

# Stochastic string motion above and below the world sheet horizon

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# Stochastic string motion above and below the world sheet horizon

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**ABSTRACT:** We study the stochastic motion of a relativistic trailing string in black hole  $\text{AdS}_5$ . The classical string solution develops a world-sheet horizon and we determine the associated Hawking radiation spectrum. The emitted radiation causes fluctuations on the string both above and below the world-sheet horizon. In contrast to standard black hole physics, the fluctuations below the horizon are causally connected with the boundary of  $\text{AdS}$ . We derive a bulk stochastic equation of motion for the dual string and use the  $\text{AdS/CFT}$  correspondence to determine the evolution of a fast heavy quark in the strongly coupled  $\mathcal{N} = 4$  plasma. We find that the kinetic mass of the quark decreases by  $\Delta M = -\sqrt{\gamma\lambda}T/2$  while the correlation time of world sheet fluctuations increases by  $\sqrt{\gamma}$ .

**KEYWORDS:**  $\text{AdS-CFT}$  Correspondence, Thermal Field Theory

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## Contents

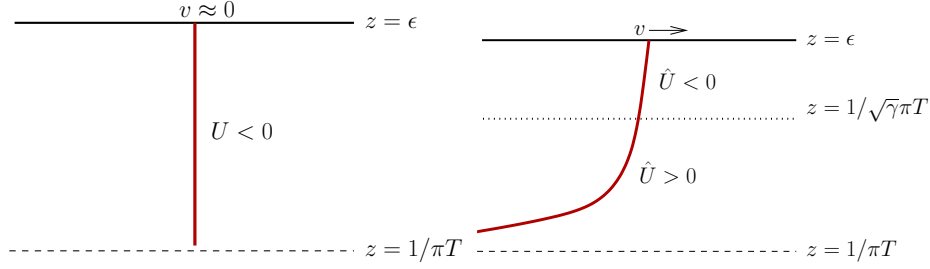
<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>3</b>
2.1	Finite temperature in AdS/CFT	3
2.2	Review of the trailing string	4
<b>3</b>	<b>String fluctuations from the stretched world-sheet horizon</b>	<b>7</b>
3.1	The partition function	7
3.2	Stochastic string motion	9
3.3	Heavy quark partition function	11
3.4	Extension to the longitudinal fluctuations	12
<b>4</b>	<b>Summary and outlook</b>	<b>13</b>
4.1	Summary of the main results	13
4.2	The low frequency limit and the physical picture	16
4.3	Limits on the validity of the approach	18
4.4	Outlook	19
<b>A</b>	<b>Green's function derivation of the fluctuation pattern</b>	<b>20</b>
<b>B</b>	<b>Momentum flux at the world sheet horizon</b>	<b>21</b>

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## 1 Introduction

In recent years the dynamics of strongly coupled non-abelian plasmas has been investigated vigorously [1]. This interest in plasma physics was motivated in part by the heavy ion program at RHIC [2–4] and the upcoming program at the LHC. For the experimentally accessible temperature range, the medium is close to the deconfinement transition and the QCD coupling constant is not small [5]. Thus it is important to compare perturbative expectations for the QCD plasma to all available strong coupling results.

The AdS/CFT correspondence relates  $\mathcal{N} = 4$  Super Yang Mills (SYM) at strong coupling and large  $N_c$  to type IIB supergravity on a curved background,  $\text{AdS}_5 \times S^5$  [6–8]. Although  $\mathcal{N} = 4$  SYM is not QCD, the AdS/CFT correspondence has provided new insight into the strongly coupled regime and the RHIC experiments [9, 10]. In particular, the calculation of  $\eta/s$  in  $\mathcal{N} = 4$  [11–13], provided a concrete example of a strongly coupled plasma which realizes the small shear viscosity needed to explain the elliptic flow observed at RHIC [5]. Since this work on shear viscosity many other transport properties of strongly coupled plasmas have been computed using the correspondence. Of particular relevance to this work is the energy loss of heavy quarks [14–20].



**Figure 1.** (a) An approximately static heavy quark string. Hawking radiation induces a random force close to the horizon which is ultimately transmitted to the boundary [21, 22]. (b) The rapidly moving trailing string analyzed in this work. At  $z = 1/\sqrt{\gamma}\pi T$  the speed of the quark equals the local speed of light and the ws-horizon develops. Fluctuations are induced above and below the world sheet horizon. In addition there is a cross correlation between the fluctuations above and below the ws-horizon.

The steady state motion of a heavy quark traversing the  $\mathcal{N} = 4$  plasma under the influence of an external electric field is represented in the gravitational theory by a semi-classical trailing string [14, 16] — see figure 1. The momentum carried down the string determines the energy loss of the heavy quark in the field theory

$$\frac{dp}{dt} = -\eta\gamma v, \quad \eta \equiv \frac{1}{2}\sqrt{\lambda}\pi T^2, \quad (1.1)$$

where  $v$  is the velocity and  $\gamma = 1/\sqrt{1-v^2}$  is the Lorentz factor. The momentum broadening of the quark probe was determined by studying small fluctuations around the drag string solution [15, 17, 18]. There is a point in the bulk where the quark velocity equals and then exceeds the local speed of light parallel to the brane. At this point the dual string develops a world-sheet event horizon (ws-horizon) which is distinct from the black hole horizon. The associated Hawking radiation from this ws-horizon determines the fluctuations on the string. At small velocities the fluctuations computed in this way are consistent with the drag in eq. (1.1) and the fluctuation-dissipation theorem [15].

The momentum broadening discussed above was computed by disturbing the string with an external force, and subsequently following the ring down of this disturbance with the classical equations of motion. In reality external forces are unnecessary since Hawking radiation emanates from the ws-horizon and induces stochastic motion in the bulk. This picture was clarified recently in refs. [21, 22] which determined the stochastic equation of motion for a slowly moving quark string. (In this limit the string is nearly straight and the ws-horizon coincides with the black-hole horizon.) The effect of Hawking radiation is to supplement the dissipative boundary conditions at the black hole event horizon with an associated random force. The classical string equations of motion transmit this random force to the boundary, leading to the stochastic motion of the heavy quark. The amplitude of the random term is suppressed by  $\lambda^{1/4}$  reflecting the quantum mechanical nature of black hole irradiance.

In this work we extend the analysis of ref. [22] to the rapidly moving trailing string. The random fluctuations of the string are generated at the world-sheet horizon and prop-

agate both upward toward the AdS boundary, and downward toward the horizon of black hole. Our study of the world sheet radiation determines the statistics of these fluctuations both above and below the world-sheet horizon. In addition there is cross correlation, i.e. fluctuations above the ws-horizon are correlated with fluctuations below the ws-horizon. This completes the analysis begun in [19] of the fluctuations above the horizon. Since the fluctuations below the horizon are causally connected with the boundary, these random vibrations will be reflected in the field theory by the stress tensor fluctuations induced by the heavy quark. We have also re-derived the heavy quark effective equation of motion [19] and computed a velocity dependent mass shift.

The paper is organized as follows. In section 2 we briefly review the trailing string solution. In section 3 we compute the stochastic equation of motion of the semiclassical string in bulk by integrating out the fluctuations in a strip around the world-sheet horizon. This bulk evolution ultimately determines the equation of motion for string end point which is dual to the heavy quark. The heavy quark partition function and its equation of motion are derived in section 3.3. Finally, the bulk fluctuations are re-derived in terms of the bulk to bulk Green's functions in appendix A. A discussion of the physical picture which emerges from this analysis is provided in section 4.

## 2 Preliminaries

### 2.1 Finite temperature in AdS/CFT

According to the AdS/CFT correspondence,  $\mathcal{N} = 4$  SYM at finite temperature and infinite 't Hooft coupling is dual to a black brane solution which is asymptotically  $\text{AdS}_5 \times S^5$ . The metric of this solution can be written as

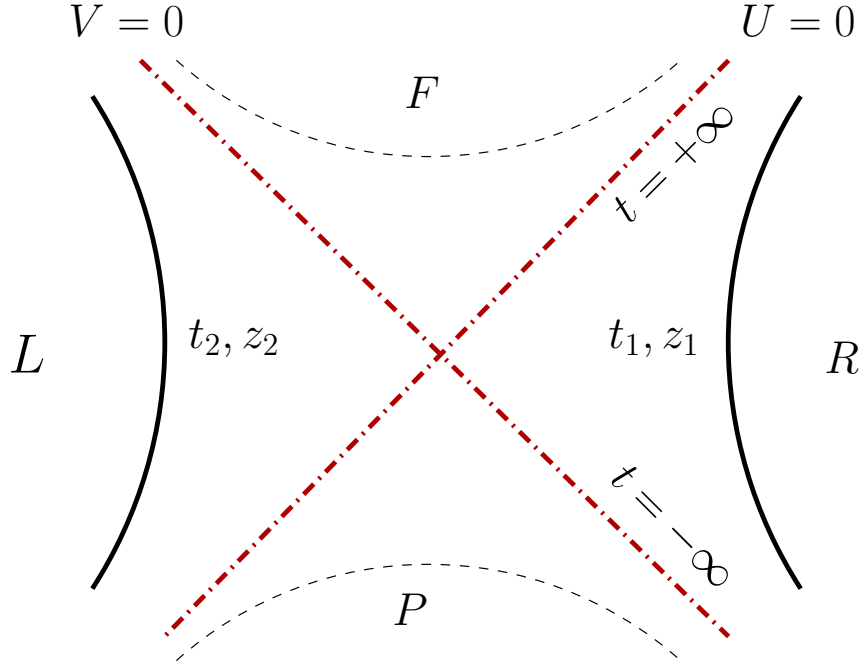
$$ds^2 = \frac{L^2}{z^2} \left( -f(z)dt^2 + d\mathbf{x}^2 + \frac{1}{f(z)}dz^2 \right), \quad (2.1)$$

where  $L$  is the AdS radius,  $f(z) = 1 - (z\pi T)^4$ , and  $T$  is the Hawking temperature of the black brane.

As is well known, the singularity of the metric at  $z = 1/\pi T$  in eq. (2.1) is only a coordinate singularity reflecting the fact that the metric does not describe the entire space-time of the black brane solution. A global set of coordinates, known as Kruskal coordinates, can be introduced which removes this singularity [23]. The Kruskal map is shown in figure 2. The set of coordinates eq. (2.1) describes only the right quadrant of the Kruskal plane — see figure 2. An identical set of coordinates may be introduced to describe the left quadrant of the Kruskal map. We will denote by

$$\{t_i, x_i, z_i\} \quad \text{with } i = 1, 2, \quad (2.2)$$

the coordinates of the right and the left quadrant respectively. The doubling of the fields in the presence of the black branes corresponds [24] to the appearance of type 1 and 2 fields in finite temperature field theory [25].



**Figure 2.** Kruskal diagram for the AdS black hole. The coordinates  $(t_1, z_1)$  span the right (R) quadrant while  $(t_2, z_2)$  span the left quadrant (L). The dash-dotted lines and the dashed hyperbolas represent the future and past horizons and the singularities, respectively. The thick hyperbolas on the sides of the two quadrants are the boundaries at  $r = \infty$  (or  $z = 0$ ).

## 2.2 Review of the trailing string

The dynamics of a heavy external quark in the fundamental color representation of the gauge theory are determined, according to the correspondence, by a semiclassical string stretching from the boundary of AdS to the horizon [26]. The forced motion of the quark under a constant external electric field in the  $\hat{x}$  direction is dual to the classical solution to the Nambu-Goto action plus an electric field acting on the boundary endpoint

$$S = -\frac{1}{2\pi\ell_s^2} \int dt dz \sqrt{-\det h_{ab}} + \int dt \mathcal{E} \bar{x}_b(t), \quad (2.3)$$

where  $h_{ab}$  is the induced metric,  $\mathcal{E}$  is the external field and  $\bar{x}_b(t) = \bar{X}(t, \epsilon)$  the position of the boundary end point of the string.

The demand that the solution is stationary (i.e. that the action is time independent and real) fixes the velocity of the quark in terms of the external field.

$$v = \frac{2\mathcal{E}}{\sqrt{\lambda\pi T^2}} \frac{1}{\sqrt{1 + \frac{4\mathcal{E}^2}{\lambda\pi^2 T^4}}}, \quad (2.4)$$

and the string solution is given by

$$\bar{X}(t, z) = vt + v\Delta X_{TS}(z), \quad (2.5)$$

with the trailing string profile [14, 16]

$$\Delta X_{TS}(z) = \frac{1}{2\pi T} [\tan^{-1}(z\pi T) - \tanh^{-1}(z\pi T)] . \quad (2.6)$$

In these coordinates, the induced metric on the string is non-diagonal. The induced metric is diagonalized by the set of coordinates

$$\hat{t} = \frac{t + \zeta(z)}{\sqrt{\gamma}} , \quad (2.7)$$

$$\begin{aligned} \zeta(z) &\equiv \frac{1}{2\pi T} (\tan^{-1}(z\pi T) - \tanh^{-1}(z\pi T)) \\ &\quad - \frac{\sqrt{\gamma}}{2\pi T} \left( \tan^{-1}(z\sqrt{\gamma}\pi T) - \frac{1}{2} \ln \left| \frac{1 + z\sqrt{\gamma}\pi T}{1 - z\sqrt{\gamma}\pi T} \right| \right) , \\ \hat{z} &= \sqrt{\gamma} z , \end{aligned} \quad (2.8)$$

$$\hat{\mathbf{x}} = \sqrt{\gamma} \mathbf{x} , \quad (2.9)$$

This change of variables is singular at the position  $z = 1/\sqrt{\gamma}\pi T$  which is the world-sheet horizon. The induced string metric in these coordinates is given by

$$h_{\hat{t}\hat{t}} = -\frac{L^2}{\hat{z}^2} f(\hat{z}) , \quad h_{\hat{z}\hat{z}} = \frac{L^2}{\hat{z}^2} \frac{1}{f(\hat{z})} , \quad h_{\hat{t}\hat{z}} = 0 , \quad (2.10)$$

which shows a horizon at  $\hat{z} = 1/\pi T$ . The singular change of variables eq. (2.7) should be understood as two different local charts describing the space above and below the world-sheet horizon.

The fluctuations around the solution eq. (2.5) are described by the Nambu-Goto action. For small fluctuations it is enough to expand this action to quadratic order. Denoting by  $\hat{X}_L(t, z)$  and  $\hat{X}_T(t, z)$  the longitudinal and transverse fluctuations, the action is

$$S_2[\hat{X}_L, \hat{X}_T] = S_L[\hat{X}_L] + S_T[\hat{X}_T] , \quad (2.11)$$

with

$$S_T[\hat{X}] = -\frac{1}{2} \int d\hat{t} d\hat{z} \left[ \left( T_o(\hat{z}) (\partial_{\hat{z}} \hat{X})^2 - \frac{m}{\pi T \hat{z}^2 f(\hat{z})} (\partial_{\hat{t}} \hat{X})^2 \right) \right] , \quad (2.12)$$

$$S_L[\hat{X}] = \gamma^2 S_T[\hat{X}] , \quad (2.13)$$

where the subscripts  $T, L$  denote transverse and longitudinal fluctuations and

$$T_o(\hat{z}) = \frac{L^2}{2\pi \ell_s^2} \frac{f(\hat{z})}{\hat{z}^2} , \quad m = \frac{\pi T L^2}{2\pi \ell_s^2} . \quad (2.14)$$

We remark that the dynamics of the longitudinal and transverse fluctuations are the same and they only differ by an overall  $\gamma^2$  factor in the action. The classical equations of motion are, thus, the same for longitudinal and transverse modes and, in Fourier space, are given by

$$\partial_{\hat{z}} \left( T_o(\hat{z}) \partial_{\hat{z}} \hat{X}(\hat{\omega}, \hat{z}) \right) + \frac{m \hat{\omega}^2}{\pi T \hat{z}^2 f(\hat{z})} \hat{X}(\hat{\omega}, \hat{z}) = 0 . \quad (2.15)$$

Classical solutions to this equation can be expressed in terms of the in-falling solution

$$F_\omega(\hat{z}) = |1 - \hat{z}\pi T|^{-i\hat{\omega}/4\pi T} \mathcal{F}_\omega(\hat{z}), \quad (2.16)$$

with  $\mathcal{F}_\omega(\hat{z})$  a regular function. For arbitrary  $\hat{z}$  and  $\hat{\omega}$ , the function  $\mathcal{F}_\omega(\hat{z})$  is only known numerically; however an analytical expression for the solution can be obtained in the low frequency limit

$$F_\omega(\hat{z}) = 1 - \frac{i\hat{\omega}}{2\pi T} \left( \tan^{-1}(\hat{z}) - \frac{1}{2} \ln \left| \frac{1 + \hat{z}\pi T}{1 - \hat{z}\pi T} \right| \right) + \mathcal{O}(\hat{\omega}^2), \quad (2.17)$$

which is valid as long as  $z$  is not exponentially close to the world-sheet horizon ( $\hat{\omega} \ln(1 - \hat{z}\pi T) \ll 1$ ). We would like to remark that, even though the  $z$ -derivative of this expression has a kink at  $\hat{z} = 1/\pi T$  the tension force of the string close to the horizon is the same on both sides of the horizon

$$T_o(\hat{z})\partial_{\hat{z}}F_\omega(\hat{z})|_{\hat{z}=(1+\epsilon)/\pi T} \approx T_o(\hat{z})\partial_{\hat{z}}F_\omega(\hat{z})|_{\hat{z}=(1-\epsilon)/\pi T}, \quad (2.18)$$

The in-falling solution  $F_\omega$  determines the retarded correlator of the random force acting on the quark as it propagates through the strongly coupled thermal medium [15]

$$G_R(\hat{\omega}) = -T_o(\hat{z})F_\omega^*(\hat{z})\partial_{\hat{z}}F_\omega(\hat{z}). \quad (2.19)$$

As noted in [22] the imaginary part of  $G_R$  coincides with the Wronskian of eq. (2.15) and, thus, it is independent of  $\hat{z}$ .

A general solution of the equation of motion can be expressed as a linear combination of  $F_\omega(z)$  and  $F_\omega^*(z)$ . Analyticity demands that the solutions below and above the world sheet horizon are connected by [17]

$$\hat{X}_1(\hat{\omega}, \hat{z}) = a_1(\hat{\omega})e^{\theta(\hat{z}-1)\hat{\omega}/2T}F_\omega^*(\hat{z}) + b_1(\hat{\omega})F_\omega(\hat{z}), \quad (2.20)$$

where we have added the subscript 1 to denote that these string fluctuations take place in the right quadrant of figure 2. A similar analysis may be performed in the left quadrant yielding

$$\hat{X}_i(\hat{\omega}, \hat{z}) = a_i(\hat{\omega})e^{\delta_i\theta(\hat{z}-1)\hat{\omega}/2T}F_\omega^*(\hat{z}) + b_i(\hat{\omega})F_\omega(\hat{z}), \quad (2.21)$$

with  $i = 1, 2$  and  $\delta_1=1$  and  $\delta_2 = -1$ . The fluctuations in both quadrants are also related by analyticity. Following [17]

$$a_2(\hat{\omega}) = e^{-\omega\sigma}e^{\hat{\omega}/T}a_1(\hat{\omega}), \quad b_2(\hat{\omega}) = e^{-\omega\sigma}b_1(\hat{\omega}), \quad (2.22)$$

with  $\omega = \hat{\omega}/\sqrt{\gamma}$  the boundary frequency in the  $t$  coordinate and  $\sigma = 1/2T$ .

As noted in [24], the identification of  $X_2$  with the type 2 fields of the Schwinger-Keldysh contour corresponds to the choice of  $\sigma = 1/2T$ . And arbitrary choice of  $\sigma$  may be found by performing the change of variables [22]

$$X_2^{(\sigma)}(\hat{\omega}, \hat{z}) = e^{\omega(1/2T-\sigma)}X_2(\hat{\omega}, \hat{z}). \quad (2.23)$$

With this choice, the relation eq. (2.22) still holds if we now identify  $\sigma$  with the  $\sigma$ -contour. From now on we will focus on  $\sigma = 0$  and drop the  $\sigma$  superscript.



### 3 String fluctuations from the stretched world-sheet horizon

#### 3.1 The partition function

The discussion in the section 2 was concentrated in the strict  $\lambda \rightarrow \infty$  limit in which the string partition function is saturated by the classical action and the motion of the string is purely classical. String (quantum) fluctuations are suppressed by  $\lambda$ . As shown in [22], in order to recover the stochastic motion of the heavy quark, we must take  $\lambda$  large but finite and consider the quantum fluctuations induced by the world-sheet horizon (Hawking radiation).

Since  $\lambda$  is large, we shall concentrate on small fluctuations around the local minimum of the classical action given by the solution eq. (2.5). The partition function is then given by

$$\mathcal{Z}_s = \mathcal{Z}_T \mathcal{Z}_L, \quad (3.1)$$

$$\mathcal{Z}_\alpha = \int \mathcal{D}_s \hat{X}_1(\hat{t}, \hat{z}) \mathcal{D}_s \hat{X}_2(\hat{t}, \hat{z}) e^{iS_\alpha[\hat{X}_1] - iS_\alpha[\hat{X}_2]}, \quad (3.2)$$

with  $\alpha = L, T$ . The measure  $\mathcal{D}_s$  is a complicated object which we will not need to specify. Since the partition function factorizes and the classical action for longitudinal fluctuations is proportional to the transverse ones, we will concentrate in the latter and we will drop the subindex  $T$ . The extension to longitudinal fluctuations is straightforward and we will quote the main results at the end.

The partition function integrates over all values of  $\hat{z}$  and, in particular, across the world-sheet horizon. Since the set of coordinates eq. (2.7) do not cover the hyper-plane  $\hat{z}^w = 1/\pi T$  we must integrate out the region surrounding this hyper-plane. We introduce

$$\hat{x}_{i\pm}^w(\hat{t}) = \hat{X}(\hat{t}, \hat{z}_\pm), \quad \hat{z}_\pm = (1 \pm \epsilon)\hat{z}^w. \quad (3.3)$$

and express the partition function as

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\hat{x}_{1-}^w \mathcal{D}\hat{x}_{1+}^w \mathcal{D}\hat{x}_{2-}^w \mathcal{D}\hat{x}_{2+}^w \\ &\quad \mathcal{Z}^{<}[\hat{x}_{1-}^w, \hat{x}_{2-}^w] \mathcal{Z}^w[\hat{x}_{i-}^w, \hat{x}_{i+}^w] \mathcal{Z}^{>}[\hat{x}_{1+}^w, \hat{x}_{2+}^w], \end{aligned} \quad (3.4)$$

with  $\mathcal{Z}^{<} (\mathcal{Z}^{>})$  the string partition function restricted to  $\hat{z} < \hat{z}_w$  ( $\hat{z} > \hat{z}_w$ ) while  $\mathcal{Z}^w$  is the partition function in the neighborhood of  $\hat{z} = \hat{z}_w$ .

Since the action is quadratic, the partition function is given by the classical action up to a constant independent of the endpoint

$$\begin{aligned} \mathcal{Z}^w[\hat{x}_{i-}^w, \hat{x}_{i+}^w] &\propto e^{-i\frac{1}{2} \int \frac{d\hat{\omega}}{2\pi} [T_o(\hat{z}_+) \hat{x}_{1+}^w(-\hat{\omega}) \partial_{\hat{z}} \hat{X}_{1+}(\hat{\omega}, \hat{z}_+) - T_o(\hat{z}_-) \hat{x}_{1-}^w(-\hat{\omega}) \partial_{\hat{z}} \hat{X}_{1-}(\hat{\omega}, \hat{z}_-)]} \\ &\quad \times e^{i\frac{1}{2} \int \frac{d\hat{\omega}}{2\pi} [T_o(\hat{z}_+) \hat{x}_{2+}^w(-\hat{\omega}) \partial_{\hat{z}} \hat{X}_{2+}(\hat{\omega}, \hat{z}_+) - T_o(\hat{z}_-) \hat{x}_{2-}^w(-\hat{\omega}) \partial_{\hat{z}} \hat{X}_{2-}(\hat{\omega}, \hat{z}_-)]}. \end{aligned} \quad (3.5)$$

For sufficiently small  $\epsilon$  we have  $T_o(\hat{z}_-) \approx -T_o(\hat{z}_+) = T_o(z_w)$ .

The classical solution eq. (2.20) may be expressed in terms of the  $\hat{x}_{i\pm}^w$ .

$$a_i(\hat{\omega}) = \frac{1}{e^{\delta_i \hat{\omega}/2T} - 1} \frac{\hat{x}_{i+}^w(\hat{\omega}) - \hat{x}_{i-}^w(\hat{\omega})}{F_{\hat{\omega}}^{w*}}, \quad (3.6)$$

$$b_i(\hat{\omega}) = \frac{1}{e^{\delta_i \hat{\omega}/2T} - 1} \frac{e^{\delta_i \hat{\omega}/2T} \hat{x}_{i-}^w(\hat{\omega}) - \hat{x}_{i+}^w(\hat{\omega})}{F_{\hat{\omega}}^w}, \quad (3.7)$$

where, again,  $F_{\hat{\omega}}(\hat{z}_+) \approx F_{\hat{\omega}}(\hat{z}_-) = F_{\hat{\omega}}^w$

Following [22] we introduce the “*ra*” basis

$$\hat{X}_r(\hat{\omega}, \hat{z}) = \frac{1}{2} \left( \hat{X}_1(\hat{\omega}, \hat{z}) + \hat{X}_2(\hat{\omega}, \hat{z}) \right), \quad \hat{X}_a(\hat{\omega}, \hat{z}) = \left( \hat{X}_1(\hat{\omega}, \hat{z}) - \hat{X}_2(\hat{\omega}, \hat{z}) \right). \quad (3.8)$$

In this basis, the exponent of eq. (3.5) is given by

$$\begin{aligned} iS^{\text{eff}} = & i\frac{1}{2}T_o(z_w) \int \frac{d\hat{\omega}}{2\pi} \times \\ & \left[ 2i\text{Im} \left\{ \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \right\} \frac{e^{\hat{\omega}/2T} + 1}{e^{\hat{\omega}/2T} - 1} (\hat{x}_{r-}^w(-\hat{\omega}) - \hat{x}_{r+}^w(-\hat{\omega})) (\hat{x}_{r-}^w(\hat{\omega}) - \hat{x}_{r+}^w(\hat{\omega})) \right. \\ & + \frac{i}{2}\text{Im} \left\{ \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \right\} \frac{e^{\hat{\omega}/2T} + 1}{e^{\hat{\omega}/2T} - 1} (\hat{x}_{a-}^w(-\hat{\omega}) - \hat{x}_{a+}^w(-\hat{\omega})) (\hat{x}_{a-}^w(\hat{\omega}) - \hat{x}_{a+}^w(\hat{\omega})) \\ & + \hat{x}_{a-}^w(-\hat{\omega}) \left( \hat{x}_{r-}^w(\hat{\omega}) \left( \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} + \frac{\partial F_{\hat{\omega}}^{w*}}{F_{\hat{\omega}}^{w*}} \right) + \hat{x}_{r+}^w(\hat{\omega}) \left( \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} - \frac{\partial F_{\hat{\omega}}^{w*}}{F_{\hat{\omega}}^{w*}} \right) \right) \\ & \left. - \hat{x}_{a+}^w(-\hat{\omega}) \left( \hat{x}_{r+}^w(\hat{\omega}) \left( \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} + \frac{\partial F_{\hat{\omega}}^{w*}}{F_{\hat{\omega}}^{w*}} \right) + \hat{x}_{r-}^w(\hat{\omega}) \left( \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} - \frac{\partial F_{\hat{\omega}}^{w*}}{F_{\hat{\omega}}^{w*}} \right) \right) \right], \end{aligned} \quad (3.9)$$

where eq. (2.18) has been used. The action can be simplified further since

$$T_o(z_w) \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} = i\hat{\omega}\eta, \quad \eta = \frac{1}{2}\sqrt{\lambda}\pi T^2, \quad (3.10)$$

as it can be easily seen from eq. (2.16) and eq. (2.14).

The partition function  $\mathcal{Z}^>[\hat{x}_{i+}^w]$  depends on the fluctuations at  $\hat{z} > \hat{z}^w$  and, in particular at the AdS horizon  $\hat{z}^h = (1 - \epsilon)\sqrt{\gamma}/\pi T$ .

$$\mathcal{Z}^>[\hat{x}_{1+}^w, \hat{x}_{2+}^w] = \int \mathcal{D}\hat{x}_1^h(\hat{\omega})\hat{x}_2^h(\hat{\omega}) \mathcal{Z}_b^>[\hat{x}_{1+}^w, \hat{x}_{2+}^w, \hat{x}_1^h, \hat{x}_2^h] \mathcal{Z}^h[\hat{x}_1^h, \hat{x}_2^h], \quad (3.11)$$

where we have made explicit the dependence on the coordinates at the horizon  $\hat{x}_i^h(\hat{\omega}) = \hat{X}_i(\hat{\omega}, \hat{z}^h)$ . Since we want to describe the string dynamics at all scales causally connected with the boundary, we perform the matching between the fluctuations on the left and right sides of the Kruskal plane, figure 2, at  $\hat{z}^h$ . As in eq. (3.5) the partition function at the horizon is given by the classical action

$$\mathcal{Z}^h[\hat{x}_1^h, \hat{x}_2^h] \propto e^{-i\frac{T_o(\hat{z}^h)}{2} \int \frac{d\hat{\omega}}{2\pi} [\hat{x}_2^h(\hat{\omega})\partial_z \hat{X}_2(\hat{\omega}, \hat{z}^h) - \hat{x}_1^h(\hat{\omega})\partial_z \hat{X}_1(\hat{\omega}, \hat{z}^h)]}. \quad (3.12)$$

An analysis similar to the one performed at  $\hat{z}^w$  must be performed at  $\hat{z}^h$ . The classical solution eq. (2.20) close to the horizon sets

$$\hat{X}_a(\hat{\omega}, \hat{z}^h) = 0. \quad (3.13)$$

Thus, the effective action  $\mathcal{Z}^h$  forces the string fluctuations to fulfill eq. (3.13). To show this, we perform the matching slightly above the horizon in the left universe and identify

$$\hat{x}_2^h(\omega) = \hat{X}_2(\hat{z}^h + \epsilon, \omega), \quad (3.14)$$

and we will take  $\epsilon \rightarrow 0$  at the end.<sup>1</sup> To leading order in  $\epsilon$  we find

$$a(\hat{\omega}) = \frac{x_a(\omega)}{2\epsilon \partial_{\hat{z}} F_{\hat{\omega}}^*(\hat{z}^h)}, \quad (3.15)$$

$$b(\hat{\omega}) = \frac{x_a(\omega)}{2\epsilon \partial_{\hat{z}} F_{\hat{\omega}}(\hat{z}^h)}. \quad (3.16)$$

Substituting the classical solution eq. (2.20), eq. (3.15) and eq. (3.16) in eq. (3.12) and taking the the small  $\epsilon$  limit we obtain

$$\mathcal{Z}^h[\hat{x}_1^h, \hat{x}_2^h] = \lim_{\epsilon \rightarrow 0} \exp \left\{ -iT_o(\hat{z}^h) \int \frac{d\hat{\omega}}{2\pi} \frac{\hat{x}_a^h(-\hat{\omega})\hat{x}_a^h(\hat{\omega})}{4\epsilon} \right\}, \quad (3.17)$$

$$= \mathcal{N} \delta(\hat{x}_a^h(\hat{\omega})), \quad (3.18)$$

with  $\mathcal{N}$  a divergent constant (independent of the string coordinates) which can be absorbed into the normalization of the partition function. Note also that, unlike the  $v = 0$  case [22], the partition function does not depend on the string coordinates at  $\hat{z}^h$ . The horizon effective action eq. (3.18) breaks the symmetry between the  $r$  and  $a$  coordinates in the string action.

The partition function in the bulk is also conveniently expressed in the  $ra$  basis. Integrating the classical action by parts, we find

$$\mathcal{Z}^<[\hat{x}_{1-}^w, \hat{x}_{2-}^w] = \int \mathcal{D}\hat{x}_a^b(\hat{\omega}) \mathcal{D}\hat{x}_r^b(\hat{\omega}) \int_{\hat{X}_i(\hat{\omega}, \hat{\epsilon})=\hat{x}_i^b}^{\hat{X}_i(\hat{\omega}, \hat{z}_-)=\hat{x}_{i-}^w} \mathcal{D}_s \hat{X}_a(\hat{\omega}, \hat{z}) \mathcal{D}_s \hat{X}_r(\hat{\omega}, \hat{z}) \quad (3.19)$$

$$\left[ e^{iT_o(z_b) \int \frac{d\hat{\omega}}{2\pi} \hat{x}_a^b(-\hat{\omega}) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{\epsilon})} e^{-iT_o(\hat{z}_-) \int \frac{d\hat{\omega}}{2\pi} \hat{x}_{a-}^w(-\hat{\omega}) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z}_-)} \times \right. \\ \left. e^{i \int \frac{d\hat{\omega}}{2\pi} \int_{\hat{\epsilon}}^{\hat{z}_-} d\hat{z} \hat{X}_a(-\hat{\omega}, \hat{z}) \left[ \partial_{\hat{z}} (T_o(\hat{z}) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z})) + \frac{m\hat{\omega}^2}{\pi T \hat{z}^2 f(\hat{z})} \hat{X}_r(\hat{\omega}, \hat{z}) \right]} \right],$$

$$\mathcal{Z}^>[\hat{x}_{1+}^w, \hat{x}_{2+}^w] = \int_{\hat{X}_i(\hat{\omega}, \hat{z}_+)=\hat{x}_{i+}^w}^{\hat{X}_i(\hat{\omega}, \hat{z})=\hat{x}_i^w} \mathcal{D}_s \hat{X}_a(\hat{\omega}, \hat{z}) \mathcal{D}_s \hat{X}_r(\hat{\omega}, \hat{z}) \\ \left[ e^{iT_o(\hat{z}_+) \int \frac{d\hat{\omega}}{2\pi} \hat{x}_{a+}^w(-\hat{\omega}) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z}_+)} \times \right. \\ \left. e^{i \int \frac{d\hat{\omega}}{2\pi} \int_{\hat{z}_+}^{\hat{z}^h} d\hat{z} \hat{X}_a(-\hat{\omega}, \hat{z}) \left[ \partial_{\hat{z}} (T_o(\hat{z}) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z})) + \frac{m\hat{\omega}^2}{\pi T \hat{z}^2 f(\hat{z})} \hat{X}_r(\hat{\omega}, \hat{z}) \right]} \right], \quad (3.20)$$

where we have introduced the notation  $T_o(z_b) = T_o(\hat{z} = \hat{\epsilon})$ . In writing  $\mathcal{Z}^>[\hat{x}_{1+}^w, \hat{x}_{2+}^w]$  we have used eq. (3.13).

### 3.2 Stochastic string motion

The “ $a$ ” coordinates in the partition function eq. (3.5) can be integrated out, after introducing a random force term in eq. (3.9)

$$e^{-\frac{1}{2} \int \frac{d\hat{\omega}}{2\pi} \frac{T_o(z_w)}{2} \text{Im} \left\{ \frac{\partial_{\hat{z}} F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \right\} \frac{e^{\hat{\omega}/2T} + 1}{e^{\hat{\omega}/2T} - 1} (\hat{x}_{a-}^w(-\hat{\omega}) - \hat{x}_{a+}^w(-\hat{\omega})) (\hat{x}_{a-}^w(\hat{\omega}) - \hat{x}_{a+}^w(\hat{\omega}))} \\ = \int \mathcal{D}\hat{\xi}^w e^{i \int \frac{d\hat{\omega}}{2\pi} (\hat{x}_{a-}^w(-\hat{\omega}) - \hat{x}_{a+}^w(-\hat{\omega})) \hat{\xi}^w(\hat{\omega})} e^{-\frac{1}{2} \int \frac{d\hat{\omega}}{2\pi} \frac{2}{T_o(z_w)} \text{Im} \left\{ \frac{F_{\hat{\omega}}^w}{\partial_{\hat{z}} F_{\hat{\omega}}^w} \right\} \frac{e^{\hat{\omega}/2T} - 1}{e^{\hat{\omega}/2T} + 1} \hat{\xi}^w(-\hat{\omega}) \hat{\xi}^w(\hat{\omega})}.$$

<sup>1</sup>This procedure avoids the complication that in eq. (2.20) the solutions in the left and right universe above the world-sheet horizon are proportional.

The quadratic term in  $\hat{x}_{r\pm}^w$  in eq. (3.9) can be understood as a discontinuity in the average string position across the world-sheet horizon

$$\hat{x}_{r+}^w = \hat{x}_{r-}^w + \Delta. \quad (3.22)$$

The discontinuity  $\Delta$  is random, with a Gaussian distribution given by eq. (3.9). This discontinuity is natural, since the correlation between the two sides of the horizon can be viewed as a tunneling process across a barrier. Close to the horizon, the fluctuations are large and the procedure of analytically continuing across the horizon is similar to the matching of WKB solutions in quantum mechanics in the vicinity of a turning point.

Integrating over the “ $a$ ” coordinates, the partition function is expressed as a product of  $\delta$ -functions.

$$\begin{aligned} \mathcal{Z} = & \int \mathcal{D}\hat{x}_r^b \mathcal{D}\hat{x}_a^b \mathcal{D}\hat{x}_{r-}^w \mathcal{D}_s \hat{X}_r \left\langle e^{i \int \frac{d\hat{\omega}}{2\pi} T_o(z_b) \hat{x}_a^b(-\hat{\omega}) \partial_z \hat{X}_r(\hat{\omega}, \hat{\epsilon})} \right. \\ & \times \delta \left( \partial_{\hat{z}} \left( T_o(\hat{z}) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z}) \right) + \frac{m\hat{\omega}^2}{\pi T \hat{z}^2 f(\hat{z})} \hat{X}_r(\hat{\omega}, \hat{z}) \right) \\ & \times \delta \left( T_o(\hat{z}_-) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z}_-) - \hat{\xi}^w(\hat{\omega}) - T_o(z_w) \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} (\hat{x}_{r-}^w(\hat{\omega}) + \Delta(\hat{\omega})) \right) \\ & \times \delta \left( T_o(\hat{z}_+) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z}_+) - \hat{\xi}^w(\hat{\omega}) - T_o(z_w) \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \hat{x}_{r-}^w(\hat{\omega}) \right) \Bigg\rangle_{\hat{\xi}, \Delta}, \end{aligned} \quad (3.23)$$

where the subscripts  $\hat{\xi}$ ,  $\Delta$  denote average with respect to the random force and discontinuity variables. These are given by a gaussian distribution such that <sup>2</sup>

$$\langle \hat{\xi}^w(\hat{\omega}) \hat{\xi}^w(-\hat{\omega}) \rangle = \frac{1}{2} T_o(z_w) \text{Im} \left\{ \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \right\} \frac{e^{\hat{\omega}/2T} + 1}{e^{\hat{\omega}/2T} - 1}, \quad (3.24)$$

$$\langle \Delta(\hat{\omega}) \Delta(-\hat{\omega}) \rangle = \frac{1}{2} \frac{1}{T_o(z_w) \text{Im} \left\{ \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \right\}} \frac{e^{\hat{\omega}/2T} - 1}{e^{\hat{\omega}/2T} + 1}. \quad (3.25)$$

The partition function is determined by the classical solutions to the string equations of motion with von Neumann boundary conditions at the boundary, the boundary conditions at the ws-horizon

$$T_o(\hat{z}_-) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z}_-) = i\hat{\omega} \eta (\hat{x}_{r-}^w(\hat{\omega}) + \Delta(\hat{\omega})) + \hat{\xi}(\hat{\omega}), \quad (3.26)$$

$$T_o(\hat{z}_+) \partial_{\hat{z}} \hat{X}_r(\hat{\omega}, \hat{z}_+) = i\hat{\omega} \eta \hat{x}_{r-}^w(\hat{\omega}) + \hat{\xi}(\hat{\omega}), \quad (3.27)$$

and a discontinuity at the ws-horizon eq. (3.22). The classical solution is given by

$$\begin{aligned} \hat{X}_r(\hat{\omega}, \hat{z}) = & \hat{x}_r^b(\hat{\omega}) F_{\hat{\omega}}(\hat{z}) + \frac{\text{Im} F_{\hat{\omega}}(\hat{z})}{-\text{Im} G_R(\hat{\omega})} F_{\hat{\omega}}^w \left( \hat{\xi}^w(\hat{\omega}) + T_o(z_w) \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \Delta(\hat{\omega}) \right) \\ & + \theta(\hat{z} \pi T - 1) \Delta(\hat{\omega}) \frac{F_{\hat{\omega}}^*(\hat{z})}{F_{\hat{\omega}}^{w*}}, \end{aligned} \quad (3.28)$$

where  $\text{Im} G_R$  has been defined in eq. (2.19).

<sup>2</sup>Note that in the  $\lambda \rightarrow \infty$  the  $\langle \Delta(-\hat{\omega}) \Delta(\hat{\omega}) \rangle$  vanishes and the discontinuity disappears.

The equations of motion propagate the stochasticity at the world-sheet horizon to all scales. From eq. (3.28) the bulk two point function for transverse fluctuations is given by

$$\begin{aligned}
 \hat{G}_{T\text{sym}}(\hat{\omega}, \hat{z}, \hat{z}') &\equiv \left\langle \Delta \hat{X}_r(\hat{\omega}, \hat{z}) \Delta \hat{X}_r(-\hat{\omega}, \hat{z}') \right\rangle \\
 &= -\frac{\text{Im} F_{\hat{\omega}}(\hat{z}) \text{Im} F_{\hat{\omega}}(\hat{z}')}{\text{Im} G_R(\hat{\omega})} (1 + 2\hat{n}) \\
 &\quad + \theta(\hat{z}\pi T - 1) \frac{1}{2} \frac{\text{Im} F_{\hat{\omega}}(\hat{z}') i F_{\hat{\omega}}^*(\hat{z})}{\text{Im} G_R(\hat{\omega})} \frac{e^{\hat{\omega}/2T} - 1}{e^{\hat{\omega}/2T} + 1} \\
 &\quad + \theta(\hat{z}'\pi T - 1) \frac{1}{2} \frac{\text{Im} F_{\hat{\omega}}(\hat{z}) (-i) F_{\hat{\omega}}(\hat{z}')}{\text{Im} G_R(\hat{\omega})} \frac{e^{\hat{\omega}/2T} - 1}{e^{\hat{\omega}/2T} + 1} \\
 &\quad - \theta(\hat{z}'\pi T - 1) \theta(\hat{z}\pi T - 1) \frac{1}{2} \frac{F_{\hat{\omega}}^*(\hat{z}) F_{\hat{\omega}}(\hat{z}')}{\text{Im} G_R(\hat{\omega})} \frac{e^{\hat{\omega}/2T} - 1}{e^{\hat{\omega}/2T} + 1},
 \end{aligned} \tag{3.29}$$

where  $\hat{n} = 1/(e^{\hat{\omega}/T} - 1)$  and we have subtracted the average string position

$$\Delta \hat{X}_r(\hat{\omega}, \hat{z}) = \hat{X}_r(\hat{\omega}, \hat{z}) - \hat{x}_r^b(\hat{\omega}) F_{\hat{\omega}}(\hat{z}). \tag{3.30}$$

Finally, we may express the two point correlator in the original  $t, z$  coordinates. Undoing the change of variables eq. (2.7),

$$G_{T\text{sym}}(t, z; t', z') = \frac{1}{\sqrt{\gamma}} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-i\omega(\zeta(z)-\zeta(z'))} \hat{G}_{T\text{sym}}(\omega\sqrt{\gamma}; z\sqrt{\gamma}, z'\sqrt{\gamma}), \tag{3.31}$$

where the function  $\zeta(z)$  has been defined in eq. (2.7) and relates the AdS  $t$ -coordinate to  $\hat{t}$ .

### 3.3 Heavy quark partition function

After integrating out  $\hat{x}_{r-}^w$  in the string partition function eq. (3.23) we get the partition function for the string boundary end point. This is the partition function for the heavy quark. Since  $\hat{z}^b < 1$  we find

$$\mathcal{Z}_T^Q = \int \mathcal{D}\hat{x}_a^b \mathcal{D}\hat{x}_r^b \left\langle e^{i \int \frac{d\hat{\omega}}{2\pi} \hat{x}_a^b(-\hat{\omega}) (\hat{x}_r^b(\hat{\omega}) T_o(z_b) \partial_{\hat{z}} F_{\hat{\omega}}(\hat{\epsilon}) + \hat{\xi}^b(\hat{\omega}))} \right\rangle_{\hat{\xi}^w, \Delta}, \tag{3.32}$$

where we have used  $\text{Im} G_R = -T_o(z_b) \text{Im} \partial_{\hat{z}} F_{\hat{\omega}}(\hat{\epsilon})$  and we have defined the boundary force as

$$\hat{\xi}^b(\hat{\omega}) = F_{\hat{\omega}}^w \left( \hat{\xi}^w(\hat{\omega}) + T_o(z_w) \frac{\partial_{\hat{z}} F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \Delta(\hat{\omega}) \right). \tag{3.33}$$

It is easy to show by integration that the boundary force distribution is also Gaussian with the second moment given by

$$\left\langle \hat{\xi}^b(\hat{\omega}) \hat{\xi}^b(-\hat{\omega}) \right\rangle = -(1 + 2\hat{n}) \text{Im} G_R(\hat{\omega}). \tag{3.34}$$

The first term in the exponent of eq. (3.32) is divergent [22]

$$T_o(z_b) \partial_{\hat{z}} F_{\hat{\omega}}(\hat{\epsilon}) = \frac{\sqrt{\lambda}}{2\hat{\epsilon}} \hat{\omega}^2 - G_R(\hat{\omega}), \tag{3.35}$$

however, this divergence can be understood as the contribution of the (large) quark mass  $M_o = \sqrt{\lambda}/2\epsilon$  with  $\hat{\epsilon} = \sqrt{\gamma}\epsilon$ .

Using eq. (3.35) and undoing the change of variables eq. (2.7) we find

$$\mathcal{Z}_T^Q = \int \mathcal{D}x_a^b \mathcal{D}x_r^b \left\langle e^{i \int \frac{d\omega}{2\pi} x_a^b(-\omega) (\gamma M_o \omega^2 x_r^b(\omega) - \sqrt{\gamma} G_R(\sqrt{\gamma}\omega) x_r^b(\omega) + \xi(\omega))} \right\rangle_{\xi, \Delta}, \quad (3.36)$$

with  $\xi(\omega) \equiv \sqrt{\gamma} \hat{\xi}^b(\sqrt{\gamma}\omega)$ . Integrating out the  $x_a^b$  coordinate we find that the average position of the quark follows the equation

$$\gamma M_o \frac{d^2 x_r^b}{dt^2} + \int dt' G_R \left( \frac{t-t'}{\sqrt{\gamma}} \right) x_r^b(t') = \xi(t), \quad (3.37)$$

as previously derived in [19].

The effective equation of motion may be further clarified by studying the long time dynamics  $\omega \rightarrow 0$  limit of eq. (3.37). Expanding the retarded correlator to second order

$$G_R(\hat{\omega}) = -i\eta\hat{\omega} + \frac{\Delta M}{\sqrt{\gamma}}\hat{\omega}^2, \quad (3.38)$$

with

$$\Delta M = \frac{\sqrt{\gamma\lambda}T}{2}. \quad (3.39)$$

Denoting  $M_{\text{kin}} = M - \Delta M$ , we obtain

$$\gamma M_{\text{kin}} \frac{d^2 x_{Tr}^b}{dt^2} + \eta\gamma \frac{dx_{Tr}^b}{dt} = \xi(t), \quad (3.40)$$

where we have recovered the subscript  $T$  to denote transverse fluctuations. Note that, due to the mass shift eq. (3.39), the heavy quark becomes effectively lighter as the velocity increases. This reduction of the kinetic mass coincides with the result argued in [28].

### 3.4 Extension to the longitudinal fluctuations

The analysis of longitudinal fluctuations around the trailing string solution is completely analogous to the one performed above for transverse modes. Thus, we will not repeat the computation here but we will simply state the main results. From the observation that the action is multiplied by a factor  $\gamma^2$  a simple rule can be given to obtain the longitudinal expressions from the transverse ones; it is sufficient to replace  $T_o \rightarrow \gamma^2 T_o$ .

The stochastic longitudinal fluctuations may be expressed as

$$\begin{aligned} \hat{X}_{Lr} = & \hat{x}_{Lr}^b F_{\hat{\omega}}(\hat{z}) + \frac{\text{Im} F_{\hat{\omega}}(\hat{z})}{-\gamma^2 \text{Im} G_R(\hat{\omega})} F_{\hat{\omega}}^w \left( \xi_L^w + \gamma^2 T_o(z_w) \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \Delta_L(\hat{\omega}) \right) \\ & + \theta(\hat{z} - 1) \Delta_L(\hat{\omega}) \frac{\partial_z F_{\hat{\omega}}^*(\hat{z})}{F^{w*}}, \end{aligned} \quad (3.41)$$

with  $\hat{x}_{Lr}^b$  is the boundary value of the fluctuation and  $\xi_L^w$  and  $\Delta_L$  are random variables distributed according to the two point functions

$$\langle \xi_L^w(\hat{\omega}) \xi_L^w(-\hat{\omega}) \rangle = \gamma^2 \langle \xi^w(\hat{\omega}) \xi^w(-\hat{\omega}) \rangle, \quad (3.42)$$

$$\langle \Delta_L(\hat{\omega}) \Delta_L(-\hat{\omega}) \rangle = \frac{1}{\gamma^2} \langle \Delta(\hat{\omega}) \Delta(-\hat{\omega}) \rangle, \quad (3.43)$$

with the transverse correlators defined in eq. (3.24) and eq. (3.25).

The correlation function for string fluctuations around the average value is also proportional to the longitudinal correlator

$$\hat{G}_{L\text{sym}}(\hat{\omega}, \hat{z}, \hat{z}') \equiv \left\langle \Delta \hat{X}_{Lr}(\hat{\omega}, \hat{z}) \Delta \hat{X}_{Lr}(-\hat{\omega}, \hat{z}') \right\rangle = \frac{1}{\gamma^2} \hat{G}_{T,\text{sym}}(\hat{\omega}, \hat{z}, \hat{z}') \quad (3.44)$$

On the boundary, the stochastic string motion leads to a random longitudinal force which is related to the random variables  $\xi_L^w$ ,  $\Delta_L$  as

$$\xi_L(\omega) = \sqrt{\gamma} F_{\hat{\omega}}^w \left( \xi_L^w(\hat{\omega}) + \gamma^2 T_o(z_w) \frac{\partial_z F_{\hat{\omega}}^w}{F_{\hat{\omega}}^w} \Delta_L(\hat{\omega}) \right). \quad (3.45)$$

The boundary force-force correlator is

$$\langle \xi_L(\omega) \xi_L(-\omega) \rangle = -\gamma^{5/2} (1 + 2n(\sqrt{\gamma}\omega)) \text{Im} G_R(\sqrt{\gamma}\omega) \quad (3.46)$$

Finally, the effective equation of motion for the boundary end-point of the string is given by

$$\gamma^3 M_o \frac{d^2 x_{Lr}^b}{dt^2} + \gamma^2 \int dt' G_R \left( \frac{t-t'}{\sqrt{\gamma}} \right) x_{Lr}^b(t') = \xi_L(t), \quad (3.47)$$

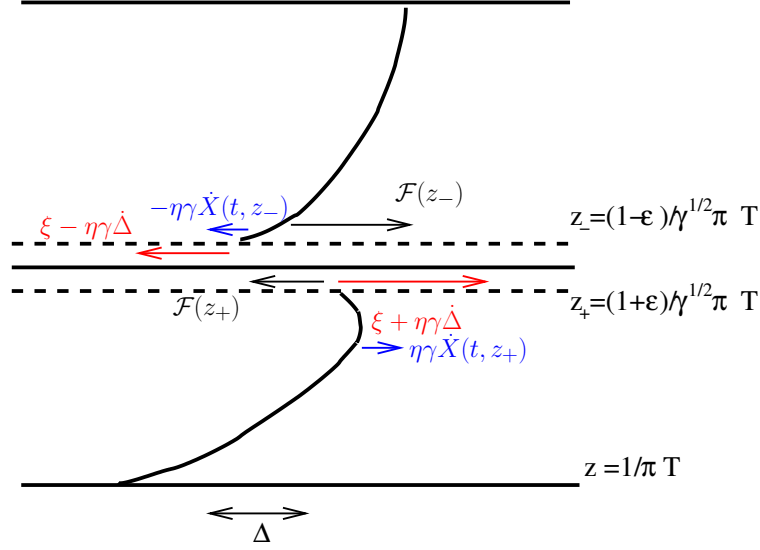
which in the low frequency limit is

$$\gamma^3 M_{\text{kin}} \frac{d^2 x_{Lr}^b}{dt^2} + \gamma^3 \eta \frac{dx_{Lr}^b}{dt} = \xi_L(t). \quad (3.48)$$

## 4 Summary and outlook

### 4.1 Summary of the main results

In this work we have analyzed the stochastic motion of the trailing string generated at the world-sheet horizon by integrating out the near horizon string fluctuation. In contrast to the static case, the string fluctuations below the world-sheet horizon are causally connected to the boundary. The integration of these modes leads to a discontinuity in the string across the world-sheet horizon  $\Delta_\alpha$  and the generation of a random force acting on the string above and below the horizon  $\xi_\alpha^w$ . The subindex  $\alpha = L, T$  indicates that the force is different for the longitudinal and transverse components. The discontinuity  $\Delta_\alpha$  should not be understood as a discontinuity of the world-sheet since our description in term of small string fluctuations is only valid outside of a small region around the world-sheet horizon; the discontinuity refers to the different positions of the string above and below the stretched horizon. We have sketched these dynamics in figure 3. The computations in this work have been performed in the set of coordinates eq. (2.7). In this discussion we will undo this change of variables and restore the AdS coordinates eq. (2.1). For simplicity of notation, we will only summarize the transverse fluctuations and drop the subscript  $T$ , since the extension to the longitudinal case is straight forward, section 3.4. From eq. (3.23) and



**Figure 3.** Sketch of the string profile across the world-sheet horizon. The string fluctuation within the stretched horizon lead to discontinuity in string position  $\Delta$  and the appearance of a random force  $\xi$  which acts on both ends of the string. The tension force across the world-sheet horizon is discontinuous due to the different velocities of the string endpoints. There is a balance of forces on each side of the world-sheet horizon among the tension force, the random force and a resistive force  $\pm \eta \gamma \dot{X}(t, z_{\pm})$  and a force which depends on the separation of the string endpoint  $\eta \gamma \dot{\Delta}(t)$ . Note that we plot the longitudinal fluctuations for clarity; there are also fluctuations transverse to the string motion.

undoing the change of variables eq. (2.7) the forced motion of the world sheet end points of the string is given by<sup>3</sup>

$$T_o(\hat{z}_{\pm})\partial_z X_r(t, z_{\pm}) - \xi^w(t) = \pm \eta \gamma \dot{\Delta} \quad (4.1)$$

$$\langle \xi^w(t) \xi^w(t') \rangle = G_{\text{sym}}^w(t - t') \quad \langle \Delta(t) \Delta(t') \rangle = G_{\Delta}(t - t') \quad (4.2)$$

with the discontinuity variable  $\Delta = X(t, z_+) - X(t, z_-)$  and the correlators are obtained from eq. (3.24) and eq. (3.25)

$$G_{\text{sym}}^w(\omega) = \frac{1}{2} \gamma \eta \omega \frac{e^{\omega \sqrt{\gamma}/2T} + 1}{e^{\omega \sqrt{\gamma}/2T} - 1}, \quad G_{\Delta}(\omega) = \frac{1}{2} \frac{1}{\eta \gamma \omega} \frac{e^{\omega \sqrt{\gamma}/2T} - 1}{e^{\omega \sqrt{\gamma}/2T} + 1}. \quad (4.3)$$

In deriving this expression we have assumed that  $\epsilon \ll 1 - 1/\sqrt{\gamma}$  so that the strip does not overlap with the world-sheet horizon. Thus, we shall not take  $v \rightarrow 0$  below<sup>4</sup>

The horizon equations of motion eq. (3.26) and eq. (3.27) can be interpreted as the balance of the random force at the horizon with the tension force,  $\mathcal{F}$ , and a drag like force

<sup>3</sup>We use that  $X(\omega, z) = e^{-i\omega\zeta(z)} \hat{X}(\omega\sqrt{\gamma}, z\sqrt{\gamma})$  with  $\zeta$  given in eq. (2.7).

<sup>4</sup>This is a consequence of exponentiating eq. (2.7),  $e^{-i\omega\hat{\xi}}$ , which leads to two poles at  $\hat{z} = 1/\pi T$  and  $z = 1/\pi T$ ; these poles coincide at  $v = 0$ . We have explicitly checked that we reproduce the  $v = 0$  limit. In this limit, the transformation eq. (2.7) is trivial and the equations of motion are the same as in [22]



on the string endpoints.

$$\mathcal{F}(z_-) + \xi - \eta\gamma\dot{\Delta} = \eta\gamma\dot{X}(t, z_-), \quad (4.4)$$

$$\mathcal{F}(z_+) - \xi - \eta\gamma\dot{\Delta} = -\eta\gamma\dot{X}(t, z_+), \quad (4.5)$$

where we have identified the tension force  $\mathcal{F}(z_{\pm}) = \pm T_o(\hat{z}_{\pm})\partial_z\hat{X}(t, \hat{z}_{\pm})$  and  $\hat{\omega} = \sqrt{\gamma}\omega$  (see appendix B for a derivation of this expression). The string is discontinuous across the world-sheet horizon and the point above and below the horizon move with different speeds which leads to a discontinuity of the tension force. The equation of motion on each side of the horizon depends on the endpoint on the other side via  $\Delta$  which might be interpreted as a force of magnitude  $\pm\eta\gamma\dot{X}(t, z_{\pm})$  due to the motion of the string on the other side of the horizon. The net force introduced by the horizon is  $\mathcal{F}(z_-) + \mathcal{F}(z_+) = \eta\gamma\dot{\Delta}$  and grows as the separation of the two sides of the string grows.

The classical string equations propagate the stochastic motion on the world-sheet horizon endpoint to all scales.

Of particular interest is the motion of the boundary endpoint, since it is dual to the position of the heavy quark. For these, the fluctuations below the world sheet horizon may be integrated out leading to a second random force at the world sheet horizon. The effective force acting on the boundary is given by eq. (3.33)

$$\xi(t) = \int^t dt' F\left(\frac{t-t'}{\sqrt{\gamma}}, z_-^w\right) \left(\xi^w(t') - \eta\gamma\dot{\Delta}(t')\right), \quad (4.6)$$

where  $F(t, z_-^w)$  is the retarded boundary to horizon propagator eq. (2.16). Note that the time it takes the noise to propagate to the boundary  $\sim \sqrt{\gamma}/\pi T$  grows with the velocity of the quark. The force correlator at the boundary is given by

$$\langle \xi(t)\xi(t') \rangle = G_{\text{sym}}^{w=0}\left(\frac{t-t'}{\sqrt{\gamma}}\right), \quad (4.7)$$

with the zero velocity symmetrized correlator

$$G_{\text{sym}}^{w=0}(\omega) = -(1+2n)\text{Im } G_R(\omega). \quad (4.8)$$

From eq. (4.7) is easy to conclude that the noise correlation time  $\tau_C$  grows with the velocity of the quark

$$\tau_C \sim \frac{\sqrt{\gamma}}{\pi T} \quad (4.9)$$

The combination of the random force eq. (4.6) and von Neumann boundary conditions at the boundary leads to the equation of motion for the boundary end point eq. (3.37), which in the low frequency limit may be expressed as

$$\frac{d\mathbf{p}}{dt} = \mathcal{E} - \mu\mathbf{p} + \xi(t), \quad (4.10)$$

with  $\mu = \eta/M_{\text{kin}}$ . This equation describes the forced motion of the particle in a dissipative medium with the random force correlators

$$\langle \xi_T(t) \xi_T(t') \rangle = \sqrt{\gamma} \kappa \delta(t - t'), \quad (4.11)$$

$$\langle \xi_L(t) \xi_L(t') \rangle = \gamma^{5/2} \kappa \delta(t - t'), \quad (4.12)$$

$$\kappa = \sqrt{\lambda} \pi T^3. \quad (4.13)$$

These equations have been previously derived in [19].

As in [22], the string position at an arbitrary point  $z$  may be expressed as the reaction of the in-falling string to the motion at the boundary  $x_0(t)$  plus a random piece.

$$X(t, z) = \int dt' F^v(t - t', z) x_0(t') + \Delta X(t, z). \quad (4.14)$$

where the in-falling function  $F^v(\omega, z) = e^{-i\omega\zeta(z)} F(\sqrt{\gamma}\omega, \sqrt{\gamma}z)$  and  $\zeta(z)$  is given in eq. (2.7). The random function  $\Delta X(t, z)$  vanishes at the boundary and it is given by the distribution

$$\langle \Delta X(t, z) \Delta X(t', z') \rangle = G_{\text{sym}}(t, z; t', z'), \quad (4.15)$$

with the explicit expression for  $G_{\text{sym}}(t, z; t', z')$  given by eq. (3.31).

## 4.2 The low frequency limit and the physical picture

To further clarify the dynamics of the string, we consider its motion over a long time interval  $\tau$  such that

$$\tau \gg \tau_C \sim \frac{\sqrt{\gamma}}{\pi T}, \quad (4.16)$$

*i. e.*, much larger than the correlation time of the noise. Over this time interval, the dynamics are well approximated by the low frequency limit of the string motion.

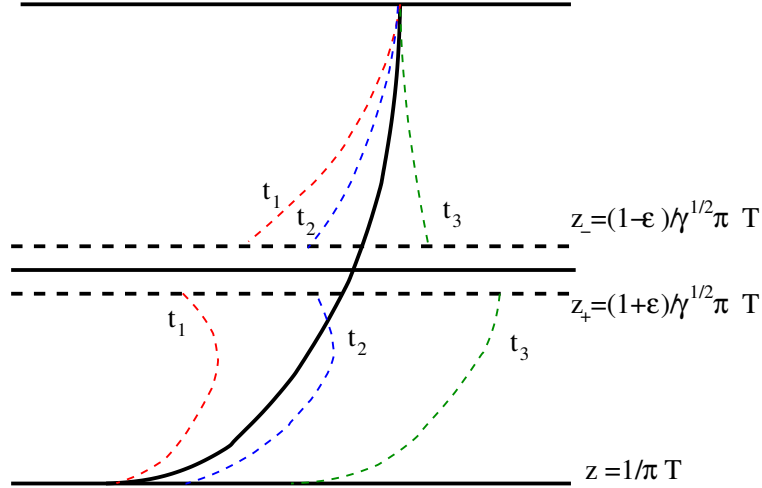
We study the reaction of the string to a small fluctuation  $x_0(t)$  on the quark motion over the stationary trajectory induced by the external electric field. This fluctuation is relaxed in a time given by the inverse drag coefficient  $\mu$

$$\tau_R \sim \frac{M_{\text{kin}}}{\sqrt{\lambda} T^2} \quad (4.17)$$

which, for sufficiently heavy quarks, is large compared to  $\tau_C$ . We will demand the time interval  $\tau \ll \tau_R$  so that the velocity can be considered as stationary. In this time interval, the average motion of the string is obtained from the low frequency expansion of eq. (4.14)

$$\langle X(t, z) \rangle = x_0(t) + \dot{x}_0(t) \Delta X_{TS}(z), \quad (4.18)$$

with the trailing string profile given by eq. (2.6). Thus, the reaction of the string to the perturbation leads to a trailing string solution with a velocity given by  $\mathbf{v} = v\hat{x} + \dot{\mathbf{x}}_0(t)$ .



**Figure 4.** Schematic view of the string fluctuations induced by the world-sheet horizon at three different times. On average the string is given by the trailing string solution eq. (2.5) with the velocity given by the motion of the boundary end point. At any given time, the string deviates from the average by a drag-like string but with a characteristic higher effective temperature  $\sqrt{\gamma}T$  and with a random amplitude  $-\xi/\eta$ . This stochastic ensemble of strings generates a random force of the boundary  $\xi$ . The string above and below the horizon are separated by a random variable  $\Delta$ . As in figure 3 we only plot the longitudinal fluctuations for clarity.

On top of the average value eq. (4.18), there is a random string profile which is given in eq. (3.28) which in the low frequency limit is<sup>5</sup>

$$\Delta X(t, z) = -\frac{\xi(t)}{\eta} \frac{1}{2\pi\sqrt{\gamma}T} \left( \tan^{-1}(z\pi\sqrt{\gamma}T) + \frac{1}{2} \ln \left| \frac{1 - z\pi\sqrt{\gamma}T}{1 + z\pi\sqrt{\gamma}T} \right| \right) + \theta(\sqrt{\gamma}\pi T z - 1) \Delta(t), \quad (4.19)$$

$$\langle \xi(t)\xi(t') \rangle = 2\eta\sqrt{\gamma}T\delta(t-t'), \quad \langle \Delta(t)\Delta(t') \rangle = \frac{1}{8\eta\sqrt{\gamma}T}\delta(t-t'), \quad (4.20)$$

This result can be interpreted as follows: the random force  $\xi$  generates an ensemble of strings which fluctuate around the average string eq. (4.18). Each of the strings in the ensemble have the same  $z$ -functional form which corresponds to the trailing string profile but with a higher characteristic temperature,  $\sqrt{\gamma}T$  which is the position of the world-sheet horizon. As the string goes across the world sheet horizon, there is a discontinuity in the position  $\Delta$  which can be understood as an effect of tunneling through a barrier. This barrier is due to the fact that at the world-sheet horizon the average velocity of the string is larger than the local longitudinal speed of light.

Finally, let us remark that turning off the electric field and in the very long time limit  $\tau \gg \tau_R$  the momentum of the quark is relaxed and the velocity tends to its (small) equilibrium value. Thus,  $\gamma \rightarrow 0$  and the long time dynamics are described by the static string [22].

<sup>5</sup>Note that the discontinuity of the derivative in eq. (4.1) vanishes at  $\omega = 0$

### 4.3 Limits on the validity of the approach

The results in this paper are based on the assumption that the heavy quark maintains an approximately constant speed during a time which is large compared to all medium scales, so that the dynamics of the quark are controlled by medium averages. Since the force correlation time grows with the quark velocity, eq. (4.9), the requirement that the  $\tau_C \ll \tau_R$  demands that

$$\sqrt{\gamma\lambda} \ll \frac{M_{\text{kin}}}{T}. \quad (4.21)$$

This velocity limit coincides parametrically with the one derived in [17] by determining the maximum value of the external electric field that the a D7 brane can support. The observation above provides a more physical interpretation for this velocity constraint: at the critical value of  $\gamma$  the correlation time between the random forces acting on the quark becomes comparable to the motion time scale; thus, the structure of the interaction cannot be neglected.

We also note that there is a different constraint on the velocity coming from the fact that the kinetic mass of the quark must be large in order for the relaxation time to be large compared to the thermal wavelength. As a consequence, the medium modified mass  $\Delta M$  should be small compared to the vacuum mass  $M_o$ ,  $\Delta M \ll M_o$ . Remarkably, this consideration leads (parametrically) to the same critical  $\gamma$  factor eq. (4.21). From the field theory side it is not clear to us why these two constraints should coincide.

We would also like to understand this limit on  $\gamma$  from the point of view of the dual gauge theory. The long correlation time signals that the fluctuations due to normalizable modes become large. Indeed, the assumption that the normalizable modes are suppressed is set by the condition that their classical action is large (and thus, their spontaneous fluctuations are unlikely). Taking the average over the normalizable fluctuations of eq. (2.12) we obtain

$$\langle S_T[\Delta X] \rangle \propto \frac{\mathcal{T}}{\sqrt{\gamma}} T, \quad (4.22)$$

where  $\mathcal{T}$  is the total time of observation.<sup>6</sup> To derive this expression we have used that, in *hat* coordinates, the only dimensionful quantity in the solution is  $T$ . Since the fluctuations are suppressed by  $\sqrt{\lambda}$ , the factor  $\sqrt{\lambda}$  in front of the action cancels. Demanding that the action is large in the typical time scale for the motion of the quark,  $\mathcal{T} = \tau_R$  leads to

$$\frac{1}{\sqrt{\gamma}} \frac{MT}{\sqrt{\lambda} T^2} \gg 1, \quad (4.23)$$

which coincides with the limit eq. (4.21). Above the critical velocity, the emission of excited string modes cannot be neglected in the description of the heavy quark dynamics.<sup>7</sup>

To conclude this discussion, we would like to address what is the typical size of the stretched horizon  $\hat{e}$ . The full Nambu-Goto action for transverse fluctuations

$$S_{NG} = -\frac{1}{2\pi\ell_s^2} \int d\hat{t} d\hat{z} \frac{L^2}{\hat{z}^2} \sqrt{1 + f(\hat{z})(\partial_{\hat{z}}\hat{X})^2 - \frac{1}{f(\hat{z})}(\partial_{\hat{t}}\hat{X})^2}. \quad (4.24)$$

<sup>6</sup>In *hat* coordinates the action scales as  $\hat{T} = \mathcal{T}/\sqrt{\gamma}$ .

<sup>7</sup>This provides a natural explanation to why the action of the string becomes imaginary for velocities larger than the critical one [27] since new channels, such as the emission of excited string states, appear.

The implicit assumption on the expansion performed in eq. (2.12) is that the fluctuations remain small. However, the solution to the small fluctuation problem diverges logarithmically close to the horizon. Thus, the last term inside the square root in eq. (4.24) grows faster than linearly sufficiently close to the horizon. Then, we must impose

$$\frac{1}{\hat{\epsilon}} \left\langle \left( \partial_t \hat{X}(\hat{t}, \hat{\epsilon}) \right)^2 \right\rangle \ll 1. \quad (4.25)$$

Since for the typical fluctuation  $\hat{\omega} \sim T$  and since the correlation function eq. (3.29) is inversely proportional to  $\eta$  we obtain the condition

$$\frac{\epsilon}{\ln^2 \sqrt{\gamma} \epsilon} \gg \frac{1}{\sqrt{\gamma} \lambda}, \quad (4.26)$$

where we have used  $\hat{\epsilon} = \sqrt{\gamma} \epsilon$ . Thus, as long as the coupling  $\lambda$  is large, the width of the stretched horizon can be small compared to temperature. As it also decreases with  $\gamma$  it is possible to separate the world-sheet horizon from the AdS horizon as long as the velocity is not too small  $v \gg 1/\lambda^{1/4}$ .

#### 4.4 Outlook

As we have seen, the string fluctuations induced by the world-sheet horizon lead to the stochastic motion of the quark at the boundary. Since only the fluctuations above this horizon are causally connected to the quark, this correspondence between string and quark fluctuations probes only the region  $\hat{z} < \pi T$ . However, as we have stressed, one of the interesting features of the trailing string is that the effective horizon is outside the AdS event horizon. As a consequence, the fluctuations below the world-sheet horizon are causally connected to the boundary and they are reflected in the boundary theory via the correlation of fields associated to the quark. The strength of these correlations is, in essence, proportional to the correlation function eq. (3.31) which we rewrite here for longitudinal fluctuations in the low frequency limit

$$G_{L\text{sym}}(\omega, z, z') = \frac{1}{\gamma^{5/2}} \left( \frac{2}{\eta T} \Delta X_{TS}(\sqrt{\gamma} z) \Delta X_{TS}(\sqrt{\gamma} z') + \frac{1}{8\eta T} \theta(z\sqrt{\gamma}\pi T - 1) \theta(z'\sqrt{\gamma}\pi T - 1) \right). \quad (4.27)$$

Expression eq. (4.27) shows a very strong suppression of the fluctuations with the quark velocity. However, the correlation of associated fields do not need to show such a large suppression. To illustrate this point, we discuss the computation of the stress tensor associated to the quark. The calculation proceeds by solving the back-reaction of the AdS metric to the string dual to the quark [29, 30]. The small deviations from the AdS metric are sourced by the (five dimensional) stress tensor of the string which is given by

$$\mathcal{T}^{MN}(t, \mathbf{k}) = -\frac{1}{2\pi\ell_s^2} \sqrt{\frac{h}{g}} \partial X^M \partial X^N e^{-ivk_x(t+\Delta X_{TS}(z))} e^{-ik_x X_L(t,z) - i\mathbf{k}_\perp \cdot \mathbf{X}_T(t,z)}, \quad (4.28)$$

where  $\mathcal{T}^{MN}$  is the 5-dimensional stress tensor (which is different from the stress tensor in the gauge theory).

The stress tensor eq. (4.28) depends on the fluctuations not only through the exponent but also through the dependence on the coordinates. Since we have restricted our analysis to the small perturbation regime, we expand eq. (4.28) to leading order in  $X_L, X_T$ . To this accuracy, the fluctuations do not change the average value of the stress tensor, which is given by the trailing string, but they lead to a non vanishing correlator. As an example, the  $\mathcal{T}^{00}$  correlator at small momentum  $\mathbf{k} \rightarrow 0$  is given by

$$\begin{aligned} \langle \Delta \mathcal{T}_{00} \Delta \mathcal{T}_{00} \rangle = \gamma^6 \left( \frac{\sqrt{\lambda}}{2\pi} \frac{z}{L^5} \right)^2 & \left[ A(z)A(z') \partial_z \partial_{z'} \langle \Delta X_L(t, z) \Delta X_L(t', z') \rangle + \right. \\ & B(z)B(z') \partial_t \partial_{t'} \langle \Delta X_L(t, z) \Delta X_L(t', z') \rangle + \\ & A(z)B(z') \partial_z \partial_{t'} \langle \Delta X_L(t, z) \Delta X_L(t', z') \rangle + \\ & \left. B(z)A(z') \partial_t \partial_{z'} \langle \Delta X_L(t, z) \Delta X_L(t', z') \rangle \right], \end{aligned} \quad (4.29)$$

where the functions  $A(z)$  and  $B(z)$  are given by

$$A(z) = -v \left( (\pi T z)^2 (1 - 2v^2 - (\pi T z)^4 / \gamma^2) \right), \quad (4.30)$$

$$B(z) = \frac{v(1 - (\pi T z)^4 / \gamma^2)}{f}, \quad (4.31)$$

and do not vanish in the large  $\gamma$  limit. The overall  $\gamma^6$  factor compensates the explicit  $\gamma^{5/2}$  suppression of the correlator eq. (4.27) leading to a strong apparent enhancement  $\gamma^{7/2}$  of the stress tensor correlation function. However, note that the  $\sqrt{\gamma}$  dependence on the functional form of the correlation function eq. (4.27) may change this scaling with the energy; its exact functional form will be addressed elsewhere.

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## A Green's function derivation of the fluctuation pattern

In this appendix we provide a different derivation of the two point correlation function eq. (3.29). We consider the string partition function with external sources

$$\mathcal{Z}_T[J_1, J_2] = \int \mathcal{D}_s \hat{X}_1 \mathcal{D}_s \hat{X}_2 e^{i S_T[\hat{X}_1] - i S_T[\hat{X}_2] + i \int dt_1 d\hat{z}_1 J_1(t_1, \hat{z}_1) \hat{X}_1(t_1, \hat{z}_1) - i \int dt_2 d\hat{z}_2 J_2(t_2, \hat{z}_2) \hat{X}_2(t_2, \hat{z}_2)}. \quad (A.1)$$

Since in eq. (3.29) the boundary value of the fluctuation  $\Delta X$  is zero, we impose this condition eq. (A.1).

The connected correlation function is given by

$$i\hat{G}_{ij}(t-t', \hat{z}, \hat{z}') = (-1)^{(i+j)} \frac{1}{i^2} \frac{\partial^2 \ln \mathcal{Z}_T}{\delta J_i(\hat{t}, \hat{z}) \delta J_j(\hat{t}', \hat{z}')} \Big|_{J_i=0}. \quad (\text{A.2})$$

Under the presence of the source, the classical solution to the string equations of motion is given by

$$\partial_{\hat{z}} \left( T_o(\hat{z}) \partial_{\hat{z}} \hat{X}_i(\hat{\omega}, \hat{z}) \right) + \frac{m\hat{\omega}^2}{\pi T \hat{z}^2 f(\hat{z})} \hat{X}_i(\hat{\omega}, \hat{z}) + J_i(\hat{\omega}, \hat{z}) = 0. \quad (\text{A.3})$$

The connections of the type 1, 2 solutions is performed as in the case without sources.

The correlation function eq. (A.2) is then given by the Green's function [22]

$$\hat{G}_{ij}(\hat{\omega}, \hat{z}, \hat{z}') = - \frac{g_{<}(\hat{z}_i) g_{>}(\hat{z}'_j) \theta(\hat{z}'_j, \hat{z}_i) + g_{>}(\hat{z}_i) g_{<}(\hat{z}'_j) \theta(\hat{z}_i, \hat{z}'_j)}{T_o(\hat{z}'_j) W(\hat{z}'_j)}. \quad (\text{A.4})$$

with  $g_{<}(\hat{z})$  ( $g_{>}(\hat{z}')$ ) the solution of the homogeneous normalizable in the right (left) classical string solution and  $W(z) = g'_{<}(z)g_{>}(z) - g_{<}(z)g'_{>}(z)$  is the Wronskian of these solutions. The function  $\theta(\hat{z}_1, \hat{z}'_1) = \theta(\hat{z}_1 - \hat{z}'_1)$ ,  $\theta(\hat{z}_2, \hat{z}'_2) = \theta(\hat{z}'_2 - \hat{z}_2)$ ,  $\theta(\hat{z}_1, \hat{z}'_2) = 1$  and  $\theta(\hat{z}_2, \hat{z}'_1) = 0$ .

Using the analytical continuation eq. (2.20), the solutions  $g_{>}(\hat{z})$ ,  $g_{<}(\hat{z}')$  are given by

$$g_{<}(\hat{z}) = \frac{1}{2i} \left( F_{\hat{\omega}}(\hat{z}) - e^{\theta(1-\hat{z})\pi\hat{\omega}/2} F_{\hat{\omega}}^*(\hat{z}) \right) \quad (\text{right quadrant}) \quad (\text{A.5})$$

$$g_{>}(\hat{z}) = \frac{1}{2i} \left( F_{\hat{\omega}}(\hat{z}) - e^{-\pi\hat{\omega}} e^{\theta(1-\hat{z})\pi\hat{\omega}/2} F_{\hat{\omega}}^*(\hat{z}) \right) \quad (\text{right quadrant}) \quad (\text{A.6})$$

$$g_{<}(\hat{z}) = \frac{1}{2i} \left( F_{\hat{\omega}}(\hat{z}) - e^{\pi\hat{\omega}} e^{-\theta(1-\hat{z})\pi\hat{\omega}/2} F_{\hat{\omega}}^*(\hat{z}) \right) \quad (\text{left quadrant}) \quad (\text{A.7})$$

$$g_{>}(\hat{z}) = \frac{1}{2i} \left( F_{\hat{\omega}}(\hat{z}) - e^{-\theta(1-\hat{z})\pi\hat{\omega}/2} F_{\hat{\omega}}^*(\hat{z}) \right) \quad (\text{left quadrant}), \quad (\text{A.8})$$

After introducing the  $ra$  basis, the symmetrized correlator is given by

$$\hat{G}_{\text{sym}} = \frac{i}{4} \left( \hat{G}_{11} + \hat{G}_{22} + \hat{G}_{12} + \hat{G}_{21} \right). \quad (\text{A.9})$$

After a tedious but straight-forward computation, the symmetrized correlator computed in this way coincides with eq. (3.29). The extension to longitudinal fluctuations is also straight forward.

## B Momentum flux at the world sheet horizon

The canonical momentum densities associated to the string are

$$\pi_{\mu}^{\tau} = \frac{\delta \mathcal{L}_{NG}}{\delta \partial_{\tau} X^{\mu}} = - \frac{1}{2\pi\alpha'} g_{\mu\nu} \frac{h_{\tau\sigma} \partial_{\sigma} X^{\nu} - h_{\sigma\sigma} \partial_{\tau} X^{\nu}}{\sqrt{-h}}, \quad (\text{B.1})$$

$$\pi_{\mu}^{\sigma} = \frac{\delta \mathcal{L}_{NG}}{\delta \partial_{\sigma} X^{\mu}} = - \frac{1}{2\pi\alpha'} g_{\mu\nu} \frac{h_{\tau\sigma} \partial_{\tau} X^{\nu} - h_{\tau\tau} \partial_{\sigma} X^{\nu}}{\sqrt{-h}}, \quad (\text{B.2})$$

where  $\mathcal{L}_{NG}$  is the Nambu-Goto lagrangian,  $h_{ab}$  is the induced metric on the world sheet and  $g$  the AdS-metric. The string equations of motion are the continuity equations

$$\partial_{\tau} \pi_{\mu}^{\tau} + \partial_{\sigma} \pi_{\mu}^{\sigma} = 0. \quad (\text{B.3})$$

The total energy and momentum of the string are

$$E = - \int d\sigma \pi_t^0, \quad p_i = \int d\sigma \pi_i^0. \quad (\text{B.4})$$

Using eq. (B.3), the change in momentum in the string between the interval  $(\sigma_a, \sigma_b)$  is

$$\frac{dp_i}{dt} = \pi_i^\sigma(\sigma_a) - \pi_i^\sigma(\sigma_b). \quad (\text{B.5})$$

The left hand side of this equation yields the forces acting on the string. The right hand side gives the tension forces at the string endpoint  $\mathcal{F}(\sigma_a) = -\pi_i^\sigma(\sigma_a)$ ,  $\mathcal{F}(\sigma_b) = \pi_i^\sigma(\sigma_b)$ .

For the trailing string eq. (2.5) we choose as  $(\tau, \sigma)$  the AdS coordinates  $(t, z)$ . To leading order in the fluctuations, the induced metric is

$$h_{tt} = -\frac{L^2}{\gamma^2 z^2} \left( 1 - (\pi T z \sqrt{\gamma})^4 \right), \quad (\text{B.6})$$

$$h_{zz} = \frac{L^2}{z^2} \frac{1}{f(z)^2} \left( 1 - \left( \frac{\pi T z}{\sqrt{\gamma}} \right)^4 \right), \quad (\text{B.7})$$

$$h_{tz} = -\frac{L^2 v^2 (\pi T)^2}{f(z)}. \quad (\text{B.8})$$

The transverse force on the string end point at the world sheet horizon  $z_\pm = \hat{z}_\pm / \sqrt{\gamma}$  is

$$\pi_T^z(z_\pm) = - \left( T_o(\hat{z}_\pm) \partial_z X_T(t, z_\pm) - \eta \gamma \dot{X}_T(t, z_\pm) \right). \quad (\text{B.9})$$

Using eq. (4.1) we find

$$\pi_T^z(z_\pm) = - \left( \xi(t) \pm \eta \gamma \dot{\Delta}(t) - \eta \gamma \dot{X}_T(t, z_\pm) \right). \quad (\text{B.10})$$

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