \omega_b \) are proportional to the wave vector azimuthal component, \( k_2 \). Thus, it is evident from equations (16) and (17) that \( \Omega_M \) is proportional to the wave vector azimuthal component, or, equivalently, to the azimuthal wave number \( m \). It is not hard to see from equation (11) that this conclusion is valid for the general case:

\[ \Omega_M(x', m) = m \Omega_M(x'), \]

where \( \Omega_M \) is the angular azimuthal phase velocity of the drift compressional mode, which depends on plasma parameters only.

Then, the solution \( b \) of equation (10) can be written as the expansion in terms of eigenfunctions \( b_N \) as

\[ b = k_2 \sum N \frac{\Lambda_N}{\Lambda_N - \Lambda(\omega)} b_N \int_0^{l_b} \frac{B_0 b_N}{B'} \left( \frac{1}{2} \beta_k \psi \right) + \Lambda(\omega) \int_0^{l_b} \frac{B_0}{B'} K_{\psi} \psi d l' \]  

After substituting the function \( b \) from equation (19) to equation (9), the system of two equations (9) and (10) is reduced to the single integral-differential equation:

\[ \frac{\partial}{\partial l} \frac{k_2^2}{B} \frac{\partial}{\partial l} \psi + \frac{\omega b}{V_A} \psi + k_2^2 \frac{1}{2} \beta_k \psi \left( \frac{b_N}{b} \right) + k_2^2 \Lambda(\omega) \int_0^{l_b} \frac{B_0}{B'} K_{\psi} \psi d l' \]  

\[ + \frac{1}{2} \Lambda_N \sum N \left( \frac{b_N}{b} \right) + \Lambda(\omega) \int_0^{l_b} \frac{B_0}{B'} K_{\psi} \psi d l' \]  

\[ \times \int_0^{l_b} \frac{B_0}{B'} \left( \frac{1}{2} \beta_k \psi \right) + \Lambda(\omega) \int_0^{l_b} \frac{B_0}{B'} K_{\psi} \psi d l' \]  

\[ = 0. \]

Further, the case of the purely azimuthal perturbations will be considered: \( k_1/k_2 = 0 \). Then, the mode frequency will be assumed to be not much different from the eigenfrequency of fundamental harmonic (\( N = 1 \)) of the drift compressional mode designated as \( \Omega_M \). Then, the function \( \Lambda(\omega) \) in equations (19) and (20) must be close to the value \( \Lambda_1 \). When \( \omega \approx \Omega_M \), the expansion can be used

\[ \Lambda(\omega) \approx \Lambda_1 + (\omega - \Omega_M) \left. \frac{\partial \Lambda(\omega)}{\partial \omega} \right|_{\omega = \Omega_M}. \]
As a result we obtain from the equation (20)
\[
B \frac{\partial}{\partial t} \frac{1}{g_2 B} \frac{\partial}{\partial t} \psi + \frac{\omega^2}{g_2 V_A^2} \psi + \delta L \psi = - \sigma \Omega_M \dot{L} \psi = 0, 
\]
where
\[
\delta L \psi = \frac{1}{2} \beta \kappa (\kappa_B - \kappa_R) \psi + \Lambda_1 \int_0^l \frac{B_0}{B} K_1 \psi' \psi' \,dl', 
\]
\[
\dot{L} \psi = \left( \frac{1}{2} \beta \kappa b_1 + \Lambda_1 \int_0^l \frac{B_0}{B} K_1 \psi' \psi' \right) \frac{1}{2} \beta \kappa \psi + \Lambda_1 \int_0^l \frac{B_0}{B} K_1 \psi' \psi' \,dl = 0, 
\]
\[
\sigma = \sigma (\Omega_M), \quad \sigma (\omega) = \left[ \frac{\omega}{\Lambda (\omega)} \right]^{-1}. 
\]
Let us assume that \( \psi \) is a solution of equation (21) satisfying the boundary conditions on the ionosphere \( \psi (\pm l) = 0 \). After multiplying all terms equation (21) on \( \psi B_0 / B \) and integrating them along field line from 0 to \( l \) we obtain the quadratic form
\[
(\omega - \Omega_M)(\omega^2 - \Omega^2_M) - \sigma \Omega^2_M \Omega_M = 0. 
\]
Here
\[
\Omega^2_C = \int_0^l \frac{B_0}{B} \psi \dot{L} \psi \,dl / \int_0^l \frac{B_0}{B} \frac{|\psi|^2}{g_2 V_A^2} \,dl, 
\]
\[
\Omega^2_P = \Omega^2_A - \Delta \Omega^2_P, 
\]
\[
\Omega^2_A = \int_0^l \frac{B_0}{B} \frac{1}{g_2} \frac{\partial \psi}{\partial l} \frac{\partial \psi}{\partial l} \,dl / \int_0^l \frac{B_0}{B} \frac{|\psi|^2}{g_2 V_A^2} \,dl, 
\]
\[
\Delta \Omega^2_P = \int_0^l \frac{B_0}{B} \psi \delta L \psi \,dl / \int_0^l \frac{B_0}{B} \frac{|\psi|^2}{g_2 V_A^2} \,dl. 
\]
The quadratic form (25) represents the algebraic equation for the wave frequency \( \omega \). Its solutions represent the eigenfrequencies of the coupled modes corresponding to the assumed solution \( \psi \).

In a cold plasma when \( \beta = 0 \) equation (21) is reduced to the well-known equation for the poloidal Alfvén mode
\[
B \frac{\partial}{\partial t} \frac{1}{g_2 B} \frac{\partial}{\partial t} \psi + \frac{\omega^2}{g_2 V_A^2} \psi = 0. 
\]
Thus, for small \( \beta \) we can find a solution of (21) in the form of perturbing fundamental solution \( \psi_1 \) of equation (30) with eigenfrequency \( \omega = \Omega_M \):
\[
\psi = \psi_1 + \delta \psi, 
\]
where \( \delta \psi \ll \psi_1 \) and
\[
\delta \psi = \sum_{N=3} c_N \psi_N, 
\]
\( \psi_N \) are orthonormal set of equation (30) solutions normalized so that
\[
\int_0^l \frac{B_0}{B} \frac{\psi_N^2}{g_2 V_A^2} \,dl = 1. 
\]
\( c_N = \frac{1}{\Omega^2_A - \omega^2} \left( \int_0^l \frac{B_0}{B} \psi_N \delta L \psi_1 \,dl \right) - \frac{\sigma \Omega_M}{\omega - \Omega_M} \left( \int_0^l \frac{B_0}{B} \psi_N \dot{L} \psi_1 \,dl \right). 
\]
Here, \( \Omega_M \) is the eigenfrequency of the poloidal Alfvén mode in a cold plasma corresponding to eigenfunction \( \psi_N \). Accordingly, we can approximately write values \( \Omega^2_C, \Omega^2_P, \Omega^2_A \), and \( \Delta \Omega^2_P \) from (26) to (29) as
\[
\Omega^2_C \approx \left\{ \frac{1}{2} \int_0^l \frac{B_0}{B} \frac{\beta \kappa (\kappa_B - \kappa_R)}{g_2 V_A} \psi_1 \psi_1 \,dl \right\}^2, 
\]
\[
\Delta \Omega^2_P \approx \frac{1}{2} \int_0^l \frac{B_0}{B} \beta \kappa (\kappa_B - \kappa_R) \psi_1^2 \,dl + \Lambda_1 \int_0^l \frac{B_0}{B} \frac{B_0}{BB'} K_c b_1 \psi_1 \psi_1' \psi_1 \,dl' \,dl, 
\]
\[
\Omega^2_A \approx \Omega^2_M. 
\]
Thus, in equation (25) values \( \Omega_A \) and \( \Omega_P \) approximately are the fundamental eigenfrequencies of the poloidal Alfvén mode in a cold plasma and in hot plasma, respectively. Value \( \Omega_C \) controls the modes coupling in equation (25); if \( \Omega_C = 0 \) then there are two uncoupled modes, the poloidal Alfvén mode with frequencies \( \pm \Omega_P \) and the drift compressional mode with frequency \( \Omega_M \).

4. **Influence of the mode coupling on the instability**

Since the waves with frequencies close to \( \Omega_M \) are considered, one has to remove the wave with negative frequency \( -\Omega_P \) corresponding to Alfvén wave propagating in the direction opposite to the ion drift. To do this we assume that wave’s frequency is close to both \( \Omega_M \) and \( \Omega_P \):
\[
2 \Omega_P (\omega - \Omega_M) (\omega - \Omega_P) - \sigma \Omega^2_M \Omega_M = 0. 
\]
This equation has two solutions:
\[
\omega = \frac{1}{2} \left( \Omega_M + \Omega_P \right) \pm \frac{1}{2} \sqrt{\left( \Omega_M - \Omega_P \right)^2 + 2 \sigma \Omega^2_M \Omega_M \Omega_P}. 
\]
These solutions correspond to two wave modes. In figure 2 the dependence \( \omega \) on \( \Omega_M \) is shown on different \( \sigma \) signs (both \( \omega \) and \( \Omega_M \) are normalized on the \( \Omega_P \) value).

The key parameter controlling the mode coupling is \( \sigma \) value. Indeed, at the negative \( \sigma \) the expression under the root in the equation (38) can take both positive and negative values, while at the positive \( \sigma \) it can be only positive. Respectively, when \( \sigma < 0 \) the wave’s frequency can have both real and imaginary parts, while in the opposite case it must be positive. As seen from (38), the instability criterion is written as
\[
(\Omega_P - \Omega_M)^2 + 2 \sigma \Omega^2_M \Omega_M < 0. 
\]
In the $\Omega_C < \Omega_p$ case, this inequality is reduced to the form
\begin{equation}
-\sqrt{2}|\sigma|\Omega_C + \Omega_M < \Omega_p < \sqrt{2}|\sigma|\Omega_C + \Omega_M.
\end{equation}

The instability criterion in MHD can be formally found from (39) when both $\Omega_M$ and $\Omega_C$ tend to zero: $\Omega_C^2 < 0$. In the $\sigma < 0$ case, the expression under the root sign in equation (38) changes its sign when the $\Omega_M$ equals to one of the two critical values:

$$\Omega_k = \Omega_{\max} \pm \sqrt{\Omega_{\max}^2 - \Omega_p^2}, \quad (41)$$

where

$$\Omega_{\max} = \Omega_p \left(1 + |\sigma| \frac{\Omega_M^2}{\Omega_p^2}\right). \quad (42)$$

The oscillation branches in the $\sigma < 0$ case are shown in figure 2(a). When $\Omega_M < \Omega_-$, there are two different modes with real frequencies. The higher and lower frequency modes can be identified, respectively, with the (poloidal) Alfvén wave and the drift compressional mode (correspondingly, curves 1 and 2 in figure 2(a)). While the frequency of the Alfvén mode is almost independent on the $\Omega_M$ value, the frequency of the drift compressional mode grows with $\Omega_M$. As a result, these two modes merge when $\Omega_M$ becomes equal the value $\Omega_-$. Then, two modes appear with the same real part of the wave frequency (labeled 3 in figure 2(a)) but mutually opposite imaginary parts (labeled 4 and 5). Further, with growing $\Omega_M$ up to the other critical value $\Omega_+$, the Alfvén and drift compressional mode reappear, but at $\Omega_M > \Omega_+$, the Alfvén mode has lower frequency than the drift compressional mode.

The mode with positive imaginary parts of frequency (labeled 3–4) and the corresponding instability can be called the drift coupling mode and the drift coupling instability. The instability exists if $\Omega_- < \Omega_M < \Omega_+$. It follows from equation (38) that the growth rate
\begin{equation}
\gamma = \frac{1}{2} \sqrt{2|\sigma|\Omega_C^2 \Omega_M}{\Omega_p} - (\Omega_p - \Omega_M)^2
\end{equation}

reaches its maximal value when $\Omega_M$ equals to value $\Omega_{\max}$ determined by equation (42). The maximal growth rate is
\begin{equation}
\gamma_{\max} = \Omega_C \sqrt{\frac{|\sigma|}{2}} \left(1 + \frac{|\sigma| \Omega_C^2}{2 \Omega_p^2}\right),
\end{equation}

which is zero when there is no mode coupling, i.e. at $\Omega_C = 0$, and

$\gamma_{\max} \approx \Omega_C \sqrt{|\sigma|/2}$ in the case of $\Omega_C \ll \Omega_p$.

In the opposite case, $\sigma > 0$, the wave frequency is real. There is no instability in this case. As shown in figure 2(b), the mode branches converge around $\Omega_M = \Omega_p$ do not merge and then diverge with growing $\Omega_M$. It should be noted, that when $\Omega_M < \Omega_p$ the Alfvén branch (labeled 1) is higher than drift compressional branch (labeled 2), but when $\Omega_M > \Omega_p$ the branches switch places.

We have already noted in section 3 that the drift compressional mode eigenfrequency $\Omega_M$ is directly proportional to the azimuthal wave number $m$, see equation (18), (hereafter we shall use $m$ instead $k_2$, that is convenient for geophysical applications). The quantities $\Omega_p$, $\Omega_C$ and $\sigma$ in equation (25) are independent of $m$. Hence, equation (37) can be considered as a dispersion relation $\omega(m)$. This function is depicted in figure 3(a) for the unstable case and in figure 3(b) for the stable case. In the case of $\sigma < 0$ the growth rate is maximal at the value of the azimuthal wave number
\begin{equation}
|m_{\max}| = \frac{\Omega_p}{|\Omega_M|} \left(1 + |\sigma| \frac{\Omega_C^2}{\Omega_p^2}\right) \quad (45)
\end{equation}

and the instability exists if the $m$ is confined between two critical values,
\begin{equation}
|m_{\pm}| = |m_{\max}| \pm \sqrt{m_{\max}^2 - (\Omega_p^2/\Omega_M^2)}. \quad (46)
\end{equation}
Figure 3. Coupling of the Alfvén and drift compressional modes: the frequency dependence of \( m \) at \( \sigma < 0 \) (a) and \( \sigma < 0 \) (b), where the imaginary part of the frequency is shown as a dashed line. Here case \( \Omega_p = 1 \) corresponds to the blue line color and case \( \Omega_p = 0.5 \) corresponds to the red line color, with \( |\sigma| = 1 \), \( \Omega_c = 0.1 \) and \( \Omega_M = 0.015 \) for both the cases.

Figure 4. An example of \( \sigma < 0 \) case: contour plots of the real part of \( L_b \) (a) and of the real (b) and imaginary (c) parts of \( \sigma \), where the black line corresponds to \( \omega = \Omega_M \). Here \( \omega_{ce} = -\Omega_d \) and \( \omega_{de} = 10 \Omega_d \) were chosen.
By this means the wave branches with azimuthal wave numbers \( m_+ \) and \( m_- \) must be identified with the Alfvén and drift compressional modes, whereas waves with
\[
|m_-| > |m| > |m_+|
\] (47)
constitute the single mode amplified by the instability (see figure 3(a)). Equation (47) is the instability criterion written in the terms of the azimuthal waves numbers \( m \). Moreover, if \( \Omega_C, \sigma \) and \( \Omega_d \) are fixed, then decrease of Alfvén mode frequency \( \Omega_p \) results in lowering of the critical azimuthal wave number \( m \), at which the instability develops (figure 3(a)). In the case of \( \sigma > 0 \) (figure 3(b)) decrease of \( \Omega_p \) leads to lower \( m \) when the the Alfvén and drift compressional modes converge.

5. Conditions for the instability

The necessary condition for the instability is negative \( \sigma \) value. According to equation (24), the \( \sigma \) value is determined by the slope of the \( \Lambda(\omega) \) function at the \( \omega = \Omega_d \) point. Let us consider when \( \sigma < 0 \) and \( \sigma > 0 \) cases are realized. In figures 4 and 5 the examples of both the cases are shown. The case of \( \sigma < 0 \) corresponds to the drift compressional mode with an anomalous field-aligned dispersion, when the wave length (proportional to \( \Lambda^{-1} \)) increases with increasing wave frequency \( \omega \). It is realized when drift compressional mode frequency \( \omega = \Omega_d \) shown as a black line in figure 4 is much higher than the ion drift frequency, \( \omega \gg \Omega_d \). An analytical expression for \( \sigma \) value in the \( \omega \gg \Omega_d \) case can be obtained from equation (13) under the assumption that \( \text{Re } \omega \gg |\text{Im } \omega| \):
\[
\sigma(\omega) = -\left(1 + \frac{15}{4} \frac{\beta_0}{L_b\Lambda(\omega)}\right)^{-1} < 0.
\] (48)

The situation \( \omega \gg \Omega_d \) is possible when the transverse density gradient of the hot plasma component is larger than the magnetic field gradient, i.e. \( \omega_{\perp d} \gg \Omega_d \) (in figure 4 we put \( \omega_{\perp d} = 10 \Omega_d \)).

The opposite case of \( \sigma > 0 \) corresponds to the drift compressional mode with normal field-aligned dispersion and is realized when \( \omega \ll \Omega_d \). An analytical expression for \( \sigma \) in
this case can be obtained from equation (14):
\[ \sigma(\omega) = \left[ 1 + \frac{3}{2} \frac{\beta_1}{L_0 \Lambda(\omega)} \omega_{pe} + \omega_{pi} \right]^{-1} > 0 \]  \hspace{1cm} (49)

(again, it is assumed that Re $\omega \gg |\text{Im} \omega|$). The situation $\omega < \Omega_d$ is possible when the transverse density gradient of the hot plasma component is the same order as the magnetic field gradient or smaller.

Let us estimate a range of azimuthal wave numbers $m$ required for the onset of the instability at the characteristic parameters of the inner magnetosphere. In calculation we will use magnetic sell $L = 6.6$ Earth’s radii, energy of hot protons $e = 10–50$ keV, and frequencies of Pc5 pulsations (2.5–10 min periods). The instability growth rate is maximal at the value of the azimuthal wave number $m$ defined by the expression (45), in which values $\Omega_c$ and $\Omega_p$ have the same order of magnitude, $\Omega_c \sim \Omega_p \sim V_d / a$, where $a$ is the characteristic scale of variation of magnetospheric parameters, and value $\sigma \sim 1$. Accordingly, the instability is realized at $m \sim \Omega_p / \Omega_d$. The angular azimuthal phase velocity of the drift compressional mode $\Omega_d$ is the same order of magnitude as angular proton drift velocity $\Omega_p$, which is approximately 0.5–2.5° min$^{-1}$ for the chosen above magnetospheric parameters. The poloidal frequency $\Omega_p$ corresponds to the angular wave frequency of Pc5 pulsations, which is $36–144^\circ$ min$^{-1}$. Thus we find that $m$ required for the instability is 14–288. This estimated range of $m$ is in agreement with observations of the azimuthal small-scale waves in Pc5 range.

6. Conclusions

Here, we present a theory of the kinetic instability in the finite $\beta$ inhomogeneous plasma. This instability is not connected with the wave-particle resonance. Rather, it develops due to the coupling between the Alfvén and drift compressional modes in inhomogeneous plasma. This instability can be responsible for the generation of the azimuthal small scale ULF waves in the magnetosphere, such as compressional Pc5 pulsations [32, 33] as well as various kinds of the near magnetotail’s perturbations [17, 34, 35].

The theory takes into account two-dimensional plasma inhomogeneity with the dipole-like field lines. A quadratic equation (25) was obtained that determined both mode frequency and the growth rate. The solution of this equation yields the instability criterion (39) in the presence of the coupling of the Alfvén and drift compressional modes in the dipole-like field. It was found that the instability developed during the merging of the Alfvén and drift compressional oscillation branches even if the poloidal Alfvén frequency squared is positive. This merging can appear under the condition that the drift compressional mode has an anomalous field-aligned dispersion: the parallel wave length increases with increasing wave frequency $\omega$. It is possible if the wave frequency is larger than the ion drift frequency. The instability develops for the azimuthal wave number to lie between two critical values, $m_\pm$, determined by equation (46).

The unstable mode has both the real and imaginary parts of the frequency. On the order of magnitude, the real part equals to the diamagnetic drift frequency. The imaginary part (i.e., the growth rate) is determined by the term $\Omega_c$ in the quadratic equation (25) which appears due to the mode coupling. Thus, the instability found in this paper can be coined as the drift coupling instability. It belongs to the the reactive-type plasma instabilities [7, pp 19–20]. Its close analogue is the well-known beam-plasma instability.

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References

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