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Hidden symmetries in *N*-layer dielectric stacks

Haihao Liu[1], M Shoufie Ukhtary[2] and Riichiro Saito[2]

[1] Department of Materials Science and NanoEngineering, Rice University, Houston, TX 77005-1892, United States of America

[2] Department of Physics, Tohoku University, Sendai 980-8578, Japan

E-mail: hl46@rice.edu and shoufie@flex.phys.tohoku.ac.jp

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Abstract

The optical properties of a multilayer system with arbitrary N layers of dielectric media are investigated. Each layer is one of two dielectric media, with a thickness one-quarter the wavelength of light in that medium, corresponding to a central frequency f_0. Using the transfer matrix method, the transmittance T is calculated for all possible 2^N sequences for small N. Unexpectedly, it is found that instead of 2^N different values of T at f_0 (T_0), there are only $(N/2+1)$ discrete values of T_0, for even N, and $(N+1)$ for odd N. We explain this high degeneracy in T_0 values by finding symmetry operations on the sequences that do not change T_0. Analytical formulae were derived for the T_0 values and their degeneracies as functions of N and an integer parameter for each sequence we call 'charge'. Additionally, the bandwidth at f_0 and filter response of the transmission spectra are investigated, revealing asymptotic behavior at large N.

Keywords: transfer matrix, electromagnetic wave, dielectric stack, multilayer system

(Some figures may appear in colour only in the online journal)

1. Introduction

Advances in electronics have enabled us to control electron transport through materials, allowing us to develop electronic devices such as transistors and diodes. However, due to the temporal and spatial limitations of electrons, transporting information using electrons for long distance is not efficient. Light can be used for such purposes as the carrier of information instead of electrons [1, 2]. Nowadays, controlling the propagation of light has been the main subject in optical engineering, which is known as photonics [1–4]. Similar to electronics, in photonics, one tries to modify how light propagates through materials, including how one can allow or prevent the propagation of light, or localize the light [2, 5, 6], which can be useful to amplify the electric field. To achieve this, one can use a multilayer system consisting of N layers of dielectric media varying in one dimension. We will refer to this system as a multilayer stack. By manipulating the sequence of dielectric media in one dimension, one can control how light propagates through the stack [1, 2, 6, 7].

Previous studies have shown that sequences of dielectric media with a periodic structure, known as photonic crystals (PCs), or generated based on fractal patterns can control the propagation properties of light through the multilayer stack, such as transmission (T) spectra, group velocity and dispersion [4, 8–11]. One can expect some desirable optical properties, such as a very sharp and localized peak in the T spectrum or high electric field enhancement at a specific point, which allow us to develop optical filters, widely known as Fabry–Pérot resonators [6, 12, 13], and optical switches [9]. One can also realise perfect dielectric mirrors based on a multilayer stack, which are known as Bragg reflectors [6, 14–16], as well as structures based on them called Bragg-grating filters [17–20]. Furthermore, if one adds a conducting layer inside the multilayer stack, such as metal or graphene, one can also control the absorption of light intensity as a function of Fermi energy of the metal layer [21–24], and we can expect enhancement of absorption due to the high electric field enhancement inside the multilayer system. However, these previous studies focus only on certain specific sequences of dielectric media, such as alternating and periodic sequences, or Fibonacci and Cantor sequences [8–11, 25]. The general optical properties for *any arbitrary* sequence in a multilayer stack have not yet been discussed as far as we know, simply because there was not a systematic analysis to understand the phenomena for any arbitrary sequence.

In this study, we investigate the optical properties of a multilayer system consisting of arbitrary sequences of N layers, in particular the transmittance of light T through the system. In this work, the N layers are made of two kinds of dielectric media. In contrast to the previous studies, which discussed only very specific sequences, we calculate T for all possible 2^N sequences. Hence, our system includes all previously mentioned sequences. One might think that there is no pattern in T for an arbitrary sequence of the N-layer stack. In this work, we found that instead of 2^N different T values at a particular central (or resonant) frequency f_0, there are only $(N/2+1)$ and $(N+1)$ discrete values, for even and odd N respectively, provided we select the thickness of each layer to be one-quarter the wavelength of light in that layer corresponding to f_0. This high degeneracy generally implies the existence of hidden symmetry operations for exchanging the dielectric layers in the stack without changing T at f_0. In particular, we will define a new integer parameter called 'charge' which is invariant under the operations. We will show that all T values at f_0 are given by the 'charge' and understanding the origin of these hidden symmetries and patterns can be useful for finding and designing optimal sequences, especially for systems with large N.

Our paper is organized as follows. In section 2 we will describe our method to calculate the T of a multilayer system with arbitrary N layers of dielectric media. In section 3 we will show our results and explain the symmetry operations of the multilayer system. We will also provide the analytical formula of T at f_0 as a function of 'charge' in section 3, as well as briefly discuss the bandwidth of the T spectra (i.e. how sharp the peak at f_0 is) and the filter response. We will give our conclusion in section 4. All the mathematical proofs are given in the appendix.

2. Method

In figure 1, we show a schematic picture of our multilayer system consisting of N layers of dielectric media where the i-th layer, L_i, is one of two dielectric media that are labeled by A and B, with refraction indices n_H and n_L, respectively. We will use the convention that A is always the high-index medium, and B the low-index medium. The thickness of L_i is selected as $\ell_i = \lambda_0/4n_i$, where $\lambda_0 = c/f_0$ is the wavelength of light in vacuum with frequency f_0, which is chosen as the central frequency, and n_i is either n_H or n_L. Hence, we have 2^N possible sequences of $L_1L_2...L_N$, for example with $N=6$, we have 64 different possible sequences.

We assume that the incident light I is normal to the surface of the layer. The reflectance and transmittance of light, R and T respectively, can be calculated by the transfer matrix method [22, 26]. By using the transfer matrix method, we can relate the electromagnetic (EM) fields of light between any two different positions without knowing the multiple reflection processes between them in detail. This method has been used in previous studies of propagation of a wave inside varying media [8–10, 22, 27, 28]. We will briefly show the transfer matrix method as below.

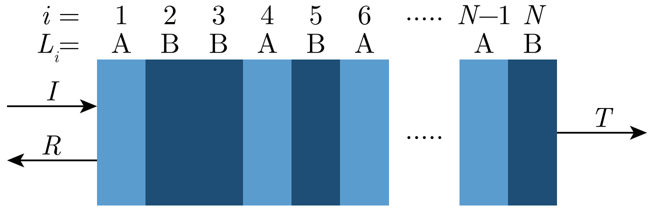

Figure 1. Schematic picture of dielectric stack. The multilayer system consists of N layers of dielectric media where the i-th layer, L_i, is one of two media, A and B, shown as two different colors; therefore there are 2^N possible sequences. Reflectance and transmittance of light are shown as R and T respectively.

In figure 2 we define the electric field of left- $(-)$ and right- $(+)$ going waves from $z=0$ to $z=\sum_{i=1}^{N}\ell_i$. The light propagates in the z-direction and the electric field is chosen to be in the x-direction. In this case, the magnetic field lies in the positive (negative) y-direction which we show as red dots (crosses) in figure 2. $E_+^{(i)}$ and $E_-^{(i)}$ are electric field amplitude at the leftmost edge of L_i for right- and left-going waves respectively. In this paper, L_0 and L_{N+1} are taken as vacuum. Therefore, $E_+^{(0)}$ and $E_-^{(0)}$ denote the incident and reflected electric fields, respectively, while $E_+^{(N+1)}$ is the transmitted field.

The electric field in L_i as a function of z_i (local z-coordinate, $z_i=0$ at the leftmost edge of L_i) is given by

$$E^{(i)}(z_i) = E_+^{(i)}\mathrm{e}^{\mathrm{i}k_iz_i} + E_-^{(i)}\mathrm{e}^{-\mathrm{i}k_iz_i}, \tag{1}$$

where $k_i = 2\pi n_i/\lambda$ is the wavevector of light with wavelength λ in L_i. Equation (1) means that the electric field can be written as a superposition of right- and left-going electromagnetic waves. The magnetic field is related to the electric field by the following equation: $H^{(i)}(z_i) = i\omega\varepsilon_0 n_i^2 \int E^{(i)} dz_i$. Thus the magnetic field in L_i as a function of z_i is given by

$$H^{(i)}(z_i) = \frac{\omega\varepsilon_0 n_i^2}{k_i}\left(E_+^{(i)}\mathrm{e}^{\mathrm{i}k_iz_i} - E_-^{(i)}\mathrm{e}^{-\mathrm{i}k_iz_i}\right). \tag{2}$$

The total electric and magnetic fields are continuous at the interface between L_i and L_{i+1}, so in terms of amplitude, $E^{(i)} = E^{(i+1)}$ and $H^{(i)} = H^{(i+1)}$. Let us take for example the interface between layer 0 and layer 1 shown in figure 2. Using equations (1) and (2) and the above, we get

$$E_+^{(0)} + E_-^{(0)} = E_+^{(1)} + E_-^{(1)}, \tag{3}$$

$$E_+^{(0)} - E_-^{(0)} = \frac{k_0}{k_1}\left(\frac{n_1}{n_2}\right)^2\left(E_+^{(1)} - E_-^{(1)}\right). \tag{4}$$

From equations (3) and (4), we can form a matrix that relates the electric fields across the interface,

$$\begin{bmatrix} E_+^{(0)} \\ E_-^{(0)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\beta_0 & 1-\beta_0 \\ 1-\beta_0 & 1+\beta_0 \end{bmatrix}\begin{bmatrix} E_+^{(1)} \\ E_-^{(1)} \end{bmatrix}, \tag{5}$$

where β_i denotes

$$\beta_i = \frac{k_i}{k_{i+1}}\left(\frac{n_{i+1}}{n_i}\right)^2 = \frac{n_{i+1}}{n_i}. \tag{6}$$

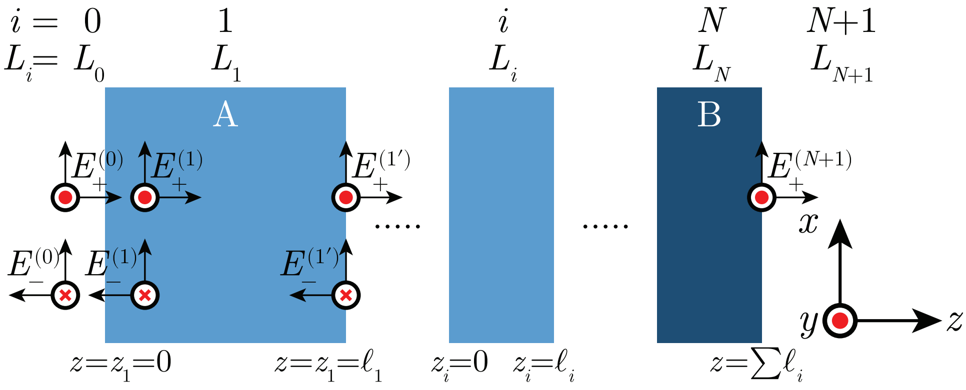

Figure 2. Schematic of light propagation from vacuum (0) through N-layer dielectric stack. $E_\pm^{(0)}$ and $E_\pm^{(1)}$ (in general $E_\pm^{(i')}$ and $E_\pm^{(i+1)}$) should both be exactly on the boundary, which cannot be easily represented.

After entering L_1, the right-going light propagates through L_1 until it hits another interface with L_2. During the propagation inside L_1 (from $z=0$ to $z=\ell_1$), the electric field changes only by its phase. From equation (1), we can form another matrix that relate the electric field at $z=0$ and $z=\ell_1$ inside L_1,

$$\begin{bmatrix} E_+^{(1)} \\ E_-^{(1)} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}k_1\ell_1} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}k_1\ell_1} \end{bmatrix} \begin{bmatrix} E_+^{(1')} \\ E_-^{(1')} \end{bmatrix}, \tag{7}$$

where $E_+^{(1')}$ and $E_-^{(1')}$ are the electric fields in L_1 at $z=\ell_1$. We can combine the matrices of equations (5) and (7) to get

$$\begin{bmatrix} E_+^{(0)} \\ E_-^{(0)} \end{bmatrix} = M_0 P_1 \begin{bmatrix} E_+^{(1')} \\ E_-^{(1')} \end{bmatrix}, \tag{8}$$

where

$$M_i = \frac{1}{2} \begin{bmatrix} 1+\beta_i & 1-\beta_i \\ 1-\beta_i & 1+\beta_i \end{bmatrix}, \tag{9}$$

$$P_i = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}k_i\ell_i} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}k_i\ell_i} \end{bmatrix}. \tag{10}$$

M_i and P_i are called matching and propagation matrices, respectively. The product of M_0P_1 in equation (8) is known as the transfer matrix [22].

This transfer matrix describes the propagation of incident light from vacuum L_0 through L_1. If we have multiple layers, we can continue the multiplication of M_{i-1} and P_i for $i=2,\ldots,N$. We can write the transfer matrix for an N-layer system as follows,

$$\begin{bmatrix} E_+^{(0)} \\ E_-^{(0)} \end{bmatrix} = M_0 P_1 M_1 \cdots P_N M_N \begin{bmatrix} E_+^{(N+1)} \\ 0 \end{bmatrix}, \tag{11}$$

where we do not expect any left-going light coming to the system at L_{N+1}. The product of M_{i-1} and P_i in equation (11) can be expressed by a 2×2 matrix as follows,

$$\begin{bmatrix} E_+^{(0)} \\ E_-^{(0)} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} E_+^{(N+1)} \\ 0 \end{bmatrix} \tag{12}$$

which gives us the transmittance of light T,

$$T = \left| \frac{E_+^{(N+1)}}{E_+^{(0)}} \right|^2 = \left| \frac{1}{a} \right|^2. \tag{13}$$

Using the transfer matrix method, we can calculate the T for any arbitrary sequence. In the next section we show numerically calculated T's for all 2^N different sequences.

3. Results and discussion

3.1. Transmittance of light

The transmittance T as a function of incident frequency f was calculated by MATLAB. We choose dielectric constants (or relative permittivities) ε_H and ε_L of the two dielectric media to be 4 and 2.25 respectively. Index of refraction is $n=\sqrt{\mu\varepsilon}$, with relative permeability $\mu \approx 1$, so n_H and n_L are taken to be $\sqrt{\varepsilon_H}=2$ and $\sqrt{\varepsilon_L}=1.5$, respectively, and used throughout this paper for simplicity. Examples of common real materials with refractive indices very close to these include silicon nitride (Si_3N_4) for n_H [29] and silica or acrylic glass for n_L. As mentioned before, the thickness of each layer is one-quarter the central wavelength λ_0 in that layer, i.e. $\ell_i=\lambda_0/4n_i$. If we choose $f_0=2$ THz, then $\ell_\mathrm{H}=18.75$ μm and $\ell_\mathrm{L}=25$ μm. Note as a matter of convention that in this paper, the sequence L_i's is represented by a string of A's and B's, such as ABAABBA.

In figure 3, T is plotted as a function of frequency normalized to central frequency f_0 for all 16 possible four-layer sequences. We also show the same plot for six-layer sequences in figure 4. It is noted that the shape of the spectra does not change for different f_0's, since ℓ_i's also change accordingly, hence why we plot T as a function of f/f_0.

The first thing that is noticed in figure 3 is that there are not actually 16 unique spectra, but only ten, by counting the number of curves on the graph. Upon investigation, it is realised that sequences that are mirrored versions of each other, e.g. AABA and ABAA, would produce identical T spectra. This is not too surprising, since light propagating through

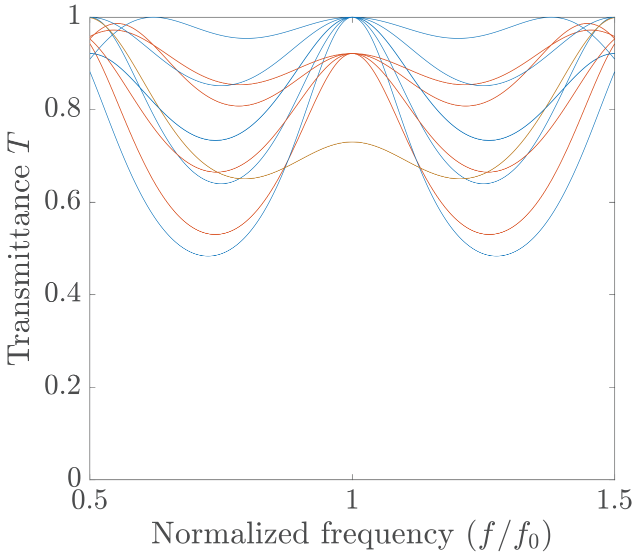

Figure 3. For $N = 4$, T spectra for all $2^4 = 16$ possible sequences. Due to the mirror symmetry, there are only ten unique spectra.

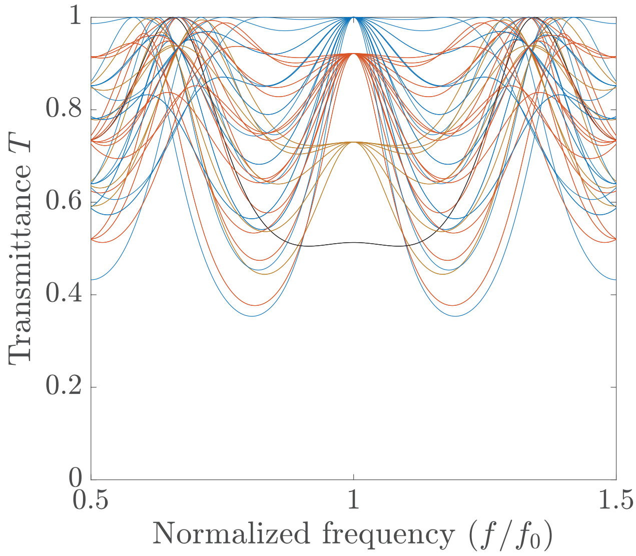

Figure 4. For $N = 6$, T spectra for all $2^6 = 64$ possible sequences. Due to the mirror symmetry, there are only 36 unique spectra.

the sequence one way is essentially equivalent to light propagating through the mirrored sequence the other way (or the time-reversal symmetry of T [30]). Detailed proofs of this mirror symmetry can be found in appendix B.1.

It is also noticed that the curves seem to converge at three points at f_0. To investigate this further, the T spectra for all 64 possible six-layer sequences are calculated, and can be seen in figure 4. We found that due to mirror symmetry, there are only 36 unique spectra. By considering the number of symmetric or 'palindromic' sequences, which are invariant under mirror symmetry, we determine the number of unique spectra for an N-layer system to be:

Even N

$$2^{\frac{N}{2}} + \frac{1}{2}\left(2^N - 2^{\frac{N}{2}}\right) \tag{14}$$

Odd N

$$2^{\frac{N+1}{2}} + \frac{1}{2}\left(2^N - 2^{\frac{N+1}{2}}\right). \tag{15}$$

With the greater number of spectra, it is clear that they are converging to 4 T's at f_0, in the case of $N = 6$. This cannot be accounted for only by 'mirror symmetry', because different spectra give the same T at f_0, hereafter denoted T_0. This is a surprising result, because by changing one layer from A to B, for example, one would expect the complex interactions of internal multiple reflections to completely change, and thus have a completely different T. Indeed, this is the behavior at frequencies other than f_0, where we see many non-degenerate spectra. The high degeneracy at f_0 implies there are hidden symmetries besides simple mirror symmetry to be found in the sequences, giving rise to the degeneracy.

In order to begin finding patterns and understanding this phenomenon, the number of unique T_0 values as a function of N is calculated and listed for $N = 1$ through 12, and shown in table 1.

There is a clear pattern in the number of T_0 values, but different patterns for even and odd N. It was conjectured that the number of T_0 values for even N is $N/2 + 1$, and for odd N, $N + 1$, which are also shown in table 1. These numbers are proved in section 3.3 formula for T_0.

Table 1. Number of unique T_0 values for $N = 1$ to 12.

N	1	2	3	4	5	6	7	8	9	10	11	12
No. of T_0 values	2	2	4	3	6	4	8	5	10	6	12	7
$N/2 + 1$ (Even N)		2		3		4		5		6		7
$N + 1$ (Odd N)	2		4		6		8		10		12	

The sequences which all give the same T_0 are manually tabulated in table 2 for $N = 6$, with all 64 sequences. The number of sequences at each T_0, which we may call the degeneracy, is listed as well. As an example, there are 20 six-layer sequences that gave a T_0 of 1.0 (perfect transmittance). These are listed in the first part of table 2.

It is not at all obvious what these sequences with the same T_0 have in common, i.e. how they are related by symmetry operations like mirroring, hence hidden symmetries is an apt name. As we begin to find these symmetries, it is clear that even and odd N do not have the same symmetries. Since the symmetries for even N seemed less elusive, we focus our efforts on finding all the hidden symmetries for even N in the next section that can explain how sequences give the same T_0. The 20 six-layer sequences with perfect transmittance also serve as a prototypical example demonstrating why all the symmetry operations are needed.

3.2. Symmetry for even N

In the previous section, we discuss how the mirror of a sequence produces the same T spectrum as the original sequence for all frequency, and so in particular, it also produces the same T_0 value. Mirror symmetry is schematically represented in figure 5(a), and can explain how 12 of the 20 sequences (in six pairs) are related. This symmetry exists in

Table 2. Sequences that give the same T_0 values for $N = 6$. $|q|$ is the total charge of the sequence (see section 3.3). The degeneracy is at f_0.

$T_0 = 1.0$		Degeneracy = 20		$\lvert q\rvert = 0$
AAAAAA	AAAABB	AAABBA	AABAAB	AABBAA
AABBBB	ABAABA	ABBAAA	ABBABB	ABBBBA
BAAAAB	BAABAA	BAABBB	BABBAB	BBAAAA
BBAABB	BBABBA	BBBAAB	BBBBAA	BBBBBB
$T_0 = 0.9216$		Degeneracy = 30		$\lvert q\rvert = 1$
AAAAAB	AAAABA	AAABAA	AAABBB	AABAAA
AABABB	AABBAB	AABBBA	ABAAAA	ABAABB
ABABBA	ABBAAB	ABBABA	ABBBAA	ABBBBB
BAAAAA	BAAABB	BAABAB	BAABBA	BABAAB
BABBAA	BABBBB	BBAAAB	BBAABA	BBABAA
BBABBB	BBBAAA	BBBABB	BBBBAB	BBBBBA
$T_0 = 0.7303$		Degeneracy = 12		$\lvert q\rvert = 2$
AAABAB	AABABA	ABAAAB	ABABAA	
ABABBB	ABBBAB	BAAABA	BABAAA	
BABABB	BABBBA	BBABAB	BBBABA	
$T_0 = 0.5130$		Degeneracy = 2		$\lvert q\rvert = 3$
ABABAB	BABABA			

both even and odd N. The rest of the symmetries are valid only at f_0 and for even N.

The first of the hidden symmetries is conjectured by looking at groups of sequences like AAAABB, AAABBA, and AABBAA. These are cyclic permutations of one another, so we call this 'cyclic symmetry'. A schematic representation is seen in figure 5(b). Straight away, this is a more complex form of symmetry than mirror, because it's not a single symmetry operation. Rather, it's a set of symmetries dependent on the number of layers you cycle, from 1 to $N - 1$ (cycling N layers would be the identity operation). The invariance of T_0 under cyclic symmetry can be proven directly and rather elegantly from properties of the transfer matrix, as shown in appendix B.2.

The next symmetry is swapping all A's with B's and vice versa, which we call 'inversion symmetry', and schematically represented in figure 5(c). Inversion symmetry can explain how sequences like AABAAB and BBABBA, which seem very different at first glance, are related to each other. It turns out that this symmetry is very difficult to prove directly from operations on the transfer matrices, which is how mirror and other symmetries are proven, and we are not able to do it. Instead, the proof of inversion symmetry becomes trivial with the formula for T_0 given in section 3.3.

Having cyclic symmetry, along with mirror and inversion, the 20 sequences that gives $T_0 = 1.0$ can be split into three 'cycles', represented by AAAAAA, AAAABB, and AABAAB, such that any two sequences in the same cycle can be related by at most two of these symmetry operations. An example would be AAABBA and BBAABB, both in the cycle AAAABB, and related by an inversion and cyclic permutation. However, we still have no symmetry operation to related sequences from different cycles.

The last two symmetries essentially explain how to connect between these 'cycles'. They are rather unusual, in that there

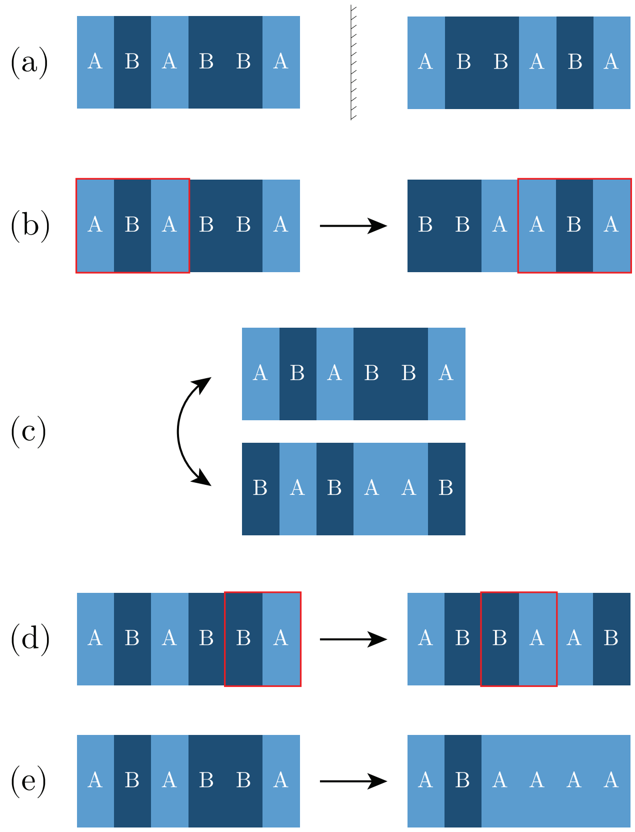

Figure 5. Hidden symmetry operations that do not change T_0: (a) mirror, (b) cyclic, (c) inversion, (d) double permutation and (e) pair inversion.

are not analogous operations in discussing, for example, point symmetry group of molecules, which does not change numbers of A and B. The first is arbitrary permutations of double layers, that is, break up the sequence into $N/2$ two-layer segments (remember this symmetry is only for even N), such as (AA)(BA)(AB), and permute those segments arbitrarily, shown in figure 5(d). We may call this 'double permutation'. This along with cyclic permutation allows us to get between the AAAABB and AABAAB cycles as follows: we permute (AA)(BA)(AB) to (AA)(AB)(BA), then cyclical permute one place to the right to AAAABB. The proof that double permutation does not change T_0 is given in appendix B.3.

The last symmetry needed is inversion of a pair of layers that are the same, i.e. either (AA) or (BB), which we call 'pair inversion', shown in figure 5(e). This is easy to understand, and allows us to get between AAAABB and AAAAAA, where the BB inverted to AA. AABAAB can also be turned in AAAABB by inverting the second AA in the first sequence so that it becomes AABBBB, inverting the whole sequence to BBAAAA, the cyclically permuting two places to the left. Like for inversion symmetry, the proof that pair inversion does not change T_0 is trivial once we define the formula for T_0 seen in section 3.3.

These symmetry operations and their products can now relate any two sequences with even N that have the same T_0 value. However, we have not yet investigated what those actual T_0 values are. In the next section, an analytical formula for T_0 shall be derived, for both even and odd N, as functions

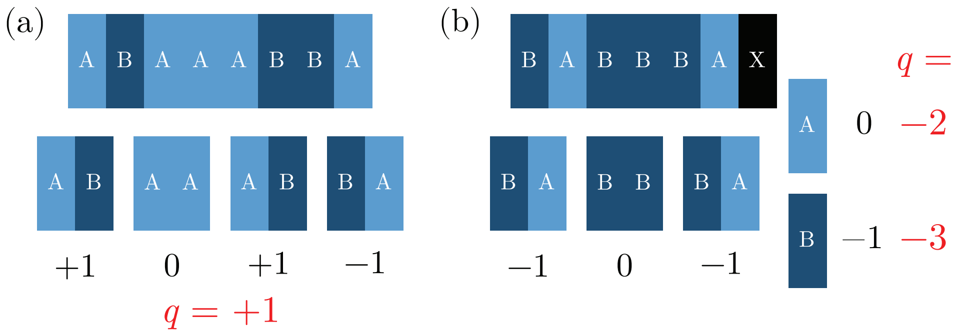

Figure 6. Concept of 'charge'. (a) We assign a charge $+1, 0, -1$ for AB, AA (or BB), BA sequences. Total charge q is defined by the sum of charge. (b) In the case of odd number N, we add 0 or -1 for the last L_i of A or B to get q.

of a parameter associated with each sequence we shall call 'charge'.

3.3. Formula for T_0

Looking at all the sequences for even N, with different T_0 values, two obvious patterns immediately jump out. The two unvarying sequences (AAAA... and BBBB...) always have the highest $T_0 = 1.0$. On the other hand, the two alternating sequences (ABAB... and BABA...), and only those, always have the lowest T_0, decreasing as N increases. This structure is known as a dielectric mirror or Bragg reflector, since having the lowest T_0 means it has the highest $R = 1 - T$ [6]. This is the starting point and clue that lead to the theory of 'charge' for sequences in general. The definition of 'charge' can be drawn by considering that the unvarying sequence can be thought of as being composed of blocks of AA or BB repeated. Similarly, the alternating sequences are blocks of AB or BA repeated. These are the two extremes, and every sequence can be thought of as being composed of some combination of these four blocks, observing that their T_0 values fall somewhere in between as well.

The basic idea is that we assign a 'charge' to each of these blocks: AB is $+1$, BA is -1, and AA and BB are both 0 as shown in figure 6(a). Note that this is not inherently related to electrical charge (though we are investigating a potential link), but the trichotomy of values and, as will be seen, the behavior of charges 'cancelling' is entirely analogous, so 'charge' is an apt name. Given these assignments, the total 'charge' of an even N sequence is straightforwardly defined by adding together the charge of each two-layer block, as illustrated in figure 6(a). We shall denote the total 'charge' of a sequence as q.

Then it can be seen that all sequences with the same T_0 also have the same $|q|$ as shown in table 2. For example, the 20 prototypical six-layer sequences with $T_0 = 1.0$ have $|q| = 0$. Moreover, each $|q|$ value corresponds of only one T_0 value. We now see that all of the symmetries operations discussed in the previous section simply preserve $|q|$. In fact, any operation defined in section 3.2 on a sequence that does not change $|q|$ would be a symmetry operation that does not change T_0.

We know that the lowest $|q| = 0$ gives the highest $T_0 = 1.0$. We also know that the highest $|q| = N/2$, corresponding to the alternating sequences, gives the lowest T_0. Through some intuition and careful algebraic manipulation, the following formula for T_0 as a function of $|q|$ (and fixed ε_H, ε_L) was derived. From equation (13), T_0 is given by

$$T_0 = \left|\frac{1}{a}\right|^2 = \frac{4(\varepsilon_H\varepsilon_L)^{|q|}}{\left(\varepsilon_H^{|q|} + \varepsilon_L^{|q|}\right)^2}, \tag{16}$$

where $a = (-1)^{\frac{N}{2}}(n_H^{2|q|} + n_L^{2|q|})/(2(n_H n_L)^{|q|})$ for even N. A full proof is given in appendix C. This equation can be considered a generalization of the equation for reflectance given by Orfanidis in section 6 of [6], wherein only the max $|q|$ is considered. With $|q| = 0$, $T_0 = 4(1)/(1+1)^2 = 1.0$, as expected. As $|q|$ increases, the square in the denominator makes it increase faster than the numerator, meaning T_0 decreases.

Proof of inversion symmetry follows as a direct consequence, since equation (16) is symmetric with respect to ε_H and ε_L. Pair inversion is simply replacing AA with BB and vice versa, which has no effect on q and thus T_0, since both have 0 'charge'. It is also clear now why there are $(N/2+1)$ T_0 values for even N, as conjectured. Given even N layers, each sequence is composed of $N/2$ blocks, so the maximum $|q|$ is $N/2$. Every integer from 0 to $N/2$ is a possible $|q|$ value, so there are $N/2+1$ values, each corresponding to a different T_0 value.

We finally tackle the question of odd N sequences. We found that the patterns are too difficult and non-obvious to study in terms of symmetry. However, the theory of 'charge' offers a simpler yet more powerful tool to understand the patterns. First, we extend the definition of total 'charge' q to odd N. The first $N-1$ even number of layers can have 'charge' assigned exactly as for even N sequences. All that remains is one extra layer, which is either A or B. We assign a 'charge' of 0 to A and -1 to B, which is added on to the 'charge' of the first $N-1$ layers, to get the total 'charge' q as shown in figure 6(b). (Note that the assignment of 0 and -1 was somewhat arbitrary, it could also work with 1 and 0, but the formula below would be slightly different.)

The formula for T_0 for odd N is somewhat trickier, but we get the following expression, where proof is given in appendix C. From equation (13), T_0 is given by

$$T_0 = \left|\frac{1}{a}\right|^2 = \frac{4\varepsilon_H^{|q+1|}\varepsilon_L^{|q|}}{\left(\varepsilon_H^{|q+1|} + \varepsilon_L^{|q|}\right)^2}, \tag{17}$$

where $a = (-1)^{\frac{N+1}{2}}(n_H^{2|q+1|} + n_L^{2|q|})i/(2n_H^{|q+1|}n_L^{|q|})$. We first note that it is not symmetric with respect to ε_H and ε_L, explaining the lack of inversion symmetry that was noticed when initially looking for patterns. Second, there is no value of q for which the expression reduces to 1 for any ε_H and ε_L except for $\varepsilon_H = \varepsilon_L = 1$ (vacuum), like $|q| = 0$ with even N, meaning perfect transmittance is not guaranteed. T_0 does eventually decrease as $|q|$ increases, for the same reason as for even N, though not monotonically.

Third, the function is not even in q, i.e. $+|q|$ and $-|q|$ give different T_0 values. Finally, we can explain why there are $N+1$ T_0 values for odd N, as conjectured. The first $N-1$ layers have can have 'charge' of $0, \pm 1, \ldots, \pm\frac{N-1}{2}$, a total of $2\frac{N-1}{2} + 1 = N$ values. The final layer either does not change charge (if A) or decreases it by 1 (if B). For almost all of them, decreasing by 1 simply gives the charge below, no extra q values, *except* for the lowest charge, $-\frac{N-1}{2}$, where decreasing by 1 produces $q = -\frac{N+1}{2}$. Hence there are $(N+1)$ q values, each corresponding to a different T_0 value.

3.4. Degeneracy

The last unsolved question is, for a given N, how many sequences there are for each T_0 value, which we call the degeneracy at that T_0 (d_N for even N and d'_N for odd N). The first step to understanding the pattern is noticing a connection to Pascal's triangle and the binomial coefficients. In particular, the number of sequences at $T_0 = 1.0$ for every even N seemed to be a central binomial coefficient: 1, 2, 6, 20, 70, …. With this as the starting point, the following combinatorial formulae were inductively derived, by calculating degeneracies at each T_0 for increasing N.

$$d_N(|q|) = \begin{cases} \binom{N}{\frac{N}{2}} & \text{for } |q| = 0 \\ 2\binom{N}{\frac{N}{2}+|q|} & \text{for } |q| = 1, 2, \ldots, \frac{N}{2} \end{cases} \tag{18}$$

$$d'_N(q) = \binom{N}{\frac{N+1}{2}+q} \text{ for } q = 0, \pm 1, \ldots, \pm\frac{N-1}{2}, -\frac{N+1}{2}. \tag{19}$$

Armed with our understanding of 'charge', the proof of this becomes a problem of combinatorics. Essentially, we can count the number of ways to get q in N layers given the number of ways to get $q-1$, q, and $q+1$ in $N-2$ layers, form a recurrence relation, then relate this to the binomial coefficients. A full proof is given in appendix D.

3.5. Bandwidth and filtering application

In this section, we investigate how the sequence affects the bandwidth of T spectra at f_0. Finding sequences with the sharpest peak (i.e. narrowest bandwidth) at f_0 is useful for application such as optical filter. However, we will not discuss how the optical filter is realised, rather we will discuss the behavior of the bandwidth and the pattern that gives the sharpest T spectrum. We will also briefly discuss the filtering frequency response of the structure.

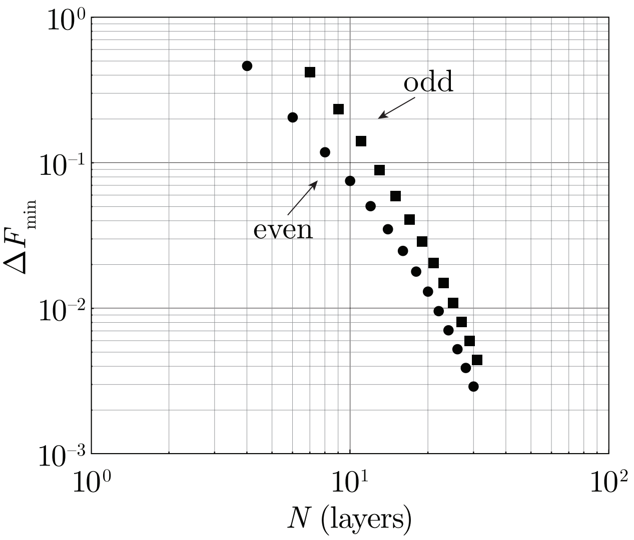

Figure 7. The minimum ΔF amongst T spectra that give the highest T_0, as a function of N for even (dot) and odd (box) N.

Firstly, we are only looking at sequences with the highest T_0, that is, with $|q| = 0$ for even N-layer sequences, where $T_0 = 1.0$. For odd N, there is in general no value of q that gives $T_0 = 1.0$. The 'charge' q that gives the highest T_0 varies as a function of ε_H and ε_L. To find this q, we differentiate equation (17) (T_0 for odd N) to find the maximum, and get

$$q_{\max} = \text{round}\left(\frac{\log(\varepsilon_H)}{\log(\varepsilon_L/\varepsilon_H)}\right), \tag{20}$$

rounding because q can only take integer values. Then after setting ε_H and ε_L, we consider only sequences with this $q_{\max}$ for odd N.

We define fractional bandwidth of a spectrum normalized to f_0 as $\Delta F = \Delta f / f_0$, where Δf is the full width at half maximum (FWHM). In figure 7 we plot on log–log scale the minimum ΔF among the T spectra for a given N that give the highest T_0, that is $q = 0$ for even N or $q = q_{\max}$ for odd N, as a function of N. The first thing to notice is that $\Delta F_{\min}$ (the narrowest spectrum) decreases with increasing N. So on a very general level, to get a sharper peak in T spectrum at f_0, we need to have more layers, as can be expected.

At first, the $\Delta F_{\min}$ values appear to follow roughly a straight line on the log–log plot, indicating a power law relationship. However, even in figure 7, the points clearly start to curve. So $\Delta F_{\min}$ is calculated for larger N and shown in figure 8 as a log-linear plot. The values of ε_H and ε_L were also varied to see how $\Delta F_{\min}$ changes. As seen in figure 8, the linear relationship between $\Delta F_{\min}$ and N in log-linear plot immediately jumped out, indicating exponential relationship in a linear plot. We plot only even N for clarity, as odd N has the same long-term linear behavior, parallel to even N but shifted upwards slightly.

Although the points clearly do not follow a straight line for small N, they do show very regular behavior as N gets larger. The exponential fit (with $\varepsilon_H = n_H^2$ and $\varepsilon_L = n_L^2$ fixed) for the asymptotic behavior for even N was found to be the very simple equation:

$$\Delta F_{\min}(N) \simeq e^{-\rho N - 2}, \tag{21}$$

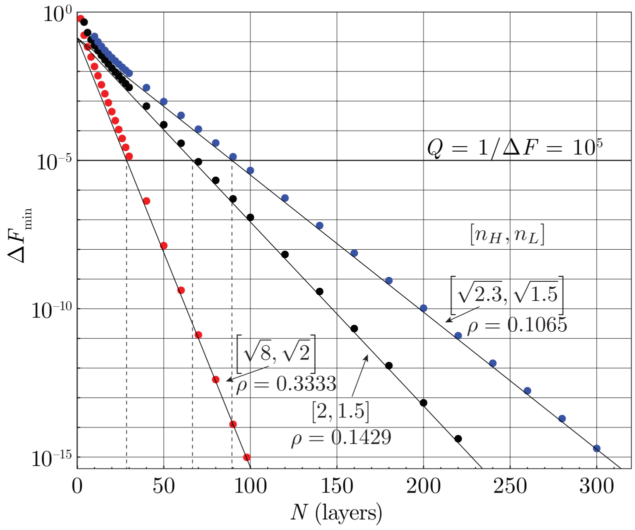

Figure 8. The long-term behavior of minimum $\Delta F_{\min}$ as N increases. Note the change from log–log scale in figure 7 to log-linear scale. The different colors indicate different choices of ε_H and ε_L shown in the figure. Quality factor Q, a more commonly used measure of bandwidth, is defined as $f_0/\Delta f = 1/\Delta F$.

where

$$\rho = \frac{n_H - n_L}{n_H + n_L} \tag{22}$$

is called the (elementary) reflection coefficient between two media [6]. It is noted that equation (21) is valid only for even N and $q = 0$ ($T_0 = 1.0$).

Several things may be taken from this. First, it is useful in the design of optical filters. For example, if we need a filter with a specified Q factor of 10^5 (i.e. $\Delta F = 10^{-5}$), and knowing the refractive indices n_H and n_L of the materials we have available and thus ρ, we can easily solve equation (21) for an estimate of the minimum number of layers N required. This is graphically represented by the dotted lines in figure 8. Second, it demonstrates and moreover explains why having materials with large $(n_H - n_L)$ is better for narrower filters, since that maximizes ρ, thereby minimizing ΔF for a given N.

It is worth pointing out that an expression for bandwidth for this type of filter is given by Macleod in [31]. However, our expression is considerably simpler, making it much easier and quicker to solve for N, the only tradeoff being a worse fit for small N. We would like to emphasize that equation (21) is an 'empirical' fit, however, its simplicity and accuracy suggests it should be possible to derive analytically with some suitable approximations to account for its asymptotic nature. Though we offer no such derivation in this paper, we would conjecture it can be derived from the expressions given in [31].

We also found the pattern for which sequence gives the narrowest Δf for any given even N. It is explained in table 3. This pattern holds for any ε_H and ε_L.

It may be of interest to note that the second narrowest sequence for any given even N follows a very simple pattern too. Simply replace the middle two layers with AA if it is BB, and vice versa. For example, for $N = 8$, the narrowest

Table 3. Pattern for the sequence that gives the narrowest ΔF for even N.

Even N	
If $\frac{N}{2}$ is even:	If $\frac{N}{2}$ is odd:
$\underbrace{\text{AB...AB}}_{\frac{N}{4}\times\text{AB}}\underbrace{\text{BA...BA}}_{\frac{N}{4}\times\text{BA}}$	$\underbrace{\text{AB...AB}}_{\frac{N-2}{4}\times\text{AB}}\text{AA}\underbrace{\text{BA...BA}}_{\frac{N-2}{4}\times\text{BA}}$
E.g. $N = 8$: $\frac{N}{2} = 4$ $\underbrace{\text{ABAB}}_{2\times\text{AB}}\underbrace{\text{BABA}}_{2\times\text{BA}}$	E.g. $N = 14$: $\frac{N}{2} = 7$ $\underbrace{\text{ABABAB}}_{3\times\text{AB}}\text{AA}\underbrace{\text{BABABA}}_{3\times\text{BA}}$

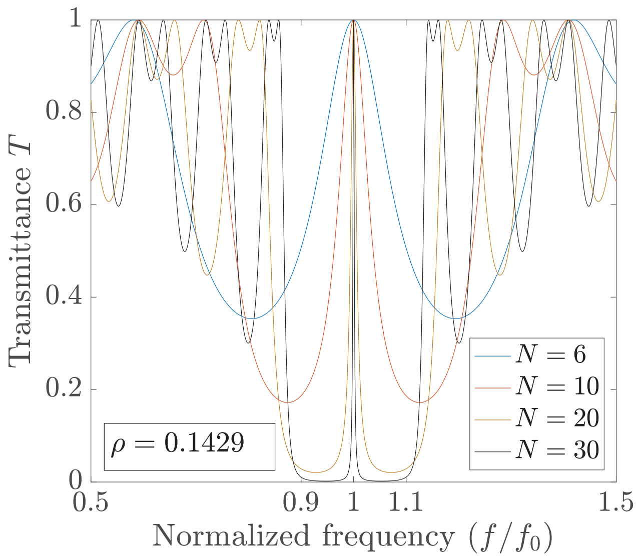

Figure 9. The transmission spectra of sequences which give $\Delta F_{\min}$ for several N.

spectrum is given by the sequence ABA**BB**ABA, the second narrowest is given by ABA**AA**ABA. These in fact exactly correspond to the high-index and low-index cavity all-dielectric filters described by Macleod in [31], and we have now conclusively shown, by calculating all 2^N sequences, that they are the 'best' possible filters (in terms of bandwidth) for a given N. These structures may also be considered 1D photonic crystals with a point defect (either one extra or one missing layer) in the center, forming what is known as a defect mode [32].

A similar albeit more complicated pattern was found for odd N. However, because the q that gives the highest T_0 varies as a function of ε_H and ε_L, so too does this pattern. Thus, we feel it is not worth describing here the rule for odd N, since it only works for some particular values of ε_H and ε_L, along with the fact that the narrowest bandwidth for any odd N is larger than that for the even $N - 1$.

Figure 9 shows the transmission spectra of sequences which give $\Delta F_{\min}$ for several N. As we can see, the minimum transmittance of each spectrum decreases with increasing N. As N increases, we can see clearly that the transmission peak at f_0 appears at the center of the reflecting band. Now we define the filtering frequency response of our structure. The practical frequency response of the filter is the frequency range in which the filter will reflect nearly all the incident light ($R = 1$), except at one particular frequency, where it transmits all the

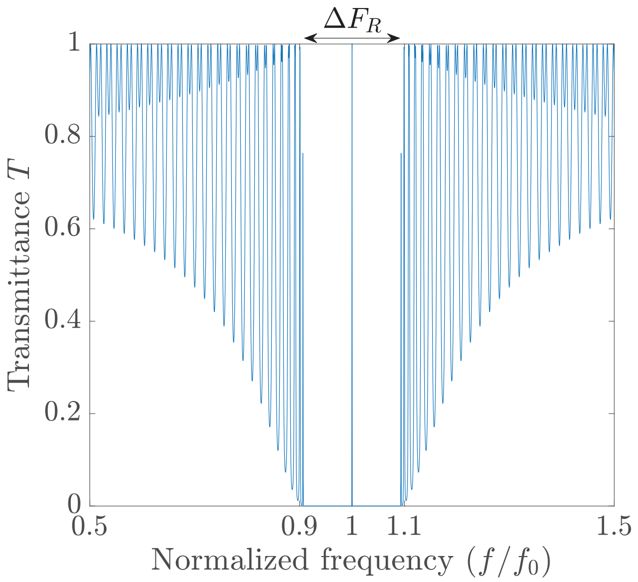

Figure 10. The transmission spectrum of sequence which gives $\Delta F_{\min}$ for $N = 200$.

light ($T = 1$); in other words, the contrast in T should be high. The contrast in T is defined as $(T_{\max} - T_{\min})/(T_{\max} + T_{\min})$, where $T_{\max} = 1$. It is noted that only the transmission peak at f_0, which originates from the interference phenomena of the multilayer system, can be used for practical filter application, while other peaks are not useful, because minimum transmittance surrounding these peaks is not zero; they are not inside the reflecting band, but instead come from a similar phenomenon of air column resonance.

The minimum transmittance $T_{\min}$ in the reflecting band is given by an asymptotic equation similar to equation (21):

$$T_{\min}(N) \simeq \mathrm{e}^{-2(\rho+\rho^4)N+5}. \tag{23}$$

From equation (23), we know that $T_{\min}$ exponentially decreases with increasing N. At $N = 20$, where the contrast is already close to 1 (contrast = 0.96, which means that $T_{\min}$ is close to zero), we can already see that the filtering frequency response of the structure will be around $0.9f_0 < f < 1.1f_0$. The frequency response can be seen more clearly for large N as shown in figure 10. The reflecting band can also be called the propagation bandgap of a photonic crystal, with the transmission peak being a defect mode introduced by the extra (or missing) layer in the center of the stack [32].

The fractional bandwidth of the reflecting band (ΔF_{R}), or useful filtering frequency response, asymptotically approaches, in the limit of large N, the following value [6]:

$$\Delta F_{\mathrm{R}} \to \frac{4}{\pi}\arcsin(\rho), \tag{24}$$

where ρ is given by equation (22). For $n_H = 2$ and $n_L = 1.5$, and $\rho = 0.1429$, $\Delta F_{\mathrm{R}} = 0.182$, i.e. the reflecting band or frequency response of the filter is $0.909f_0 < f < 1.091f_0$, corroborating what we see in figures 9 and 10. From equation (24), we see that increasing the difference between n_H and n_L will result in an increase of the frequency response bandwidth.

4. Conclusion

In conclusion, we have found that, somewhat unexpectedly, the transmittance T of N-layer dielectric stacks are highly degenerate and discrete at the central frequency f_0. We have found all hidden symmetry operations to sufficiently explain how all even N sequences with the same T_0 are related. Furthermore, T_0 depends only on the 'charge' q of a sequence, with formulae for T_0, for both even and odd N, derived as functions of q. This is a simpler, more elegant way to explain why different sequences have the same T_0 value. The degeneracy at each T_0 is explained by combinatorics, again with formulae derived as functions of q.

There is a lot of potential for future work, in various directions stemming from this initial discovery and investigation. A well-established mathematical tool to analyze and understand symmetries is group theory. In fact, we have already started in this endeavor, trying to form a group of symmetry operations both for $N = 2$ and for $N = 4$, then analyzing the structure using representation theory to extract the degeneracies. However, we ran into issues such as not being able to include some of the more exotic operations in the group, and the predicted degeneracies of irreducible representation do not match the degeneracies that we calculated. These problem might be related to the fact that we discuss transmittance but not transmission coefficient or any eigenvalue of linear operators that commute with symmetry operations. Alternatively, the symmetry operations could potentially have the structure of a groupoid, a generalization of a group.

Recalling that PCs may be used for optical filters, we want sequences with both high T *and* a sharp peak at f_0. Now that we understand how to find T_0 just by looking at the sequence, we only have to consider a much smaller subset of sequences, those with low q and high T_0. The next big step is to continue our preliminary investigation into how bandwidth depends on the sequence, e.g. we would want a sharp peak for a filter. If we can fully understand how that changes under the symmetry operations as well, we could imagine creating an algorithm to find the optimal sequence for any kind of T spectrum desired for a given N, or designing sequences satisfying some given requirements (e.g. Q factor > some value).

An interesting and potentially fruitful area to investigate is whether there is any physical meaning to this artificial value associated with a sequence we call 'charge'. Again bringing it back to physical applications, PCs can also have E field enhancement, which is useful in the enhanced Raman spectroscopy to get a stronger signal. Preliminary investigations suggest that there may be a relationship between 'charge' or 'cumulative charge' in the sequence, and the E field within the PC. For example, a sequence like ABABAB...BABABA has overall $|q| = 0$ so $T_0 = 1.0$. But right at the middle of the sequence, it has very high 'cumulative charge', and correspondingly, a very high E field at the middle point. Further investigation and understanding could allow us to design PC sequences with E field enhancement at any position we desire.

Before finishing the story, we would like to point out the similarities of the present story to a general physics in which an odd and even number of particles give a different symmetry (or statistics). Although it is beyond our ability, it is our pleasure if the reader has an interest in such hidden symmetries for application to general physics.

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Appendix A. Definitions of notation

First we define additional notation for the proofs. All other notation is defined in section 2 method.

$$
\begin{aligned}
seq &: L_1L_2...L_N\\
\overline{seq} &: L_NL_{N-1}...L_1\\
T_{seq} &: \text{ total transfer matrix for sequence } seq\\
M_{i,j} &: \text{matching matrix at interface from } L_i \text{ to } L_j\\
&= \frac{1}{2}\begin{bmatrix}1+\beta_{i,j} & 1-\beta_{i,j}\\ 1-\beta_{i,j} & 1+\beta_{i,j}\end{bmatrix}, \text{ where } \beta_{i,j}=\frac{n_j}{n_i}\\
(A)_{ij} &: (i,j) \text{ entry of any matrix } A\in\mathbb{C}^{m\times n},\\
&\quad \text{where } 1\leqslant i\leqslant m, 1\leqslant j\leqslant n
\end{aligned}
$$

Appendix B. Proofs of symmetries

Mirror symmetry (B.1) is valid for all frequencies. Cyclic (B.2) and double permutation (B.3) symmetries are valid only at the central frequency f_0.

B.1. Proof of mirror symmetry

B.1.1. Intuitive derivation. By considering the symmetry of the physical system, we can easily derive a formula for the transfer matrix of $\overline{seq}$ ($T_{\overline{seq}}$) in terms of that of seq (T_{seq}). The total transfer matrices may be written as:

$$
\begin{aligned}
T_{seq} &= M_{0,1}P_1M_{1,2}\cdots P_NM_{N,N+1}\\
&= M_{0,1}\left(\prod_{i=1}^{N}P_iM_{i,i+1}\right),
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
T_{\overline{seq}} &= M_{N+1,N}P_NM_{N,N-1}\cdots P_1M_{1,0}\\
&= \left(\prod_{i=0}^{N-1}M_{N-i+1,N-i}P_{N-i}\right)M_{1,0}.
\end{aligned}
\tag{B.2}
$$

Then, the original system of sequence seq may be described as:

$$
\begin{bmatrix}E_+^{(0)}\\ E_-^{(0)}\end{bmatrix} = T_{seq}\begin{bmatrix}E_+^{(N+1)}\\ E_-^{(N+1)}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}E_+^{(N+1)}\\ E_-^{(N+1)}\end{bmatrix}. \tag{B.3}
$$

Assuming T_{seq} has a non-zero determinant (will be shown in next section), it is invertible, and we can write:

$$
\begin{bmatrix}E_+^{(N+1)}\\ E_-^{(N+1)}\end{bmatrix} = T_{seq}^{-1}\begin{bmatrix}E_+^{(0)}\\ E_-^{(0)}\end{bmatrix} = \begin{bmatrix}e & f\\ g & h\end{bmatrix}\begin{bmatrix}E_+^{(0)}\\ E_-^{(0)}\end{bmatrix}. \tag{B.4}
$$

Now, since the right-(left-) going EM wave propagating through $\overline{seq}$ can be regarded as simply the left-(right-) going EM wave through seq, the above equation can be trivially rearranged as such to:

$$
\begin{aligned}
\begin{bmatrix}E_+^{\prime(0)}\\ E_-^{\prime(0)}\end{bmatrix} = \begin{bmatrix}E_-^{(N+1)}\\ E_+^{(N+1)}\end{bmatrix} &= \begin{bmatrix}h & g\\ f & e\end{bmatrix}\begin{bmatrix}E_-^{(0)}\\ E_+^{(0)}\end{bmatrix}\\
&= T_{\overline{seq}}\begin{bmatrix}E_+^{\prime(N+1)}\\ E_-^{\prime(N+1)}\end{bmatrix},
\end{aligned}
\tag{B.5}
$$

where E' is the E field amplitude of the EM waves propagating through $\overline{seq}$. In other words, $(T_{seq}^{-1})_{11}$ and $(T_{seq}^{-1})_{22}$ (diagonal elements), and $(T_{seq}^{-1})_{12}$ and $(T_{seq}^{-1})_{21}$ (anti-diagonal elements) are swapped. This can be accomplished by the exchange matrix

$$
J_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \tag{B.6}
$$

since

$$
J_2T_{seq}J_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}d & c\\ b & a\end{bmatrix}. \tag{B.7}
$$

Finally, we can write $T_{\overline{seq}}$ in terms of T_{seq} as follows:

$$
T_{\overline{seq}} = J_2T_{seq}^{-1}J_2. \tag{B.8}
$$

Equation (B.8) means that the transfer matrix of the mirrored sequence is not arbitrary, but is related to the original sequence.

B.1.2. Mathematical derivation. Since determinant of the general matching matrix,

$$
\begin{aligned}
\det M_{i,j} &= \left(\frac{1}{2}\right)^2[(1+\beta_{i,j})^2-(1-\beta_{i,j})^2]\\
&= \frac{1}{4}[1+2\beta_{i,j}+\beta_{i,j}^2-(1-2\beta_{i,j}+\beta_{i,j}^2)]\\
&= \beta_{i,j},
\end{aligned}
\tag{B.9}
$$

is non-zero, we can have an inverse matrix as follows,

$$
\begin{aligned}
M_{i,j}^{-1} &= \frac{1}{\det M_{i,j}}\frac{1}{2}\begin{bmatrix}1+\beta_{i,j} & -(1-\beta_{i,j})\\ -(1-\beta_{i,j}) & 1+\beta_{i,j}\end{bmatrix}\\
&= \frac{1}{2}\frac{1}{\beta_{i,j}}\begin{bmatrix}1+\beta_{i,j} & \beta_{i,j}-1\\ \beta_{i,j}-1 & 1+\beta_{i,j}\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}\frac{1}{\beta_{i,j}}+1 & 1-\frac{1}{\beta_{i,j}}\\ 1-\frac{1}{\beta_{i,j}} & \frac{1}{\beta_{i,j}}+1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}1+\beta_{j,i} & 1-\beta_{j,i}\\ 1-\beta_{j,i} & 1+\beta_{j,i}\end{bmatrix}\\
&= M_{j,i},
\end{aligned}
\tag{B.10}
$$

and note that it is equal to the corresponding matching matrix for light passing through the same interface in the opposite way.

Because a general propagation matrix is diagonal, with all diagonal elements non-zero, its inverse matrix is given by

$$P_i^{-1} = \begin{bmatrix} \frac{1}{\mathrm{e}^{-\mathrm{i}k_i\ell_i}} & 0 \\ 0 & \frac{1}{\mathrm{e}^{\mathrm{i}k_i\ell_i}} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}k_i\ell_i} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_i\ell_i} \end{bmatrix} = J_2 P_i J_2. \quad \text{(B.11)}$$

Definition 1. A **centrosymmetric** 2×2 matrix has both equal diagonal elements and equal anti-diagonal elements, i.e. a matrix of the form

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

Using equations (B.10) and (B.11), and the above definition, we are now ready to prove the relation between T_{seq} and $T_{\overline{seq}}$ mathematically.

$$\begin{aligned} J_2 T_{seq}^{-1} J_2 &= J_2 (M_{0,1} P_1 M_{1,2} \cdots P_N M_{N,N+1})^{-1} J_2 \\ &= J_2 M_{N,N+1}^{-1} P_N^{-1} \cdots P_1^{-1} M_{0,1}^{-1} J_2 \\ &= J_2 M_{N+1,N} P_N^{-1} \cdots P_1^{-1} M_{1,0} J_2 \\ &= J_2 M_{N+1,N} (J_2 P_N J_2) \cdots (J_2 P_1 J_2) M_{1,0} J_2 \\ &= (J_2 M_{N+1,N} J_2) P_N (J_2 \cdots J_2) P_1 (J_2 M_{1,0} J_2) \\ &= M_{N+1,N} P_N \cdots P_1 M_{1,0} \\ &= T_{\overline{seq}}, \end{aligned} \quad \text{(B.12)}$$

where in the second-to-last line we use the fact that $J_2 M_{i,j} J_2 = M_{i,j}$ since $M_{i,j}$ is centrosymmetric, and so remains invariant under exchange of the diagonal and anti-diagonal elements. Thus we get equation (B.8). □

B.1.3. Transfer matrices have unitary determinant. In this section, we show $\det P_i = 1$ and thus $\det T_{seq} = 1$. This fact means T_{seq} is an element of the special linear group $\mathrm{SL}(2, \mathbb{C})$ [30]. First, we show that $\det P_i = 1$ as below,

$$\det P_i = \det \begin{bmatrix} \mathrm{e}^{-\mathrm{i}k_i\ell_i} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}k_i\ell_i} \end{bmatrix} = \mathrm{e}^{-\mathrm{i}k_i\ell_i + \mathrm{i}k_i\ell_i} = 1. \quad \text{(B.13)}$$

Next, we show that $\det T_{seq} = 1$. By using equations (6), (B.9) and (B.13), we get

$$\begin{aligned} \det T_{seq} &= \det M_{0,1} \det P_1 \det M_{1,2} \cdots \det P_N \det M_{N,N+1} \\ &= \det M_{0,1} \det M_{1,2} \cdots \det M_{N,N+1} \\ &= \prod_{i=0}^{N} \det M_{i,i+1} \\ &= \prod_{i=0}^{N} \frac{n_{i+1}}{n_i} \\ &= \frac{n_{N+1}}{n_0} = 1, \end{aligned} \quad \text{(B.14)}$$

since n_0 and n_{N+1} both correspond to vacuum, whose index of refraction n is 1.

B.1.4. Mirror sequences have identical transmittance.

Theorem 2. *The transmittance for mirror sequences is the same.*

Finally, we prove that T (as given by equation (13)) for T_{seq} and $T_{\overline{seq}}$ are equal if $T_{\overline{seq}} = J_2 T_{seq}^{-1} J_2$.

$$\begin{aligned} T_{\overline{seq}} &= J_2 T_{seq}^{-1} J_2 \\ &= J_2 \frac{1}{\det T_{seq}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} J_2 \\ &= J_2 \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} J_2 \\ &= \begin{bmatrix} a & -c \\ -b & d \end{bmatrix}. \end{aligned} \quad \text{(B.15)}$$

Thus, $(T_{seq})_{11} = (T_{\overline{seq}})_{11} = a$. Hence, $T = \left|\frac{1}{a}\right|^2$ are also identical.

B.2. Proof of cyclic symmetry

B.2.1. The required lemmas. In this section we describe the lemmas or helping theorems that we will use to prove the cyclic symmetry.

Lemma 3. *Trace of product of N matrices is preserved under cyclic permutation.*

Proof. Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times m}$.

$$\begin{aligned} \mathrm{tr}(AB) &= \sum_{i=1}^{n} \sum_{j=1}^{m} (A)_{ij} (B)_{ji} \\ &= \sum_{j=1}^{m} \sum_{i=1}^{n} (B)_{ji} (A)_{ij} = \mathrm{tr}(BA). \end{aligned} \quad \text{(B.16)}$$

Consider a product of N matrices $A_1 A_2 A_3 \cdots A_N$. Let $A = A_1 A_2 \cdots A_n$ and $B = A_{n+1} A_{n+2} \cdots A_N$, where $1 < n < N$. Then,

$$\begin{aligned} \mathrm{tr}(A_1 A_2 A_3 \cdots A_N) &= \mathrm{tr}(A_1 A_2 \cdots A_n A_{n+1} \cdots A_N) \\ &= \mathrm{tr}(AB) = \mathrm{tr}(BA) (\text{by the above}) \\ &= \mathrm{tr}(A_{n+1} \cdots A_N A_1 A_2 \cdots A_n), \end{aligned}$$

noting that $A_{n+1} \cdots A_N A_1 A_2 \cdots A_n$ is a cyclic permutation of $A_1 A_2 A_3 \cdots A_N$. □

Lemma 4. *Product of n centrosymmetric 2×2 matrices is centrosymmetric.*

Proof. First note that the product of two general centrosymmetric 2 × 2 matrices

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{bmatrix} \quad \text{(B.17)}$$

is still centrosymmetric. Apply this fact recursively to n matrices. This is in essence the closure property required for T_{seq} to form a subgroup of SL(2, $\mathbb{C}$) [30]. □

Lemma 5. *The transfer matrix at $f = f_0$ for any even N-layer sequence is real and centrosymmetric. In particular, $(T_{seq})_{11} = (T_{seq})_{22} \in \mathbb{R}$.*

Proof. At the central frequency, $\lambda = \lambda_0$:

$$k_i \ell_i = \frac{2\pi n_i}{\lambda_0} \frac{\lambda_0}{4n_i} = \frac{\pi}{2} \quad \text{(B.18)}$$

Therefore general propagation matrix at $f = f_0$ is

$$P_i = \begin{bmatrix} \mathrm{e}^{-ik_i\ell_i} & 0 \\ 0 & \mathrm{e}^{ik_i\ell_i} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-i\frac{\pi}{2}} & 0 \\ 0 & \mathrm{e}^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \text{(B.19)}$$

where the i's in the final matrix denote the imaginary unit, not the index of P_i. Since all P_i's are this same matrix that does not depend on the index i, from here on we denote the general propagation matrix as P, to avoid ambiguity.

For simplicity, let general matching matrix $M_{i,i+1}$ be written as

$$\begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix}, \quad \text{(B.20)}$$

where $a_i = \frac{1}{2}(1+\beta_{i,i+1}), b_i = \frac{1}{2}(1-\beta_{i,i+1})$, and $a_i, b_i \in \mathbb{R}$ since $n_1, n_2 \in \mathbb{R}$.

$$\begin{aligned} PM_{i,i+1} &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix} \\ &= \begin{bmatrix} -a_i i & -b_i i \\ b_i i & a_i i \end{bmatrix}, \end{aligned} \quad \text{(B.21)}$$

$$PM_{i+1,i+2} = \begin{bmatrix} -a_{i+1} i & -b_{i+1} i \\ b_{i+1} i & a_{i+1} i \end{bmatrix}, \quad \text{(B.22)}$$

and

$$\begin{aligned} &PM_{i,i+1}PM_{i+1,i+2} \\ &\quad = \begin{bmatrix} -a_i i & -b_i i \\ b_i i & a_i i \end{bmatrix} \begin{bmatrix} -a_{i+1} i & -b_{i+1} i \\ b_{i+1} i & a_{i+1} i \end{bmatrix} \\ &\quad = \begin{bmatrix} -a_i a_{i+1} + b_i b_{i+1} & -a_i b_{i+1} + b_i a_{i+1} \\ b_i a_{i+1} - a_i b_{i+1} & b_i b_{i+1} - a_i a_{i+1} \end{bmatrix}. \end{aligned} \quad \text{(B.23)}$$

Notice that $PM_{i,i+1}PM_{i+1,i+2}$ is real and centrosymmetric, so let us write it as

$$\begin{bmatrix} c_{\frac{i+1}{2}} & d_{\frac{i+1}{2}} \\ d_{\frac{i+1}{2}} & c_{\frac{i+1}{2}} \end{bmatrix}, \quad \text{(B.24)}$$

where $c_i, d_i \in \mathbb{R}$.

Consider general transfer matrix for even N-layer sequence:

$$\begin{aligned} T_{seq} &= M_{0,1} \left(\prod_{i=1}^{N} PM_{i,i+1} \right) \\ &= M_{0,1} \left(\prod_{i=1}^{N/2} PM_{2i-1,2i} PM_{2i,2i+1} \right) \\ &= \begin{bmatrix} a_0 & b_0 \\ b_0 & a_0 \end{bmatrix} \left(\prod_{i=1}^{N/2} \begin{bmatrix} c_i & d_i \\ d_i & c_i \end{bmatrix} \right) \text{(by equation (B.24)).} \end{aligned} \quad \text{(B.25)}$$

Since all $a_i, b_i, c_i, d_i \in \mathbb{R}$, and by lemma 4, the above product is real and centrosymmetric. □

Corollary 6. *The transfer matrix at $f = f_0$ for any odd N-layer sequence is imaginary and Hermitian. Additionally, its trace is 0.*

Proof. First, write the above transfer matrix for even N'-layer sequence (where $N' = N - 1$) as

$$\begin{bmatrix} s & t \\ t & s \end{bmatrix}, \quad \text{(B.26)}$$

where $s, t \in \mathbb{R}$.

Consider transfer matrix for odd N-layer sequence, by multiplying $PM_{N,N+1}$ (as given by equation (B.21)) to the end of transfer matrix for an even N'-layer sequence.

$$\begin{aligned} T_{seq} &= M_{0,1} \left(\prod_{i=1}^{N'} PM_{i,i+1} \right) PM_{N,N+1} \\ &= \begin{bmatrix} s & t \\ t & s \end{bmatrix} \begin{bmatrix} -a_N i & -b_N i \\ b_N i & a_N i \end{bmatrix} \\ &= \begin{bmatrix} (-sa_N + tb_N)i & (-sb_N + ta_N)i \\ (sb_N - ta_N)i & (sa_N - tb_N)i \end{bmatrix} \\ &= \begin{bmatrix} s'i & -t'i \\ t'i & -s'i \end{bmatrix}. \end{aligned} \quad \text{(B.27)}$$

where $s', t' \in \mathbb{R}$. Note that $(T_{seq})_{11} = -(T_{seq})_{22}$, hence $\operatorname{tr} T_{seq} = 0$. □

Lemma 7. $M_{i,j} = M_{i,k} M_{k,j}$

Proof. By using the definition of $M_{i,j}$, we have

$$\begin{aligned} M_{i,k}M_{k,j} &= \frac{1}{2} \begin{bmatrix} 1+\beta_{i,k} & 1-\beta_{i,k} \\ 1-\beta_{i,k} & 1+\beta_{i,k} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1+\beta_{k,j} & 1-\beta_{k,j} \\ 1-\beta_{k,j} & 1+\beta_{k,j} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} a & b \\ b & a \end{bmatrix} \end{aligned} \quad \text{(B.28)}$$

where a and b are given by

$$\begin{aligned} a &= (1+\beta_{i,k})(1+\beta_{k,j}) + (1-\beta_{i,k})(1-\beta_{k,j}) \\ &= 2(1+\beta_{i,k}\beta_{k,j}) \\ &= 2(1+\frac{n_k}{n_i}\frac{n_j}{n_k}) \\ &= 2(1+\frac{n_j}{n_i}) \\ &= 2(1+\beta_{i,j}), \end{aligned} \tag{B.29}$$

$$\begin{aligned} b &= (1+\beta_{i,k})(1-\beta_{k,j}) + (1-\beta_{i,k})(1+\beta_{k,j}) \\ &= 2(1-\beta_{i,k}\beta_{k,j}) \\ &= 2(1-\beta_{i,j}). \end{aligned} \tag{B.30}$$

Hence we have

$$\begin{aligned} \frac{1}{4}\begin{bmatrix} a & b \\ b & a \end{bmatrix} &= \frac{1}{4}\begin{bmatrix} 2(1+\beta_{i,j}) & 2(1-\beta_{i,j}) \\ 2(1-\beta_{i,j}) & 2(1+\beta_{i,j}) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1+\beta_{i,j} & 1-\beta_{i,j} \\ 1-\beta_{i,j} & 1+\beta_{i,j} \end{bmatrix} \\ &= M_{i,j}. \end{aligned} \tag{B.31}$$

□

B.2.2. Proof of cyclic symmetry.

> **Theorem 8.** *The transmittance at central frequency f_0 for cyclic permutations ($\overleftarrow{seq}$) of an even N-layer sequence (seq) is the same.*

By lemma 7, $M_{n,n+1} = M_{n,N+1}M_{0,n+1}$, since L_{N+1} and L_0 both correspond to vacuum. Likewise, $M_{N,1} = M_{N,N+1}M_{0,1}$ Then,

$$\begin{aligned} T_{seq} &= M_{0,1}PM_{1,2}\cdots PM_{N,N+1} \\ &= M_{0,1}PM_{1,2}\cdots PM_{n,n+1}\cdots PM_{N,N+1} \\ &= M_{0,1}PM_{1,2}\cdots PM_{n,N+1}M_{0,n+1}\cdots PM_{N,N+1}. \end{aligned}$$

Let $\overleftarrow{seq}$ be $L_{n+1}L_{n+2}\cdots L_N L_1 ... L_n$, $1<n<N$; i.e. *seq* under a cyclic permutation. We have

$$\begin{aligned} T_{\overleftarrow{seq}} &= M_{0,n+1}\cdots PM_{N,1}PM_{1,2}\cdots PM_{n,N+1} \\ &= M_{0,n+1}\cdots PM_{N,N+1}M_{0,1}PM_{1,2}\cdots PM_{n,N+1}, \end{aligned}$$

which we note to be a cyclic permutation of the last expression for T_{seq} above, and so by lemma 3,

$$\mathrm{tr}T_{\overleftarrow{seq}} = \mathrm{tr}T_{seq}. \tag{B.32}$$

By using lemma 5, we have

$$\mathrm{tr}T_{seq} = (T_{seq})_{11} + (T_{seq})_{22} = 2(T_{seq})_{11}. \tag{B.33}$$

Thus,

$$(T_{seq})_{11} = \frac{1}{2}\mathrm{tr}T_{seq} = \frac{1}{2}\mathrm{tr}T_{\overleftarrow{seq}} = (T_{\overleftarrow{seq}})_{11} = a. \tag{B.34}$$

Hence, we have proved that $T_0 = \left|\frac{1}{a}\right|^2$ are also identical.

B.3. Proof of double permutation symmetry

B.3.1. The required lemmas. In this section, we describe the lemmas that we will use to prove the double permutation symmetry.

> **Lemma 9.** *Two symmetric $n\times n$ matrices A, B commute iff AB is symmetric. That is, given $A = A^T$ and $B = B^T$, $AB = BA \Leftrightarrow AB = (AB)^T$.*

Proof. Suppose we have two symmetric $n\times n$ matrices A, B, we can show that

$$\begin{aligned} &(\Rightarrow) AB = BA = ((BA)^T)^T = (A^TB^T)^T = (AB)^T \\ &(\Leftarrow) AB = (AB)^T = B^TA^T = BA. \end{aligned} \tag{B.35}$$

□

> **Lemma 10.** *T_{seq} is invariant under arbitrary permutation of double layers.*

Proof. By lemma 7, the general transfer matrix for even N-layer sequence may be written as

$$\begin{aligned} T_{seq} &= M_{0,1}PM_{1,2}PM_{2,3}\cdots PM_{N,N+1} \\ &= M_{0,1}PM_{1,2}PM_{2,0}M_{0,3}\cdots PM_{N,N+1} \\ &= \prod_{i=1}^{N/2} M_{0,2i-1}PM_{2i-1,2i}PM_{2i,0}. \end{aligned} \tag{B.36}$$

But each $M_{0,2i-1}PM_{2i-1,2i}PM_{2i,0}$ is nothing more than the transfer matrix for a 2-layer sequence, so by lemma 5, we know it is centrosymmetric. By lemma 4, the product of n such double layer transfer matrices is still centrosymmetric, and in particular, symmetric. Hence by lemma 9, any two groups of these double layer transfer matrices commute, and thus they can be arbitrarily permuted.

Then, T_{seq} written as some arbitrary permutation of double layers exactly corresponds to $T_{\widetilde{seq}}$, where $\widetilde{seq}$ is the original *seq* under the same permutation, since all the matching matrices are still correct (i.e. $M_{i,0}M_{0,j} = M_{i,j}$, where $i\in\{1,3,5,\ldots,N-1\}, j\in\{2,4,6,\ldots,N\}$). That is, $T_{seq} = T_{\widetilde{seq}}$. □

B.3.2. Proof of double permutation symmetry.

Lemma 11. *The transmittance at central frequency f_0 for even N sequences under double permutation is the same .*

By using lemma 10, we can have $(T_{seq})_{11} = (T_{\widetilde{seq}})_{11} = a$. Hence, $T_0 = \left|\frac{1}{a}\right|^2$ are also identical.

Appendix C. Proof of T_0 formulae

C.1. The required lemmas In this section, we describe the lemmas that we will use to obtain the T_0 formulae.

Lemma 12. *(Stronger form of lemma 5) The transfer matrix at $f = f_0$ for any even N-layer sequence is given by*

$$T_{seq}(N) = (-1)^{\frac{N}{2}} \frac{1}{2} \begin{bmatrix} p_N + q_N & p_N - q_N \\ p_N - q_N & p_N + q_N \end{bmatrix}, \tag{C.1}$$

where p_N and q_N are defined by

$$p_N = \prod_{i=0}^{N/2} \beta_{2i-1}, \tag{C.2}$$

$$q_N = \prod_{i=0}^{N/2} \beta_{2i}, \tag{C.3}$$

using β_i as given in equation (6) and defining $\beta_{-1} = 1$.

Proof. (By induction) Let $P(N)$ be the given statement, $N = 0, 2, 4, \ldots$. (All M_i below as given in equation (9).) In the case of $N = 0$:

$$\begin{aligned} T_{seq}(0) &= M_0 \\ &= \frac{1}{2} \begin{bmatrix} 1 + \beta_0 & 1 - \beta_0 \\ 1 - \beta_0 & 1 + \beta_0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \beta_{-1} + \beta_0 & \beta_{-1} - \beta_0 \\ \beta_{-1} - \beta_0 & \beta_{-1} + \beta_0 \end{bmatrix} \\ &= (-1)^0 \frac{1}{2} \begin{bmatrix} p_0 + q_0 & p_0 - q_0 \\ p_0 - q_0 & p_0 + q_0 \end{bmatrix}. \\ &\therefore P(0) \text{ is true.} \end{aligned} \tag{C.4}$$

Assume $P(k)$ is true:

$$T_{seq}(k) = (-1)^{\frac{k}{2}} \frac{1}{2} \begin{bmatrix} p_k + q_k & p_k - q_k \\ p_k - q_k & p_k + q_k \end{bmatrix}. \tag{C.5}$$

We will show that equation (C.1) is correct in the case of $N = k + 2$: (Using equations (B.23) and (C.5))

$$\begin{aligned} T_{seq}(k+2) &= T_{seq}(k) P M_{k+1} P M_{k+2} \\ &= (-1)^{\frac{k}{2}} \frac{1}{2} \begin{bmatrix} p_k + q_k & p_k - q_k \\ p_k - q_k & p_k + q_k \end{bmatrix} \\ &\times \begin{bmatrix} -a_{k+1}a_{k+2} + b_{k+1}b_{k+2} & -a_{k+1}b_{k+2} + b_{k+1}a_{k+2} \\ b_{k+1}a_{k+2} - a_{k+1}b_{k+2} & b_{k+1}b_{k+2} - a_{k+1}a_{k+2} \end{bmatrix} \\ &= (-1)^{\frac{k}{2}} \frac{1}{2} \begin{bmatrix} p_k + q_k & p_k - q_k \\ p_k - q_k & p_k + q_k \end{bmatrix} \\ &\times \begin{bmatrix} -\frac{1}{2}(\beta_{k+1} + \beta_{k+2}) & -\frac{1}{2}(\beta_{k+1} - \beta_{k+2}) \\ -\frac{1}{2}(\beta_{k+1} - \beta_{k+2}) & -\frac{1}{2}(\beta_{k+1} + \beta_{k+2}) \end{bmatrix} \\ &= (-1)^{\frac{k}{2}+1} \frac{1}{4} \begin{bmatrix} p_k + q_k & p_k - q_k \\ p_k - q_k & p_k + q_k \end{bmatrix} \\ &\times \begin{bmatrix} \beta_{k+1} + \beta_{k+2} & \beta_{k+1} - \beta_{k+2} \\ \beta_{k+1} - \beta_{k+2} & \beta_{k+1} + \beta_{k+2} \end{bmatrix} \\ &= (-1)^{\frac{k+2}{2}} \frac{1}{4} \begin{bmatrix} 2(p_k\beta_{k+1} + q_k\beta_{k+2}) & 2(p_k\beta_{k+1} - q_k\beta_{k+2}) \\ 2(p_k\beta_{k+1} - q_k\beta_{k+2}) & 2(p_k\beta_{k+1} + q_k\beta_{k+2}) \end{bmatrix} \\ &= (-1)^{\frac{k+2}{2}} \frac{1}{2} \begin{bmatrix} p_{k+2} + q_{k+2} & p_{k+2} - q_{k+2} \\ p_{k+2} - q_{k+2} & p_{k+2} + q_{k+2} \end{bmatrix}. \\ &\therefore P(k) \Rightarrow P(k+2) \end{aligned} \tag{C.6}$$

$P(0)$ and $P(k) \Rightarrow P(k+2)$ implies $P(N)$ is true for all $N = 0, 2, 4, \ldots$. □

Corollary 13. *(Stronger form of corollary 6) The transfer matrix at $f = f_0$ for any odd N-layer sequence is given by*

$$T_{seq}(N) = (-1)^{\frac{N+1}{2}} \frac{1}{2} \begin{bmatrix} p'_N + q'_N & -(p'_N - q'_N) \\ p'_N - q'_N & -(p'_N + q'_N) \end{bmatrix} i, \tag{C.7}$$

where p'_N and q'_N are defined by

$$p'_N = \prod_{i=0}^{(N-1)/2} \beta_{2i+1}, \tag{C.8}$$

$$q'_N = \prod_{i=0}^{(N-1)/2} \beta_{2i}. \tag{C.9}$$

Proof. Consider transfer matrix for odd N-layer sequence, by multiplying PM_N (as given by equation (B.21)) to the end of transfer matrix for even N'-layer sequence (where $N' = N - 1$), as given by lemma 12.

$$
\begin{aligned}
T_{seq}(N) &= T_{seq}(N')PM_N \\
&= (-1)^{\frac{N'}{2}}\frac{1}{2}\begin{bmatrix} p_{N'}+q_{N'} & p_{N'}-q_{N'} \\ p_{N'}-q_{N'} & p_{N'}+q_{N'}\end{bmatrix} \\
&\quad\times\begin{bmatrix} -a_N i & -b_N i \\ b_N i & a_N i\end{bmatrix} \\
&= (-1)^{\frac{N'}{2}}\frac{1}{2}\begin{bmatrix} p_{N'}+q_{N'} & p_{N'}-q_{N'} \\ p_{N'}-q_{N'} & p_{N'}+q_{N'}\end{bmatrix} \\
&\quad\times\begin{bmatrix} -\frac{1}{2}(1+\beta_N) & -\frac{1}{2}(1-\beta_N) \\ \frac{1}{2}(1-\beta_N) & \frac{1}{2}(1+\beta_N)\end{bmatrix} i \\
&= (-1)^{\frac{N'}{2}+1}\frac{1}{4}\begin{bmatrix} p_{N'}+q_{N'} & p_{N'}-q_{N'} \\ p_{N'}-q_{N'} & p_{N'}+q_{N'}\end{bmatrix} \\
&\quad\times\begin{bmatrix} 1+\beta_N & 1-\beta_N \\ -(1-\beta_N) & -(1+\beta_N)\end{bmatrix} i \\
&= (-1)^{\frac{N'+2}{2}}\frac{1}{4}\begin{bmatrix} 2(p_{N'}\beta_N+q_{N'}) & -2(p_{N'}\beta_N+q_{N'}) \\ 2(p_{N'}\beta_N-q_{N'} & -2(p_{N'}\beta_N+q_{N'})\end{bmatrix} i \\
&= (-1)^{\frac{N+1}{2}}\frac{1}{2}\begin{bmatrix} p'_N+q'_N & -(p'_N-q'_N) \\ p'_N-q'_N & -(p'_N+q'_N)\end{bmatrix} i. \qquad \text{(C.10)}
\end{aligned}
$$

C.2. T_0 formula for even N

> **Theorem 14.** *The transmittance at central frequency f_0 of an even N-layer sequence with charge q is given by*
>
> $$T_0 = \frac{4(\varepsilon_H\varepsilon_L)^{|q|}}{\left(\varepsilon_H^{|q|}+\varepsilon_L^{|q|}\right)^2}. \qquad \text{(C.11)}$$

In this section, we show the derivation of T_0 formula for even N, which is given in equation (16). By using lemma 12, we have

$$(T_{seq}(N))_{11} = (-1)^{\frac{N}{2}}\frac{1}{2}(p_N+q_N) = a. \qquad \text{(C.12)}$$

Let us carefully examine the expression $p_N + q_N$:

$$
\begin{aligned}
p_N+q_N &= \prod_{i=0}^{N/2}\beta_{2i-1} + \prod_{i=0}^{N/2}\beta_{2i} \\
&= \prod_{i=1}^{N/2}\frac{n_{2i}}{n_{2i-1}} + \prod_{i=0}^{N/2}\frac{n_{2i+1}}{n_{2i}} \\
&= \left(\frac{n_2}{n_1}\frac{n_4}{n_3}\cdots\frac{n_N}{n_{N-1}}\right) + \left(\frac{n_1}{n_0}\frac{n_3}{n_2}\cdots\frac{n_{N+1}}{n_N}\right) \\
&= \left(\frac{n_2}{n_1}\frac{n_4}{n_3}\cdots\frac{n_N}{n_{N-1}}\right) + \left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{N-1}}{n_N}\right). \qquad \text{(C.13)}
\end{aligned}
$$

Recalling our definitions for 'charge' q from section 3.3, each AA or BB double layer gives $\frac{n_{2i}}{n_{2i-1}} = \frac{n_{2i-1}}{n_{2i}} = 1$, thus having no contribution in either product. In the first product, each AB double layer contributes $\frac{n_L}{n_H}$, while each BA double layer contributes $\frac{n_H}{n_L}$, meaning a BA double layer would cancel with an AB double layer.

If we count up the net number of $\frac{n_L}{n_H}$ in the first product, with the number being negative if there are more $\frac{n_H}{n_L}$ than $\frac{n_L}{n_H}$, this exactly matches the definition of total 'charge' q of a sequence. Since each term in the second product is the reciprocal of the corresponding term in the first product, q is also the net number of $\frac{n_H}{n_L}$ in the second product. Thus we can write

$$
\begin{aligned}
p_N+q_N &= \left(\frac{n_2}{n_1}\frac{n_4}{n_3}\cdots\frac{n_N}{n_{N-1}}\right) + \left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{N-1}}{n_N}\right) \\
&= \left(\frac{n_L}{n_H}\right)^q + \left(\frac{n_H}{n_L}\right)^q \\
&= \frac{n_H^{2q}+n_L^{2q}}{n_H^q n_L^q} \\
&= \frac{n_H^{2q}+n_L^{2q}}{(n_H n_L)^q}. \qquad \text{(C.14)}
\end{aligned}
$$

By replacing q with $-q$ in the last expression, we find it is unchanged:

$$
\begin{aligned}
\frac{n_H^{-2q}+n_L^{-2q}}{(n_H n_L)^{-q}} &= (n_H n_L)^q\left(\frac{1}{n_H^{2q}}+\frac{1}{n_L^{2q}}\right) \\
&= n_H^q n_L^q \frac{n_H^{2q}+n_L^{2q}}{n_H^{2q}n_L^{2q}} \\
&= \frac{n_H^{2q}+n_L^{2q}}{n_H^q n_L^q} \\
&= \frac{n_H^{2q}+n_L^{2q}}{(n_H n_L)^q} \\
&= p_N+q_N. \qquad \text{(C.15)}
\end{aligned}
$$

This means we can write

$$p_N+q_N = \frac{n_H^{2|q|}+n_L^{2|q|}}{(n_H n_L)^{|q|}}. \qquad \text{(C.16)}$$

By substituting equation (C.16) into (C.12), we get

$$a = (-1)^{\frac{N}{2}}\frac{n_H^{2|q|}+n_L^{2|q|}}{2(n_H n_L)^{|q|}}. \qquad \text{(C.17)}$$

Substituting equation (C.17) into (13), and using $n_i^2 = \varepsilon_i$, we get equation (16).

$$
\begin{aligned}
T_0 &= \left|\frac{1}{a}\right|^2 \\
&= \left((-1)^{-\frac{N}{2}}\frac{2(n_H n_L)^{|q|}}{n_H^{2|q|}+n_L^{2|q|}}\right)^2 \quad (\because a\in\mathbb{R}) \\
&= 1^{-\frac{N}{2}}\frac{4(n_H^2 n_L^2)^{|q|}}{\left(n_H^{2|q|}+n_L^{2|q|}\right)^2} \\
&= \frac{4(\varepsilon_H\varepsilon_L)^{|q|}}{\left(\varepsilon_H^{|q|}+\varepsilon_L^{|q|}\right)^2} \qquad \text{(C.18)}
\end{aligned}
$$

□

C.3. T_0 formula for odd N

Theorem 15. *The transmittance at central frequency f_0 of an odd N-layer sequence with charge q is given by*

$$T_0 = \frac{4\varepsilon_H^{|q+1|}\varepsilon_L^{|q|}}{\left(\varepsilon_H^{|q+1|} + \varepsilon_L^{|q|}\right)^2}. \tag{C.19}$$

In this section, we show the derivation of T_0 formula for odd N, which is given in equation (17). By using corollary 13, we have

$$(T_{seq}(N))_{11} = (-1)^{\frac{N+1}{2}}\frac{1}{2}(p'_N + q'_N)i = a. \tag{C.20}$$

Again, we examine the expression $p'_N + q'_N$:

$$\begin{aligned} p'_N + q'_N &= \prod_{i=0}^{(N-1)/2}\beta_{2i+1} + \prod_{i=0}^{(N-1)/2}\beta_{2i} \\ &= \prod_{i=0}^{(N-1)/2}\frac{n_{2i+2}}{n_{2i+1}} + \prod_{i=0}^{(N-1)/2}\frac{n_{2i+1}}{n_{2i}} \\ &= \left(\frac{n_2}{n_1}\frac{n_4}{n_3}\cdots\frac{n_{N+1}}{n_N}\right) + \left(\frac{n_1}{n_0}\frac{n_3}{n_2}\cdots\frac{n_N}{n_{N-1}}\right) \\ &= \left(\frac{n_2}{n_1}\frac{n_4}{n_3}\cdots\frac{n_{N-1}}{n_{N-2}}\right)\frac{1}{n_N} + \left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{N-2}}{n_{N-1}}\right)n_N \end{aligned}$$

make the substitution $N' = N - 1$:

$$\begin{aligned} p'_N + q'_N &= \left(\frac{n_2}{n_1}\frac{n_4}{n_3}\cdots\frac{n_{N'}}{n_{N'-1}}\right)\frac{1}{n_N} + \left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{N'-1}}{n_{N'}}\right)n_N \\ &= p_{N'}\frac{1}{n_N} + q_{N'}n_N \\ &= \left(\frac{n_L}{n_H}\right)^{q'}\frac{1}{n_N} + \left(\frac{n_H}{n_L}\right)^{q'}n_N, \end{aligned} \tag{C.21}$$

where q' is the total charge of the first even N' layers. Now there are two cases for the last layer:

If $n_N = n_H$:

$$\begin{aligned} p'_N + q'_N &= \left(\frac{n_L}{n_H}\right)^{q'}\frac{1}{n_H} + \left(\frac{n_H}{n_L}\right)^{q'}n_H \\ &= \frac{n_L^{q'}}{n_H^{q'+1}} + \frac{n_H^{q'+1}}{n_L^{q'}} \\ &= \frac{n_H^{2(q'+1)} + n_L^{2q'}}{n_H^{q'+1}n_L^{q'}}. \end{aligned} \tag{C.22}$$

If $n_N = n_L$:

$$\begin{aligned} p'_N + q'_N &= \left(\frac{n_L}{n_H}\right)^{q'}\frac{1}{n_L} + \left(\frac{n_H}{n_L}\right)^{q'}n_L \\ &= \frac{n_L^{q'-1}}{n_H^{q'}} + \frac{n_H^{q'}}{n_L^{q'-1}} \\ &= \frac{n_H^{2q'} + n_L^{2(q'-1)}}{n_H^{q'}n_L^{q'-1}}. \end{aligned} \tag{C.23}$$

Substituting $q = q'$ in the first expression, and $q = q' - 1$ in the second, the two expressions can be made to look the same:

$$p'_N + q'_N = \frac{n_H^{2(q+1)} + n_L^{2q}}{n_H^{(q+1)}n_L^{q}}. \tag{C.24}$$

This corresponds to our definition of q for odd N, where we add 0 if the last layer is A and -1 if the last layer is B. If we replace q with $-q$ and $(q+1)$ with $-(q+1)$ in equation (C.24), we find it is unchanged:

$$\begin{aligned} \frac{n_H^{-2(q+1)} + n_L^{-2q}}{n_H^{-(q+1)}n_L^{-q}} &= n_H^{(q+1)}n_L^{q}\left(\frac{1}{n_H^{2(q+1)}} + \frac{1}{n_L^{2q}}\right) \\ &= n_H^{(q+1)}n_L^{q}\frac{n_H^{2(q+1)} + n_L^{2q}}{n_H^{2(q+1)}n_L^{2q}} \\ &= \frac{n_H^{2(q+1)} + n_L^{2q}}{n_H^{(q+1)}n_L^{q}} \\ &= p'_N + q'_N. \end{aligned} \tag{C.25}$$

This means we can write

$$p'_N + q'_N = \frac{n_H^{2|q+1|} + n_L^{2|q|}}{n_H^{|q+1|}n_L^{|q|}}. \tag{C.26}$$

By substituting equation (C.26) into (C.20), we get

$$a = (-1)^{\frac{N+1}{2}}\frac{n_H^{2|q+1|} + n_L^{2|q|}}{2n_H^{|q+1|}n_L^{|q|}}i. \tag{C.27}$$

Substituting equation (C.27) into (13), and using $n_i^2 = \varepsilon_i$, we get equation (17).

$$\begin{aligned} T_0 &= \left|\frac{1}{a}\right|^2 \\ &= \left((-1)^{-\frac{N+1}{2}}\frac{2n_H^{|q+1|}n_L^{|q|}}{n_H^{2|q+1|} + n_L^{2|q|}}\right)^2 i(-i) \\ &= (1)^{-\frac{N+1}{2}}\frac{4n_H^{2|q+1|}n_L^{2|q|}}{\left(n_H^{2|q+1|} + n_L^{2|q|}\right)^2} \\ &= \frac{4\varepsilon_H^{|q+1|}\varepsilon_L^{|q|}}{\left(\varepsilon_H^{|q+1|} + \varepsilon_L^{|q|}\right)^2}. \end{aligned} \tag{C.28}$$

□

Appendix D. Proof of degeneracy formulae

D.1. Degeneracy formula for even N

Let $d_N(q)$ be the degeneracy at q for N (i.e. number of N-layer sequences with 'charge' q), $q = 0, \pm 1, \ldots, \pm\frac{N}{2}$. Consider an $(N-2)$-layer sequence. If this sequence has 'charge' q, there are two ways to extend it by two layers so that the resulting N-layer sequence has 'charge' q, namely, appending an AA or BB, both with 0 'charge'. If this sequence has 'charge' $q - 1$, there only one way to extend it so that the N-layer sequence has 'charge' q, namely, appending an AB with charge $+1$.

Likewise, if this sequence has 'charge' $q+1$, there also only one way to extend it so that the N-layer sequence has 'charge' q, namely, appending an BA with charge -1. We thus establish the following recurrence relation:

$$d_N(q) = d_{N-2}(q-1) + 2d_{N-2}(q) + d_{N-2}(q+1). \tag{D.1}$$

We will also need the following recurrence relation for the binomial coefficients:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{D.2}$$

Using equations (D.1) and (D.2), we now prove the following degeneracy formula for even N:

$$d_N(q) = \binom{N}{\frac{N}{2}+q},\ q = 0, \pm 1, \ldots, \pm\frac{N}{2}. \tag{D.3}$$

Proof. (By induction) Let $P(N)$ be the given statement, $N = 0, 2, 4, \ldots$. In the case of $N = 0$:

$$d_0(q) = d_0(0) = 1 = \binom{0}{0}.$$
$$\therefore P(0) \text{ is true.} \tag{D.4}$$

Assume $P(k)$ is true:

$$d_k(q) = \binom{k}{\frac{k}{2}+q},\ q = 0, \pm 1, \ldots, \pm\frac{k}{2}. \tag{D.5}$$

In the case of $N = k + 2$, we have

$$\begin{aligned}
d_{k+2}(q) &= d_k(q-1) + 2d_k(q) + d_k(q+1) \\
&= \binom{k}{\frac{k}{2}+(q-1)} + 2\binom{k}{\frac{k}{2}+q} + \binom{k}{\frac{k}{2}+(q+1)} \\
&= \binom{k}{\frac{k}{2}+q-1} + \binom{k}{\frac{k}{2}+q} \\
&+ \binom{k}{\frac{k}{2}+q} + \binom{k}{\frac{k}{2}+q+1} \\
&= \binom{k+1}{\frac{k}{2}+q} + \binom{k+1}{\frac{k}{2}+q+1} \\
&= \binom{k+2}{\frac{k}{2}+q+1} \\
&= \binom{k+2}{\frac{k+2}{2}+q}.
\end{aligned} \tag{D.6}$$

$P(0)$ and $P(k) \Rightarrow P(k+2)$ implies $P(N)$ is true for all $N = 0, 2, 4, \ldots$. □

In the case of $q = 0 = |q|$, we get

$$d_N(0) = \binom{N}{\frac{N}{2}}, \tag{D.7}$$

matching the first part of equation (18). When $q = \pm 1, \ldots, \pm\frac{N}{2}$, we know from equation (16) that $\pm q$ give the same T_0. Therefore the degeneracy at $|q|$ is given by

$$\begin{aligned}
d_N(|q|) &= d_N(q) + d_N(-q) \\
&= \binom{N}{\frac{N}{2}+q} + \binom{N}{\frac{N}{2}-q} \\
&= 2\binom{N}{\frac{N}{2}+|q|},
\end{aligned} \tag{D.8}$$

where the last line follows from the symmetry of Pascal's triangle and the binomial coefficients. This matches the second part of equation (18). □

D.2. Degeneracy formula for odd N

Let $d'_N(q)$ be the degeneracy at q for N, $q = 0, \pm 1, \ldots, \pm\frac{N-1}{2}, -\frac{N+1}{2}$. Consider an even $(N-1)$-layer sequence. If this sequence has 'charge' q, there only one way to extend it by one layer so that the resulting N-layer sequence has 'charge' q, namely, appending an A with 0 'charge'. If this sequence has 'charge' $q+1$, there also one way to extend it so that the N-layer sequence has 'charge' q, namely, appending a B with charge -1. Since we have already derived an expression for $d_{N-1}(q)$ for even $N-1$, we can directly derive $d'_N(q)$ for odd N based on the following relation:

$$\begin{aligned}
d'_N(q) &= d_{N-1}(q) + d_{N-1}(q+1) \\
&= \binom{N-1}{\frac{N-1}{2}+q} + \binom{N-1}{\frac{N-1}{2}+q+1} \\
&= \binom{N}{\frac{N-1}{2}+q+1} \\
&= \binom{N}{\frac{N+1}{2}+q}.
\end{aligned} \tag{D.9}$$

This exactly matches equation (19). □

ORCID iDs

Haihao Liu http://orcid.org/0000-0002-0740-7765

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