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Abstract: We present type IIB supergravity solutions which are expected to be dual to certain Lifshitz-like fixed points with anisotropic scale invariance. They are expected to describe a class of D3-D7 systems and their finite temperature generalizations are straightforward. We show that there exist solutions that interpolate between these anisotropic solutions in the IR and the standard AdS_5 solutions in the UV. This predicts anisotropic RG flows from familiar isotropic fixed points to anisotropic ones. In our case, these RG flows are triggered by a non-zero theta-angle in Yang-Mills theories that linearly depends on one of the spatial coordinates. We study the perturbations around these backgrounds and discuss the possibility of instability. We also holographically compute their thermal entropies, viscosities, and entanglement entropies.

Keywords: Brane Dynamics in Gauge Theories, D-branes, AdS-CFT Correspondence

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1 Introduction

The AdS/CFT correspondence has provided a very powerful and successful paradigm to analyze relativistic and isotropic fixed points in various quantum field theories [1–4]. On the field theory side, they are described by $(d+1)$-dimensional conformal field theories and are invariant under the homogeneous scaling transformation $(t, x_1, x_2, \ldots, x_d) \to (\lambda t, \lambda x_1, \lambda x_2, \ldots, \lambda x_d)$. On the gravity side, they are equivalently described by gravity on a $(d+2)$-dimensional AdS space

$$ds^2 = r^2\left(-dt^2 + \sum_{i=1}^{d} dx_i^2\right) + \frac{dr^2}{r^2}. \tag{1.1}$$

It is natural to try to extend the AdS/CFT correspondence to a holography for the following anisotropic spacetime

$$ds^2 = r^{2z}\left(-dt^2 + \sum_{i=1}^{p} dx_i^2\right) + r^2 \sum_{j=p+1}^{d} dy_j^2 + \frac{dr^2}{r^2}, \tag{1.2}$$

where $0 \le p \le d-1$, and the parameter $z(\neq 1)$ measures the degree of Lorentz symmetry violation and anisotropy. Since the metric (1.2) is invariant under the scaling $(t, x_i, y_j, r) \to (\lambda^z t, \lambda^z x_i, \lambda y_j, \frac{r}{\lambda})$, we expect that on the field theory side it is dual to a fixed point which is invariant under the scaling transformation

$$(t, x_i, y_j) \to (\lambda^z t, \lambda^z x_i, \lambda y_j). \tag{1.3}$$

Notice also that by a coordinate redefinition $r^z = \rho$, we can rewrite the metric (1.2) into another illuminating form (after rescaling (t, x_i, y_i) accordingly)

$$ds^2 = \rho^2\left(-dt^2 + \sum_{i=1}^{p} dx_i^2\right) + \rho^{\frac{2}{z}} \sum_{j=p+1}^{d} dy_j^2 + \frac{d\rho^2}{\rho^2}. \tag{1.4}$$

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Thus we can equally argue that the dual background is invariant under an anisotropic scaling transformation $(t, x_i, y_j, \rho) \to (\lambda t, \lambda x_i, \lambda^{\frac{1}{z}} y_j, \frac{\rho}{\lambda})$, where the y_j directions are responsible for the Lorentz symmetry violation and anisotropy.

In general, fixed points with the anisotropic scaling property (1.3) are called Lifshitz(-like) fixed points[1] (see e.g. the textbook [7] for a brief review). This generalization of AdS/CFT correspondence to Lifshitz-like fixed points (1.2) was first proposed and analyzed by Kachru, Liu and Mulligan [8] in the particular case of $p = 0$. The simplest case with $p = 0$ represents non-relativistic fixed points with dynamical critical exponent z, which appear in many examples of quantum criticality in condensed matter physics (see the references in [8]). See [9–14] for further progress on holographic aspects of this topic.[2]

The other cases where $1 \leq p \leq d-1$ are not only generalizations of $p = 0$ case but can also be interpreted as space-like anisotropic fixed points (see also [13]) as is clear from the expression (1.4). Lifshitz fixed points with space-like anisotropic scale invariance appear in realistic magnets such as MnP and the axial next-nearest-neighbor Ising model [5]. They are also realized in models of directed percolation [7].

To understand holographic duals of such gravity backgrounds, it is the best to embed them into string theory, where microscopic interpretations are often possible by using D-branes. However, so far there has been no known embedding of (1.2) in string theory. Motivated by this circumstance, in this paper we will construct such anisotropically scaling solutions in type IIB supergravity. We mainly focus on the backgrounds generated by intersections of D3 and D7 branes. They correspond to the choice $p = 2$ and $d = 3$ and are expected to be non-supersymmetric. This restriction is imposed not only for the tractability of the supergravity analysis, but is also due to another motivation, namely to construct back-reacted D3-D7 solutions that are dual to the pure Chern-Simons gauge theory in the second setup of [22]. In the end, we find a class of solutions with the exponent $z = 3/2$. We also extend them to black brane solutions dual to finite temperature theories.

Furthermore, we show that there exist solutions which interpolate between our anisotropic solutions and the familiar $AdS_5 \times X_5$ solutions. We also construct their numerical solutions. The holography suggests that our Lifshitz-like fixed points can be obtained from various four-dimensional CFTs including $\mathcal{N} = 4$ super Yang-Mills via RG flows.[3] These flows are triggered by the relevant and anisotropic perturbation which gives a non-zero θ parameter (i.e. the coefficient in front of the topological Yang-Mills coupling $F \wedge F$) that depends linearly on one of the three spatial coordinates i.e. $\theta \propto x_3$. Notice that when x_3 is compactified, the perturbation induces the Chern-Simons coupling $\int A \wedge F + \frac{2}{3} A^3$ as in [22], which becomes relevant in the IR.

[1] The most standard example is the free scalar field theory with $z = 2$, known as the Lifshitz model. Anisotropic fixed points in interacting field theories in general can have $z \neq 2$ [5, 6]. Even though the original Lifshitz fixed points were found in anisotropic magnets where three critical lines meet, in this paper we simply define Lifshitz-like fixed points as any fixed points which have anisotropic scale invariance.

[2] Gravity duals of another types of fixed points with non-relativistic scaling symmetry [15] have also been studied especially for systems with non-relativistic conformal invariance [16–21].

[3] Here the Lifshitz-like fixed points are realized in the IR limit. It is also intriguing to consider opposite RG flows, where IR fixed points become relativistic and isotropic $z = 1$ as in [8, 10].

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This paper is organized as follows. In section 2, we present solutions dual to a class of Lifshitz-like fixed points based on D3-D7 systems with their black brane generalizations. In section 3, we show there exist interpolating solutions which approach the Lifshitz-like scaling solutions in the IR and the standard AdS_5 solutions in the UV. In section 4, we holographically calculate the shear and bulk viscosity. In section 5, we compute their holographic entanglement entropies and discuss how the scaling behaviors of the entanglement entropies depend on the direction along which the sub-systems are delineated. In section 6, we study the perturbations around these backgrounds and discuss the instabilities. In section 7, we present anisotropic solutions based on D4-D6 systems. In section 8, we summarize our conclusions.

2 Holographic duals of Lifshitz-like fixed points in type IIB string

In this section we will present the main result of this paper. We will construct new solutions in type IIB supergravity with RR 5-form and 1-form fluxes whose Einstein metrics enjoy a nice scaling property. Since their scaling is anisotropic as opposed to the well-known AdS_5 background, we argue that they are dual to Lifshitz-like fixed points described by certain D3-D7 systems.

2.1 Type IIB supergravity

The IIB supergravity action $S_{\rm IIB} = \frac{1}{2\kappa_{10}^2}\int \mathcal{L}$ in the string frame is defined by the Lagrangian (we follow the convention in [23])

$$\begin{aligned}\mathcal{L} = \sqrt{-g}e^{-2\phi}(R + 4\partial_M\phi\partial^M\phi) - \frac{e^{-2\phi}}{2}H_3\wedge *H_3 - \frac{1}{2}F_1\wedge *F_1 - \frac{1}{2}\tilde{F}_3\wedge *\tilde{F}_3 \\ -\frac{1}{4}\tilde{F}_5\wedge *\tilde{F}_5 - \frac{1}{2}C_4\wedge H_3\wedge F_3,\end{aligned} \tag{2.1}$$

where $F_1 = d\chi$, $\tilde{F}_3 \equiv F_3 - \chi H_3$, and $\tilde{F}_5 \equiv F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$. We set $\alpha' = 1$ therefore $2\kappa_{10}^2 = (2\pi)^7$.

The fluxes obey the equations of motion:

$$\begin{aligned}d*F_1 = *\tilde{F}_3\wedge H_3, \qquad d*\tilde{F}_3 = -H_3\wedge\tilde{F}_5, \qquad d*\tilde{F}_5 = H_3\wedge\tilde{F}_3, \\ d(e^{-2\phi}*H_3) = F_1\wedge *\tilde{F}_3 + \tilde{F}_3\wedge\tilde{F}_5.\end{aligned} \tag{2.2}$$

plus the Bianchi identities:

$$dH_3 = 0, \qquad dF_1 = 0, \qquad d\tilde{F}_3 = H_3\wedge F_1, \qquad d\tilde{F}_5 = H_3\wedge\tilde{F}_3, \tag{2.3}$$

and the self-dual constraint for $\tilde{F}_5$:

$$*\tilde{F}_5 = \tilde{F}_5. \tag{2.4}$$

The dilaton equation of motion is

$$R + 4\nabla_M\nabla^M\phi - \frac{1}{12}H_{MNP}H^{MNP} - 4\nabla_M\phi\nabla^M\phi = 0. \tag{2.5}$$

And the Einstein equation becomes

$$R_{MN} + 2\nabla_M\nabla_N\phi + \frac{1}{4}g_{MN}A = \frac{1}{4}H_{MAB}H_N{}^{AB} + \frac{1}{2}e^{2\phi}F_MF_N + \frac{1}{4}e^{2\phi}\tilde{F}_{MAB}\tilde{F}_N{}^{AB} + \frac{1}{4\cdot 4!}e^{2\phi}\tilde{F}_{MABCD}\tilde{F}_N{}^{ABCD}, \tag{2.6}$$

where

$$A \equiv e^{2\phi}\partial_M\chi\partial^M\chi + \frac{1}{3!}e^{2\phi}\tilde{F}_{ABC}\tilde{F}^{ABC} + \frac{1}{2\cdot 5!}e^{2\phi}\tilde{F}_{ABCDE}\tilde{F}^{ABCDE}. \tag{2.7}$$

2.2 D3-D7 ansatz

We start with the (string frame) metric ansatz that preserves the three-dimensional Lorentz symmetry SO(2, 1):

$$ds^2 = e^{2b(r)}(-dt^2 + dx^2 + dy^2) + e^{2h(r)+2a(r)}dw^2 + e^{2c(r)-2a(r)}dr^2 + e^{2c(r)}r^2ds^2_{X_5}. \tag{2.8}$$

We require the five-dimensional compact manifold X_5 to be a unit-radius Einstein manifold with the same Ricci curvature as the unit-radius S^5, i.e. it satisfies

$$R_{\alpha\beta} = 4g_{\alpha\beta}. \tag{2.9}$$

The simplest example of X_5 is obviously the unit radius sphere S^5. The self-dual 5-form and 1-from fluxes are given in terms of constants α and β by

$$F_5 = \alpha\left(\Omega_{X_5} + *\Omega_{X_5}\right), \tag{2.10}$$

$$F_1 = d\chi = \beta dw, \tag{2.11}$$

where χ is the axion field (i.e. the RR 0-form potential) and Ω_{X_5} is the volume form of X_5. The fluxes (2.10) and (2.11) satisfy the equations of motion (2.2). We also assume that the dilaton ϕ only depends on r and both 3-form fluxes (H_3 and F_3) vanish. Our ansatz, which looks rather different from [8], is motivated in part by an attempt to construct back-reacted solutions of the D3-D7 intersecting systems introduced in [22], as will be explained in detail later.

Under this ansatz, the equations of motion for the metric and the dilaton ((2.6) and (2.5)) are summarized as follows:

$$\begin{aligned}
[b'e^{2z}]' &= \frac{\beta^2}{4}e^{-2a-h+3b+6c}r^5 + \frac{\alpha^2}{4}e^{-4c+3b+h}r^{-5},\\
[(a+h)'e^{2z}]' &= -\frac{\beta^2}{4}e^{-2a-h+3b+6c}r^5 + \frac{\alpha^2}{4}e^{-4c+3b+h}r^{-5},\\
[(c+\log r)'e^{2z}]' &= \frac{4}{r^2}e^{2z-2a} + \frac{\beta^2}{4}e^{-2a-h+3b+6c}r^5 - \frac{\alpha^2}{4}e^{-4c+3b+h}r^{-5},\\
[(2z+c-a)'e^{2z}]' &= \frac{20}{r^2}e^{2z-2a} - \frac{\beta^2}{4}e^{-2a-h+3b+6c}r^5 - \frac{\alpha^2}{4}e^{-4c+3b+h}r^{-5},\\
2z'' + c'' - a'' + 2(z')^2 + \frac{1}{2}(h')^2 + a'h' + 2(c')^2 &\\
+\left(\frac{5}{r} + a'\right)c' + \frac{3}{2}(b')^2 - \frac{10e^{-2a}}{r^2} + \frac{5}{2r^2} &= 0.
\end{aligned}$$

Here we have defined

$$z \equiv \frac{3}{2}b + \frac{5}{2}\log r + a + 2c + \frac{1}{2}h - \phi. \tag{2.12}$$

The derivative of a function f with respect to r is denoted by $f'(r)$. An observation, which will be useful in the next section, is that a linear combination of the first four equations gives

$$[(2b - 2a - \phi - 2h)'e^{2z}]' = 0. \tag{2.13}$$

2.3 D3-D7 scaling solutions: holographic duals of Lifshitz-like fixed points

Since we are looking for scaling solutions (namely solutions invariant under scale transformations), we require all metric components in (2.8) to be power functions of r. In other words, the functions a, b, c, z and ϕ are all logarithmic functions of r. For such a scaling ansatz, the equations of motion (2.5) and (2.6) reduce to algebraic equations and the solution is easily found to be:

$$\begin{aligned}
&a(r) = \frac{1}{2}\log\frac{12}{11} - \log\xi_s, \qquad b(r) = \frac{7\xi_s}{6}\log r + b_0, \qquad c(r) = \left(-1 + \frac{\xi_s}{6}\right)\log r + c_0,\\
&h(r) = \frac{5\xi_s}{6}\log r + \log\xi_s + h_0, \qquad \phi(r) = \frac{2\xi_s}{3}\log r + \phi_0,\\
&\alpha = 4e^{4c_0-\phi_0}, \qquad \beta = 4\sqrt{\frac{2}{11}}e^{h_0-c_0-\phi_0},
\end{aligned} \tag{2.14}$$

where b_0, c_0, h_0, ϕ_0 and ξ_s are arbitrary constants. ξ_s corresponds to the degrees of freedom of the reparameterization of r, while b_0 and h_0 correspond to the rescaling of the (t, x, y, w) directions.

Without loss of generality, we choose

$$\xi_s = 1, \qquad b_0 = c_0 + \frac{1}{2}\log\frac{11}{12}, \qquad h_0 = c_0 + \log\frac{11}{12}, \tag{2.15}$$

and the solution in the string frame reads:

$$ds_s^2 = \tilde{R}_s^2\left[r^{\frac{7}{3}}(-dt^2 + dx^2 + dy^2) + r^{\frac{5}{3}}dw^2 + \frac{dr^2}{r^{\frac{5}{3}}}\right] + R_s^2 r^{\frac{1}{3}}ds_{X_5}^2, \tag{2.16}$$

where $R_s^2 = \frac{12}{11}\tilde{R}_s^2 = e^{2c_0}$. And the dilaton scales with r as

$$e^{\phi} = r^{\frac{2}{3}}e^{\phi_0}, \tag{2.17}$$

where $e^{\phi_0} = \frac{\sqrt{22}}{3\beta}$.

Since the dilaton depends on r non-trivially, it is helpful to discuss the metric in the Einstein frame. Indeed, later we will see explicitly that a large class of scalar fluctuations around this solution can be described by Klein-Gordon equations on curved spacetimes based on the Einstein frame metric instead of on the string frame metric. The above solution in the Einstein frame is

$$ds_E^2 = \tilde{R}^2\left[r^2(-dt^2 + dx^2 + dy^2) + r^{\frac{4}{3}}dw^2 + \frac{dr^2}{r^2}\right] + R^2 ds_{X_5}^2, \tag{2.18}$$

where the radii

$$R^2 = \frac{12}{11}\tilde{R}^2 = e^{-\frac{\phi_0}{2}+2c_0} = \frac{\sqrt{\alpha}}{2}. \tag{2.19}$$

The metric (2.18) is invariant under the scaling

$$(t, x, y, w, r) \to \left(\lambda t, \lambda x, \lambda y, \lambda^{\frac{2}{3}} w, \frac{r}{\lambda}\right), \tag{2.20}$$

and therefore is expected to be holographically dual to Lifshitz-like fixed points with space-like anisotropic scale invariance. Note that the metric (2.18) is equivalent to (1.4) with $z = 3/2$, $p = 2$ and $d = 3$.

By redefining the radius coordinate $\rho \equiv r^{\frac{2}{3}}$ and rescaling (t, x, y, w) accordingly, we can rewrite the metric (2.18) into another illuminating form

$$ds_E^2 = \tilde{R}^2 \left[\rho^3(-dt^2 + dx^2 + dy^2) + \rho^2 dw^2 + \frac{d\rho^2}{\rho^2}\right] + R^2 ds_{X_5}^2. \tag{2.21}$$

This can be regarded as gravity duals of Lifshitz-like fixed points with $z = 3/2$. It coincides with the metric (1.2) with $p = 2$ and $d = 3$.

2.4 Holographic interpretation in terms of D3-D7 system

Since our solution (2.18) is sourced by the RR 5-form (2.10) and 1-form flux (2.11), we expect it to be interpreted as a D3-D7 system in string theory. When we compactify the w direction such that $w \sim w + L$ and place N D3-brane along the (t, x, y, w) directions and k D7-branes along the (t, x, y, X_5) directions:

$\mathcal{M}_4 \times S^1 \times X_5$	t	x_1	y	r	w	s_1	s_2	s_3	s_4	s_5
N D3	×	×	×		×					
k D7	×	×	×			×	×	×	×	×

these N D3 and k D7 branes can source the desired RR 5-form and 1-form fluxes with

$$\alpha = \frac{(2\pi)^4 N}{\mathrm{Vol}(X_5)}, \qquad \beta = \frac{k}{L}. \tag{2.22}$$

This brane configuration is the same as the one constructed to model the fractional quantum Hall effect in [22].

The number of the D3-branes determines the radii R and $\tilde{R}$ in the scaling solution (2.18):

$$R^2 = \frac{12}{11}\tilde{R}^2 = 2\sqrt{\frac{\pi^4}{\mathrm{Vol}(X_5)}N}. \tag{2.23}$$

For $X_5 = S^5$ (whose volume is π^3), $R^2 = \frac{12}{11}\tilde{R}^2 = 2\sqrt{\pi N}$. The number of the D7-branes gives the string coupling at $r = 1$:

$$e^{\phi_0} = \frac{\sqrt{22}}{3}\frac{L}{k}. \tag{2.24}$$

Now we say a few words about the field theory living on this D3-D7 system. We take $X_5 = S^5$ to simplify the arguments. If we start with N D3-branes, whose low energy theory

is the four-dimensional $\mathcal{N}=4$, SU(N) super Yang-Mills theory, then the additional k D7-branes will source a non-trivial axion field $\chi = \frac{k}{L}w$, which in turn induces a w-dependent θ term (i.e. the topological term) of the Yang-Mills theory

$$\frac{1}{4\pi}\int \chi(w)\mathrm{Tr}F\wedge F. \tag{2.25}$$

For finite β and k, w-direction is compactified. After integrating over w, the 4D topological term (2.25) becomes a 3D Chern-Simons term at level k:

$$\frac{k}{4\pi}\int_{R^{1,2}} \mathrm{Tr}\left[A\wedge F+\frac{2}{3}A^3\right]. \tag{2.26}$$

Now we have two choices of the boundary condition for the w-circle: periodic or anti-periodic. If we impose the anti-periodic one, all fermions will become massive. This breaks all supersymmetries and gives masses to scalar fields through quantum corrections. In the IR limit, only a pure Yang-Mills term is left of the original 4D $\mathcal{N}=4$ super Yang-Mills part of the action. Since in the IR limit, the Chern-Simons term dominates this Yang-Mills term, the final three-dimensional theory is a pure Chern-Simons theory. In [22], this D3-D7 system was constructed to holographically model the FQHE precisely because it flows to the pure Chern-Simons gauge theory in the IR. In this model, the AdS/CFT correspondence in the IR limit manifests itself as the level-rank duality of the pure Chern-Simons gauge theory.

On the other hand, if we take $k\to\infty$ (and simultaneously $L\to\infty$) while keeping β finite, the w-direction is non-compact and the field theory is four-dimensional. Even though the interaction (2.25) looks non-local at the first sight, its contribution to the equations of motion is actually local. This remarkable property occurs only when $\chi(w)$ is a linear function of w (as is the case here).

One might still doubt any relations of our new background (2.18) to the $\mathcal{N}=4$ super Yang-Mills theory as it is not asymptotically AdS_5. One might also worry that the dilaton (2.17) blows up near the boundary $r\to\infty$. However, as we will show in the next section, we can in fact construct solutions which interpolate between the AdS_5 and our scaling solution (2.18). This interpolating solution can be considered as the dual of the RG flow between the two systems. Notice that this caps off the strongly coupled region of the scaling solution. We will also present anisotropic solutions for analogous D4-D6 systems in section 7.

2.5 Black brane solutions and entropy

One more interesting fact about our scaling solutions is that we can straightforwardly generalize them to black brane solutions which have regular event horizons. The metric in the Einstein frame is

$$ds_E^2=\tilde{R}^2\left[r^2(-F(r)dt^2+dx^2+dy^2)+r^{\frac{4}{3}}dw^2+\frac{dr^2}{r^2F(r)}\right]+R^2ds_{X_5}^2, \tag{2.27}$$

where

$$F(r)=1-\frac{\mu}{r^{\frac{11}{3}}}. \tag{2.28}$$

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The constant μ represents the mass parameter of the black brane. The dilaton and RR fields remain the same.

Requiring the smoothness of the Euclidean geometry of (2.27) gives the Hawking temperature

$$T_H = \frac{11}{12\pi}\mu^{\frac{3}{11}}. \tag{2.29}$$

The Bekenstein-Hawing entropy is then

$$S_{BH} = \gamma \cdot \left(\frac{\pi^3}{\mathrm{Vol}(X_5)}\right) \cdot N^2 \cdot T_H^{\frac{8}{3}} \cdot V_2 \cdot L, \tag{2.30}$$

where γ is a numerical factor

$$\gamma = 2^{\frac{4}{3}} \cdot 3^{\frac{7}{6}} \cdot 11^{-\frac{7}{6}} \cdot \pi^{\frac{5}{3}} \simeq 3.729 \tag{2.31}$$

and V_2 represents the area in the (x, y) direction. The entropy (2.30) is proportional to N^2 and thus is consistent with the planar limit of a certain gauge theory.

Notice that the power 8/3 of temperature in (2.30) can also be obtained from a simple dimension counting. From the metric (2.27), the coordinate w has the fractional dimension 2/3, while each of (t, x, y) carries the unit dimension.

3 RG flow in $\mathbf{AdS_5/CFT_4}$ and scaling solution

In the previous section, we find a new scaling solution of the D3-D7 system in type IIB supergravity. To clarify its physical interpretation, we will show below that we can construct interpolating solutions that approach the $AdS_5 \times X_5$ solutions in the $r \to \infty$ limit (i.e. UV limit of the holographic duals) and the scaling solutions in the opposite limit $r \to 0$. Then via the AdS/CFT correspondence, we can argue that the system dual to our scaling solution is connected to the one dual to the AdS_5 through the RG flow.

3.1 Further reduction of equations of motion

To find the interpolating solution, we start with the general form (2.8). To simplify the problem we impose some extra constraints which are consistent with both the AdS_5 and the scaling solutions.

First, we can make the function $a(r)$ vanish by a reparametrization of r:

$$a(r) = 0. \tag{3.1}$$

Secondly, recall that we showed $[(2b - 2a - \phi - 2h)'e^{2z}]' = 0$ for generic solutions. In fact, both the AdS_5 and the scaling solution satisfy a much stronger condition

$$(2b - 2a - \phi - 2h)' = 0. \tag{3.2}$$

Since we are looking for a solution that interpolates between the AdS_5 and the scaling solution, it is reasonable to impose (3.2) as a simplifying ansatz, namely

$$h(r) = b(r) - a(r) - \frac{1}{2}\phi(r) + \tilde{h}_0, \tag{3.3}$$

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where $\tilde{h}_0$ is a constant. Similarly, since both the AdS_5 and the scaling solution have the nice property that in the Einstein frame the radius of S^5 is a constant, we will also impose this condition on our interpolating solution, namely we require

$$\phi(r) = 4c(r) + 4\log r + \tilde{\phi}_0, \tag{3.4}$$

where $\tilde{\phi}_0$ is a constant. These relations (3.1), (3.3) and (3.4) are the constraints mentioned and are assumed throughout this section.

Under this ansatz, the Einstein frame metric becomes

$$ds_E^2 = \frac{e^{2b(r)-2c(r)-\frac{\tilde{\phi}_0}{2}}}{r^2}(-dt^2+dx^2+dy^2) + e^{-\frac{3}{2}\tilde{\phi}_0+2\tilde{h}_0}\frac{e^{2b(r)-6c(r)}}{r^6}dw^2 + e^{-\frac{\tilde{\phi}_0}{2}}\frac{dr^2}{r^2} + e^{-\frac{\tilde{\phi}_0}{2}}ds_{X_5}^2, \tag{3.5}$$

and the equations of motion are greatly simplified:

$$\alpha = 4e^{-\tilde{\phi}_0}, \tag{3.6}$$

$$b'' = \frac{2}{r^2} + \frac{23}{r}b' - 10b'^2 - \frac{16c'}{r} + 24b'c' - 8c'^2, \tag{3.7}$$

$$c'' = \frac{4}{r^2} + \frac{14}{r}b' - 6b'^2 - \frac{5c'}{r} + 14b'c' - 2c'^2, \tag{3.8}$$

$$\frac{1}{4}e^{-2\tilde{h}_0+3\tilde{\phi}_0-2b+14c}r^{14}\beta^2 = 6 - 8 - 6r^2b'^2 - 16rc' - 8r^2c'^2 + 18rb'(1+rc'). \tag{3.9}$$

We can confirm that the derivative of the r.h.s of (3.9) is vanishing if (3.7), (3.8) and (3.9) are satisfied. This means that the constraint (3.9) is consistent with (3.7) and (3.8).

3.2 Interpolating solution between AdS_5 and D3-D7 scaling solution

Now the problem amounts to solving the system of two coupled first-ordered nonlinear ODEs (3.7) and (3.8) under the constraint (3.9). First, notice that (3.7) and (3.8) involve only the derivatives of b and c, thus once we find a solution to them, we can simply choose the integration constants of $b(r)$ and $c(r)$ such that they satisfy the constraint (3.9) — as long as it allows the r.h.s of (3.9) to be positive.[4] Therefore essentially we only need to solve (3.7) and (3.8).

Next we redefine the radial coordinate r and the derivatives of the functions $b(r)$ and $c(r)$ as follows

$$s \equiv \log r, \quad B(s) \equiv \frac{\partial b(r)}{\partial \log r}, \quad C(s) \equiv \frac{\partial c(r)}{\partial \log r}. \tag{3.10}$$

Then the equations (3.7) and (3.8) are simply a pair of first-ordered non-linear ODEs:

$$\dot{B} = 2 + 24B - 16C - 10B^2 + 24BC - 8C^2, \tag{3.11}$$

$$\dot{C} = 4 + 14B - 4C - 6B^2 + 14BC - 2C^2. \tag{3.12}$$

where $\dot{B} \equiv \frac{dB}{ds}$. A physical solution also needs to satisfy

$$(9B - 8C - 8)^2 - (33B^2 + 48) < 0, \tag{3.13}$$

[4]However as we will show later this requirement is actually automatically satisfied by the interpolating solution we are looking for; thus it does not impose any additional constraint.

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due to the constraint (3.9).

The dynamical system (3.11) and (3.12) has four fixed points $(B,C)^*$ which can be classified into two pairs:

$$(B,C)^* = (\pm 1, -1) \qquad \text{and} \qquad (B,C)^* = \left(\pm\frac{7}{\sqrt{33}}, \pm\frac{1}{\sqrt{33}} - 1\right). \tag{3.14}$$

Inside each pair, the two fixed points are related by a coordinate redefinition $r \to \frac{1}{r}$ thus are equivalent. The fixed point

$$(B,C)^*_{AdS_5} = (1,-1) \tag{3.15}$$

corresponds the standard $AdS_5 \times X_5$ solution.[5]

Our scaling D3-D7 solution (at zero temperature) corresponds to the fixed point

$$(B,C)^*_{\text{scaling}} = \left(\frac{7}{\sqrt{33}}, \frac{1}{\sqrt{33}} - 1\right) \simeq (1.2185, -0.8259). \tag{3.16}$$

One can easily see that the metric (3.5) with (3.16) is equivalent to (2.18) via the redefinition of radial coordinate $r \to r^{\sqrt{33}/6}$. Since the two fixed points with "$-$" sign are equivalent to the two with "$+$" sign and are disconnected from them, we will not consider those any further.

Now let's study the behavior of this dynamical system (3.11) and (3.12). Near the AdS_5 fixed point (3.15), the eigensystem of the linear perturbations (defined by $B(s) = B^*_{AdS_5} + \epsilon_b(s)$ and $C(s) = C^*_{AdS_5} + \epsilon_c(s)$ for $\epsilon_b, \epsilon_c \ll 1$) is

$$\dot{\epsilon_b} = -20\epsilon_b + 24\epsilon_c, \qquad \dot{\epsilon_c} = -12\epsilon_b + 14\epsilon_c, \tag{3.17}$$

and both eigenvalues are negative: $\lambda_1 = -4$ and $\lambda_2 = -2$; therefore the AdS_5 solution is a stable fixed point as the system flows to the UV (i.e. $r \to \infty$).

On the other hand, the eigensystem of the linear perturbation near the D3-D7 scaling fixed point (defined by $B(s) = B^*_{\text{scaling}} + \eta_b(s)$ and $C(s) = C^*_{\text{scaling}} + \eta_c(s)$ for $\eta_b, \eta_c \ll 1$) is

$$\dot{\eta_b} = -\frac{116}{\sqrt{33}}\eta_b + \frac{152}{\sqrt{33}}\eta_c, \qquad \dot{\eta_c} = -\frac{70}{\sqrt{33}}\eta_b + \frac{94}{\sqrt{33}}\eta_c. \tag{3.18}$$

In contrast to the stable AdS_5 fixed point, this scaling solution fixed point has one negative eigenvalue ($\lambda_1 = -\frac{\sqrt{33}+\sqrt{105}}{3}$) and one positive one ($\lambda_2 = \frac{\sqrt{105}-\sqrt{33}}{3}$). Therefore the fixed point corresponding to the scaling solution is unstable. Near the neighborhood of this scaling fixed point, there exist two special trajectories: one corresponding to the negative eigenvalue λ_1 and one to the positive λ_2. The fixed point behaves like a UV (resp. IR) fixed point when approached along the trajectory corresponding to the negative (resp. positive) eigenvalue. When the fixed point is approached along a generic direction, the trajectory only passes near it and then turns to flow to infinity — only one fine-tuned trajectory can reach the fixed point.

[5] The fixed point $(B,C)^* = (-1,-1)$ can be considered as its conjugate since they are connected under a coordinate redefinition $r \to \frac{1}{r}$.

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Figure 1 shows the global behavior of this dynamical system (3.11) and (3.12). Figure 1(a) is generated by numerical computations and the salient features are schematically highlighted in the hand-rendered figure 1(b). The arrows point in the direction from the UV ($s=\infty$) to the IR ($s=-\infty$). It is easy to see from the direction fields that the AdS_5 fixed point (the green dot at $(1,-1)$) is a stable UV fixed point while the scaling fixed point (the blue dot at $(\frac{7}{\sqrt{33}},\frac{1}{\sqrt{33}}-1)$) is unstable. Recall that a physical solution needs to satisfy (3.13). The allowed region in the flow diagram are between two hyperbolic lines given by $9B-8C-8=\pm\sqrt{33B^2+48}$. The black curve in figure 1(a) is the one with the "+" sign; and the other one with the "−" sign is its mirror image in the upper-left corner but is out of the range of figure 1(a). It is clear that both the AdS_5 fixed point and the scaling solution fixed point are in the allowed region. And since the AdS_5 is a stable UV fixed point and there is no critical surface separating it from the scaling solution fixed point, there exists a trajectory emanating from the AdS_5 fixed point and flowing to the scaling solution fixed point. Namely there exists a solution that interpolates between the $AdS_5\times X_5$ solution in the UV ($r\to\infty$) and the D3-D7 scaling solution in the IR ($r=0$).

Now to solve the interpolating solution, we first choose the integration constant. First, $\tilde{\phi}_0$ is determined only by the 5-form flux: $\tilde{\phi}_0=-\log\frac{\alpha}{4}=-\log(\frac{4\pi^4}{\mathrm{Vol}(X_5)}N)$. Then without loss of generality, we set $\tilde{h}_0=\frac{\tilde{\phi}_0}{2}$ and choose the boundary condition for (b,c) to be

$$b(r)\to\log r,\qquad c(r)\to-\log r\qquad \text{at}\quad r\to+\infty, \tag{3.19}$$

$$b(r)\to\frac{7}{\sqrt{33}}\log r+b_0,\qquad c(r)\to\left(\frac{1}{\sqrt{33}}-1\right)\log r+b_0\qquad \text{at}\quad r\to 0, \tag{3.20}$$

where $b_0\equiv\frac{1}{6}\log(\frac{4\pi^4}{\mathrm{Vol}(X_5)}N)-\frac{1}{12}\log\frac{3}{8}-\frac{1}{6}\log\beta$. Then the dynamical system (3.11) and (3.12) plus the constraint

$$\frac{1}{8}[(9B-8C-8)^2-(33B^2+48)]=-\frac{\beta^2}{64\left(\frac{\pi^4}{\mathrm{Vol}(X_5)}\right)^2N^2}e^{-2b(r)+14c(r)}r^{14}<0, \tag{3.21}$$

and the boundary conditions (3.19) determine an interpolating solution that approaches the $AdS_5\times X_5$ solution

$$\begin{aligned} ds_E^2&=R^2\left[r^2(-dt^2+dx^2+dy^2+dw^2)+\frac{dr^2}{r^2}\right]+R^2ds_{X_5}^2,\\ e^{\phi}&=e^{\tilde{\phi}_0}, \end{aligned} \tag{3.22}$$

in the UV and the D3-D7 scaling solution

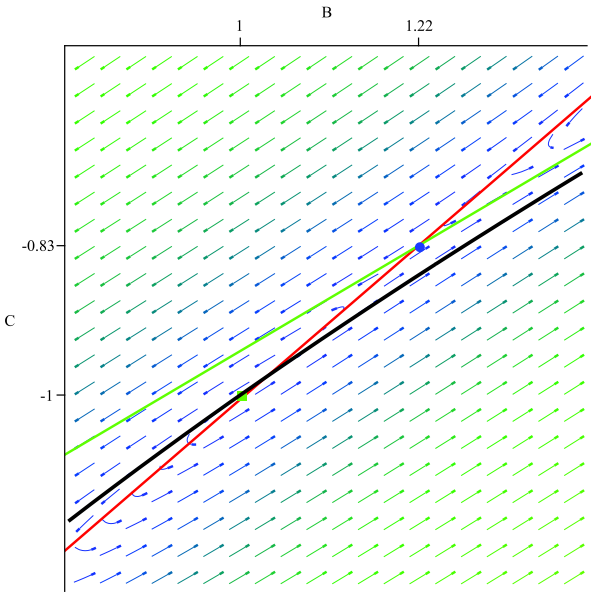
$$ds_E^2=R^2\left[r^{\frac{12}{\sqrt{33}}}(-dt^2+dx^2+dy^2)+\frac{dr^2}{r^2}\right]+\rho^2r^{\frac{8}{\sqrt{33}}}dw^2+R^2ds_{X_5}^2, \tag{3.23}$$

$$e^{\phi}=e^{4b_0}e^{\tilde{\phi}_0}r^{\frac{4}{\sqrt{33}}}, \tag{3.24}$$

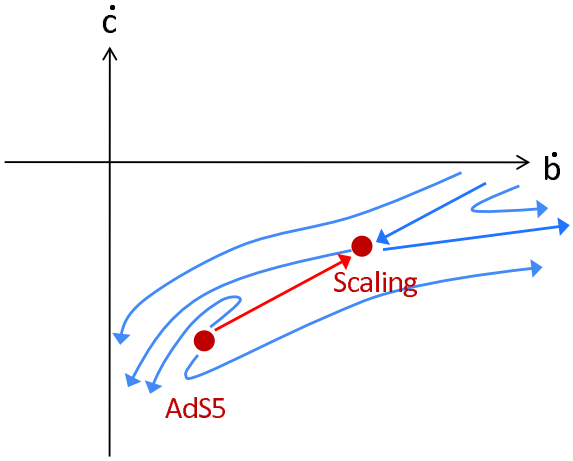
in the IR. Here $R^2=e^{-\tilde{\phi}_0/2}=2\sqrt{\frac{\pi^4}{\mathrm{Vol}(X_5)}N}$ and $\rho^2=e^{-4b_0}R^2$. And the fluxes are

$$F_5=4R^4\left(\Omega_{X_5}+*\Omega_{X_5}\right),\qquad F_1=\beta dw,$$

throughout the system.



(a)



(b)

Figure 1. The (B, C) flow diagram. The horizontal and vertical axes are $B \equiv \dot{b}$ and $C \equiv \dot{c}$, respectively. The arrows point in the direction from the UV ($r = \infty$) to the IR ($r = 0$). In the left figure, the blue dot at $(\frac{7}{\sqrt{33}}, \frac{1}{\sqrt{33}} - 1) \simeq (1.22, -0.83)$ is the unstable fixed point corresponding to the scaling solution; the green dot at $(1, -1)$ is the stable UV fixed point corresponding to the AdS_5 solution. The green line running through the scaling fixed point corresponds to the negative eigenvalue λ_1 while the red line corresponds to the positive one λ_2. The black curve is given by $9B - 8C - 8 = \sqrt{33B^2 + 48}$ and corresponds to a pure D3 solution; the allowed D3-D7 solutions are above this curve.

We can easily solve (b, c) numerically for arbitrary fluxes. Systems with different flux numbers differ only in their speeds in approaching the fixed points. Figure 2 shows the system with $\frac{\beta^2}{64(\frac{\pi^4}{\mathrm{Vol}_{(X_5)}})^2 N^2} = 1$ as an example. Figure 2(a) graphs the behavior of $(\dot{b}, \dot{c})$,

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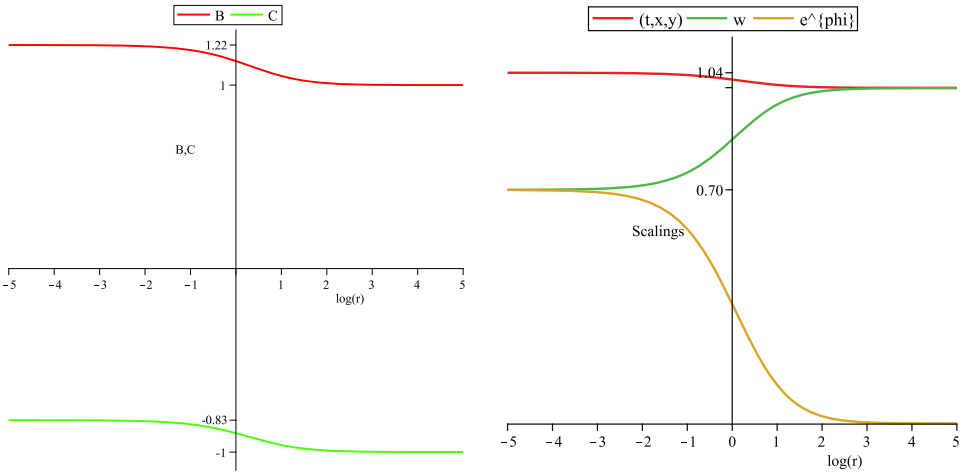

(a) (B, C) of the interpolating solution.

(b) Scalings of the interpolating solution in the Einstein Frame.

Figure 2. An interpolating solution with fluxes satisfying $\frac{\beta^2}{64(\frac{\pi^4}{\text{Vol}_{(X_5)}})^2 N^2} = 1$. In the left figure, $(B, C) \equiv (\dot{b}, \dot{c})$ flow from $(\frac{7}{\sqrt{33}}, \frac{1}{\sqrt{33}} - 1) \simeq (1.22, -0.83)$ in the IR to $(1, -1)$ in the UV. The right figure shows that, in the Einstein frame, the scalings of the (t, x, y)-directions, the w-direction, and e^{ϕ} flow from $(\frac{8}{\sqrt{33}}, \frac{4}{\sqrt{33}}, \frac{4}{\sqrt{33}}) \simeq (1.04, 0.70, 0.70)$ in the IR to $(1, 1, 0)$ in the UV.

and more directly, figure 2(b) presents the scalings of the (t, x, y)-directions, w-direction, and e^{ϕ} in the Einstein frame as the interpolating solution flows from the D3-D7 scaling solution in the IR ($r = 0$) to the AdS_5 in the UV ($r \to \infty$).

3.3 Interpolating solutions as holographic RG flows

Here we try to interpret our interpolating solutions via AdS/CFT. To make the argument simple we focus on $X_5 = S^5$. Consider the standard $\text{AdS}_5/\text{CFT}_4$ for the $\mathcal{N} = 4$ super Yang-Mills. We can perturb the $\mathcal{N} = 4$ super Yang-Mills by many relevant operators $\mathcal{O}_1, \mathcal{O}_2, \cdots$ as $\delta S = \int dx^4 [g_1 O_1 + g_2 O_2 + \cdots]$.

Our solution which flows from the $AdS_5 \times S^5$ to the scaling D3-D7 solution is dual to an anisotropic RG flow triggered by a non-supersymmetric relevant deformation (2.25), called O_1. This RG-flow eventually ends at the IR fixed point that is dual to the scaling solution. Therefore O_1 becomes irrelevant in the IR limit. However, our sketch of the holographic RG flow (see figure 1) tells us that this flow is unstable and signals the presence of another relevant operator O_2 which becomes relevant even at the IR fixed point. So if we slightly perturb this flow by O_2, then the RG flow passes near the IR fixed point and eventually goes to infinity. Its asymptotic behavior will be derived analytically in the next subsection. However, we can still fine tune such that there is no O_2 generated (i.e. $g_2 = 0$) to realize the IR fixed point governed by the scaling solution. This is exactly what experimentalists usually do to realize an unstable fixed point.[6]

[6]It is feasible because as long as the starting point is close enough to the critical line (which corresponds to the fine-tuned trajectory that hits the fixed point), the trajectory will stay for a very long time near the fixed point to allow the measurements [7].

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A similar situation occurs when we perturb the Heisenberg model by an anisotropic Ising-like interaction: $H = -J\sum_{\langle ij\rangle}(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y + \sigma_i^z\sigma_j^z) - D\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z$, where $D > 0$. In this analogy, the UV fixed point is the Heisenberg model fixed point and the IR one is the Ising model fixed point. This structure of the RG flow is rather generic and is one example of the phenomenon called crossover [7].

3.4 Exact solutions without D7-brane charge

In our interpolating solution, the flow from the UV AdS_5 fixed point reaches the IR D3-D7 scaling solution fixed point only if it starts along a specific direction. One wonders where the flow would end up if its initial direction slightly deviates from the fine-tuned one. A closer look at figure 1(a) tells us that if it deviates from the desired direction slightly to the left, it will turn back before reaching the scaling solution fixed point and asymptote to the $B < 1$ part of the black curve given by $9B - 8C - 8 = \sqrt{33B^2 + 48}$ (due to the constraint (3.13)). On the other hand, if it deviates slightly to the right, it will pass near (but not hitting) the scaling solution fixed point and then bend slightly downward and finally asymptote to the $B > 1$ part of the same black curve.

The curve given by $9B-8C-8 = \sqrt{33B^2 + 48}$ is the solution interpolating between the AdS_5 in the UV and some other solution in the IR. Since it saturates the inequality (3.21), there is no D7 brane charge ($\beta = 0$): it is a solution of the pure D3-brane system. Not only is it important because it gives the asymptotic form of what our interpolating solution decays into when perturbed by a relevant operator, it is also interesting as a pure D3-brane solution other than AdS_5. In this subsection, we will solve it analytically.

First, eliminating C from equations (3.11) and (3.12) gives

$$\frac{dB}{ds} = 4 + \frac{11}{4}B^2 - \frac{3}{4}B\sqrt{48 + 33B^2}, \tag{3.25}$$

which can actually be solved analytically. There are two solutions distinguished by the $\pm$ sign:

$$\begin{aligned} B_\pm(s) &= \frac{e^{8s} \mp 6\sqrt{\frac{3}{11}}e^{4s} + 1}{e^{8s} - 1}, \\ C_\pm(s) &= -\frac{e^{8s} \pm 4\sqrt{\frac{3}{11}}e^{4s} - 1}{e^{8s} - 1}, \end{aligned} \tag{3.26}$$

which lead to

$$\begin{aligned} b_\pm(s) &= \tilde{b}_0 - s \pm \frac{3}{4}\sqrt{\frac{3}{11}}\log\frac{e^{4s}+1}{e^{4s}-1} + \frac{1}{4}\log(e^{8s}-1), \\ c_\pm(s) &= \tilde{c}_0 - s \pm \frac{1}{2}\sqrt{\frac{3}{11}}\log\frac{e^{4s}+1}{e^{4s}-1}. \end{aligned} \tag{3.27}$$

We can choose $\tilde{h}_0 = \frac{\tilde{\phi}_0}{2}$ and $\tilde{b}_0 = \tilde{c}_0 = 0$; then the metric in the Einstein frame and the

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dilaton are simply

$$
\begin{aligned}
ds_E^2 = R^2 \Bigg[& \sqrt{\frac{r^8-1}{r^4}} \left(\frac{r^4+1}{r^4-1}\right)^{\pm\frac{1}{2}\sqrt{\frac{3}{11}}} (-dt^2+dx^2+dy^2) \\
& + \sqrt{\frac{r^8-1}{r^4}} \left(\frac{r^4+1}{r^4-1}\right)^{\mp\frac{3}{2}\sqrt{\frac{3}{11}}} dw^2 + \frac{dr^2}{r^2} + ds_{X_5}^2 \Bigg], \\
e^{\phi} = & \left(\frac{r^4+1}{r^4-1}\right)^{\pm 2\sqrt{\frac{3}{11}}} e^{\tilde{\phi}_0},
\end{aligned}
\tag{3.28}
$$

with $R^2 = 2\sqrt{\frac{\pi^4}{\mathrm{Vol}(X_5)}N}$.

Both solutions in (3.28) become $AdS_5 \times X_5$ in the UV fixed point ($r \to \infty$); however when going towards the IR, they become singular at $r = 1$. The solution with "$+$" (resp. "$-$") sign covers $B > 1$ (resp. $B < 1$) part of the curve $9B - 8C - 8 = \sqrt{33B^2 + 48}$. Even though these zero temperature solutions become singular ar $r = 1$, their corresponding black brane solutions at finite temperature are expected to be smooth. An explicit construction of such black brane solutions is left as a future problem. Also it would be very interesting to understand what anisotropic relevant deformations of the $\mathcal{N} = 4$ Yang-Mills are dual to this background.

4 Hydrodynamics

As we obtained the black brane solutions (2.27), we can consider the hydrodynamic behavior of their dual field theories from the supergravity side [24]. In this section, we especially determine the shear and bulk viscosities from dispersion relations for the corresponding quasinormal modes of fluctuations around the background (2.27), following [25, 26].

For this purpose, we first reduce S^5 part and consider fluctuations around the resulting five-dimensional background. The procedure of the reduction and derivation of the linearized equations of motion for the fluctuations are summarized in appendix A.8.1. In these equations, fluctuation of the resulting five-dimensional metric $\delta g^{(5)}_{\mu\nu} = H_{\mu\nu}$, dilaton $\delta\phi = \varphi$, axion $\delta\chi = \eta$, the trace part of the S^5 metric π and the five-form flux $\delta F_5 = f_5$ appear generally.

In this section, we assume that the w direction is compactified and the neglect the momentum in this direction so that we extract the effective $2+1$ dimensional holographic dual theories from the total $3+1$ dimensional ones. In this situation we can choose the momentum in y direction. By considering the symmetry of the background, we can decompose the fluctuation into channels decoupled from each other in the linear order:

shear channel	$H_{tx}, H_{xy}, H_{xr},$	(4.1)
scalar channel	$H_{xw},$	(4.2)
sound channel	$H_{tt}, H_{ty}, H_{tr}, H_{xx}, H_{yy}, H_{yr}, H_{ww}, H_{rr}, \varphi, \pi, f_{txywr},$	(4.3)
"axion channel"	$H_{tw}, H_{yw}, H_{rw}, \eta.$	(4.4)

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In this paper, we consider the shear and sound channel only, since we can read off the shear and bulk viscosity from dispersion relations for the quasinormal modes of these channels.

Before deriving the dispersion relation, we need to consider the gauge fixing of the fluctuations. Let us assume that the fluctuations are of the form

$$\begin{aligned} H_{\mu\nu} &= \tilde{H}_{\mu\nu}(r)e^{-i(\omega t-qy)}, & \varphi &= \varphi(r)e^{-i(\omega t-qy)}, \\ \eta &= \eta(r)e^{-i(\omega t-qy)}, & \pi &= \pi(r)e^{-i(\omega t-qy)}, \\ f_{\mu\nu\rho\sigma\lambda} &= f_{\mu\nu\rho\sigma\lambda}(r)e^{-i(\omega t-qy)}. \end{aligned} \tag{4.5}$$

Especially, for later convenience, we rewrite some of their components as

$$\tilde{H}_{tt}(r) = \tilde{R}^2 c_t^2 H_{tt}(r), \qquad \tilde{H}_{ab}(r) = \tilde{R}^2 c_x^2 H_{ab}(r), \tag{4.6}$$

$$\tilde{H}_{ta}(r) = \tilde{R}^2 c_x^2 H_{ta}(r), \qquad \tilde{H}_{ww}(r) = \tilde{R}^2 c_x^{4/3} H_{ww}(r), \tag{4.7}$$

where $a, b = x$ or y, and $c_t^2 = r^2 F(r)$ and $c_x^2 = r^2$; the function $F(r)$ is defined by (2.28). As in appendix A.8.1, we then take the radial gauge $H_{\mu r} = 0$ in the following discussion. Here we notice that there still exist residual gauge degrees of freedom under an infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ with $\xi^\mu = \xi^\mu(r)e^{-i(\omega t-qy)}$:

$$H_{\mu\nu} \to H_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \qquad \varphi \to \varphi - \xi_\mu \partial^\mu \phi, \tag{4.8}$$

$$\eta \to \eta - \xi_\mu \partial^\mu \chi, \qquad \pi \to \pi - \xi_\mu \partial^\mu g_\alpha{}^\alpha \ . \tag{4.9}$$

Here the covariant derivative is defined by the background metric. In [25], it has been noticed that one can derive the dispersion relation for the quasinormal modes by defining gauge invariant quantities, instead of performing the gauge fixing completely. Following this, we define the gauge invariant quantities as

$$Z_1(r) = qH_{tx}(r) + \omega H_{xy}(r), \tag{4.10}$$

for the shear channel and

$$\begin{aligned} Z_0(r) &= q^2 \frac{c_t^2}{c_x^2} H_{tt}(r) + 2q\omega H_{ty}(r) + \omega^2 H_{yy}(r) \\ &\quad + \frac{3}{5}\left(q^2 \frac{c_t' c_t}{c_x' c_x} - \omega^2\right)(H_{xx}(r) + H_{ww}(r)), \end{aligned} \tag{4.11}$$

$$Z_\varphi(r) = \varphi(r) - \frac{1}{2} H_{ww}(r), \tag{4.12}$$

$$Z_\pi(r) = \pi(r), \tag{4.13}$$

for the sound channel. Below, we derive equations for these quantities by using linearized equations of motion summarized in appendix B and then the dispersion relations for the quasinormal modes by imposing an appropriate boundary condition.

4.1 Shear viscosity

From the equations (B.1),(B.2) and (B.3), we obtain the equation for Z_1 as

$$Z_1'' + \frac{14F(r)(\mathfrak{q}^2 F(r) - \mathfrak{w}^2) - 3\mathfrak{w}^2 r F'(r)}{3rF(r)(\mathfrak{q}^2 F(r) - \mathfrak{w}^2)} Z_1' + \frac{121\mu^{\frac{6}{11}}(\mathfrak{w}^2 - F(r)\mathfrak{q}^2)}{36r^4 (F(r))^2} Z_1 = 0, \tag{4.14}$$

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where the dimensionless frequency $\mathfrak{w}$ and momentum $\mathfrak{q}$ are defined by

$$\mathfrak{w} = \frac{w}{2\pi T_H}, \quad \mathfrak{q} = \frac{q}{2\pi T_H}, \tag{4.15}$$

respectively.

In order to solve this equation, we consider the asymptotic behavior of Z_1 first. As $r \to \infty$, $Z_1 \sim Ar^0 + Br^{-\frac{11}{3}}$ with constants A, B generally. For the quasinormal modes, A is set to zero, or, in other words, we should impose the Dirichlet boundary condition $Z_1 = 0$ at $r = \infty$. As $r \to \mu^{\frac{3}{11}}$, $Z_1 \sim (F(r))^{\pm i\mathfrak{w}/2}$ and, in order to ensure that only incoming waves exist at the horizon, we take the one with negative sign.

By taking the hydrodynamic limit $\mathfrak{q} \ll 1$ and $\mathfrak{w} \ll 1$, we can perturbatively determine Z_1 as

$$Z_1 = C(F(r))^{-i\mathfrak{w}/2}\left(1 + \frac{i\mathfrak{q}^2}{2\mathfrak{w}}F(r) + \mathcal{O}(\mathfrak{q}^2, \mathfrak{w})\right), \tag{4.16}$$

where C is a constant. Here we assumed that $\mathfrak{q}^2$ and $\mathfrak{w}$ are of the same order as usual for the shear channel. Then by imposing the Dirichlet condition $Z_1 = 0$ at $r = \infty$, we find $\mathfrak{w} = -\frac{i\mathfrak{q}^2}{2}$. Comparing with the hydrodynamic relation for the shear channel

$$\omega = -\frac{i\eta}{T_H s}q^2, \tag{4.17}$$

we can find that the universal bound for the shear viscosity η to the entropy density s ratio [27] is saturated for the current case:

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{4.18}$$

4.2 Bulk viscosity

From the equations derived in appendix B.2, we obtain the differential equations for Z_φ, Z_π and Z_0 as

$$Z_\varphi'' + \ln'(c_t^2 c_x^{8/3})Z_\varphi' + \frac{1}{c_t^2}\left(\frac{\omega^2}{c_t^2} - \frac{q^2}{c_x^2}\right)Z_\varphi + \\ -\frac{44}{9c_t^2}Z_\varphi + \frac{11(\mu^3 - 2\mu^2 r^{11/3} + 2r^{11})}{27r^{28/3}(r^{11/3} - \mu)}Z_\pi = 0, \tag{4.19}$$

$$Z_\pi'' + \ln'(c_t^2 c_x^{8/3})Z_\pi' + \frac{1}{c_t^2}\left(\frac{\omega^2}{c_t^2} - \frac{q^2}{c_x^2}\right)Z_\pi - \frac{32}{c_t^2}Z_\pi = 0, \tag{4.20}$$

$$Z_0'' + \mathcal{F}(r)Z_0' + \mathcal{G}(r)Z_0 + \mathcal{H}(r)Z_\varphi = 0, \tag{4.21}$$

where

$$\mathcal{F}(r) = \frac{625\mu^3 q^4 - 3584(q^2-\omega^2)^2 r^{11}}{3r(\mu - r^{11/3})(5\mu q^2 - 16r^{11/3}(q^2-\omega^2))^2} + \frac{-20\mu^2 q^2(157q^2 - 112\omega^2)r^{11/3} + 32\mu(149q^2 - 24\omega^2)(q^2-\omega^2)r^{22/3}}{3r(\mu - r^{11/3})(5\mu q^2 - 16r^{11/3}(q^2-\omega^2))^2}, \tag{4.22}$$

$$\mathcal{G}(r) = \frac{3025\mu^4 q^4 - 5\mu^2 q^2 r^{16/3}(333q^2 - 1936r^2)(q^2-\omega^2) - 2304r^{38/3}(q^2-\omega^2)^3}{9r^2(\mu - r^{11/3})^2(5\mu q^2 - 16r^{11/3}(q^2-\omega^2))^2} + \frac{3744\mu r^9 q^2(q^2-\omega^2)^2 + 5\mu^3 q^2 r^{5/3}(45q^4 - 2541q^2 r^2 + 1936r^2\omega^2)}{9r^2(\mu - r^{11/3})^2(5\mu q^2 - 16r^{11/3}(q^2-\omega^2))^2}, \tag{4.23}$$

$$\mathcal{H}(r) = -\frac{22(1536r^{11}(q^2-\omega^2)^3 - 1440\mu r^{22/3}q^2(q^2-\omega^2)^2)}{45r^2(\mu - r^{11/3})(5\mu q^2 - 16r^{11/3}(q^2-\omega^2))^2} - \frac{22(-\mu^3(425q^2 - 550\omega^2)q^4 + 20\mu^2 q^2 r^{11/3}(83q^4 - 171q^2\omega^2 + 88\omega^4))}{45r^2(\mu - r^{11/3})(5\mu q^2 - 16r^{11/3}(q^2-\omega^2))^2}. \tag{4.24}$$

Then, as in the case of the shear channel, all we have to do next is to solve these equations by imposing the incoming wave boundary condition at the horizon and the Dirichlet boundary condition at infinity to derive the dispersion relation for the quasinormal mode of the sound channel. We also take the hydrodynamic limit and assume that $\mathfrak{q}$ and $\mathfrak{w}$ are of the same order, as is expected for the sound channel. For $Z_\pi = 0$, we can see that the nonsingular solution for (4.20) is a constant, which turns out to be zero due to the Dirichlet boundary condition $Z_\pi = 0$ at the boundary $r = \infty$. By substituting this into (4.19), we obtain $Z_\varphi = 0$ in a similar manner. Then from these results and (4.21), by using a similar ansatz $Z_0 = (F(r))^{-i\mathfrak{w}/2}Y_0(r)$, we can perturbatively determine Z_0 as

$$Z_0 = \tilde{C}(F(r))^{-i\mathfrak{w}/2}\left(1 - \frac{5(1+2i\mathfrak{w})\mathfrak{q}^2}{11\mathfrak{q}^2 - 16\mathfrak{w}^2}F(r) + \mathcal{O}(\mathfrak{q}^2, \mathfrak{w}^2, \mathfrak{q}\mathfrak{w})\right), \tag{4.25}$$

where $\tilde{C}$ is a normalization constant. As a result of the Dirichlet boundary condition $Z_0 = 0$ at $r = \infty$, we obtain the dispersion relation for the sound channel

$$\mathfrak{w} = \frac{1}{2}\sqrt{\frac{3}{2}}\mathfrak{q} - i\frac{5}{16}\mathfrak{q}^2 + \cdots. \tag{4.26}$$

Let us recall a hydrodynamic relation for the sound channel in the noncompact $(d+1)$-dimensional spacetime

$$\omega = c_s q - i\frac{\eta}{T_H s}\left(\frac{d-1}{d} + \frac{\zeta}{2\eta}\right)q^2 + \cdots, \tag{4.27}$$

where c_s and ζ are the sound velocity and the bulk viscosity, respectively. Since there are two noncompact spatial dimensions for the dual field theory now, $d = 2$ for the current case. Therefore, by comparing (4.26) with (4.27) with $d = 2$, we obtain

$$c_s^2 = \frac{3}{8}, \qquad \frac{\zeta}{\eta} = \frac{1}{4}. \tag{4.28}$$

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As for ζ/η and c_s of strongly coupled gauge theory plasma in d noncompact spatial dimensions, it is conjectured in [28] that they satisfy an inequality

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{d} - c_s^2\right). \tag{4.29}$$

In our case, this inequality is saturated.

5 Entanglement entropy of D3-D7 scaling solutions

When a quantum system is divided into two subsystems: A and its complement B, the von Neumann entropy $S_A = -\mathrm{Tr}\rho_A \log \rho_A$ (where ρ_A is the reduced density matrix after tracing out B) is called the entanglement entropy. The scaling behaviors and certain universal[7] coefficients of the entanglement entropy encode important information on the degrees of freedom and non-local correlations of the system [29, 30].

For an anisotropic system, an interesting question is "how does the scaling behavior of the entanglement entropy depend on the direction along which the subsystems are delineated?" In this section, we will study the entanglement entropy of various subsystems A of the (x, y, w) space at the boundary ($r \to \infty$) of the 5D part of the D3-D7 scaling solution (2.18). The field theoretical computation of the entanglement entropy is expected to be difficult as the system will be strongly coupled. We will instead compute its holographic dual on the gravity side. The holographic dual of the entanglement entropy of a subsystem A is given by

$$S_{EE} = \frac{\mathrm{Area}^{\mathrm{min}}}{4G_5}, \tag{5.1}$$

where $\mathrm{Area}^{\mathrm{min}}$ is the area of the three-dimensional *minimal surface* that lives inside the (r, x, y, w) space and borders on the boundary ∂A of the subsystem A [31].

After a coordinate transformation $r = \frac{1}{z}$, the metric of the D3-D7 scaling solution (2.18) becomes

$$ds^2 = \tilde{R}^2\left(\frac{-dt^2 + dx^2 + dy^2 + dz^2}{z^2} + \frac{dw^2}{z^{4/3}}\right) + R^2 ds^2_{X_5}. \tag{5.2}$$

For $X_5 = S^5$, $R^2 = 2\sqrt{\pi N}$ and $\tilde{R}^2 = \frac{11}{12}R^2$. We consider the full boundary system given by

$$x \in [0, L_x], \qquad y \in [0, L_y], \qquad w \in [0, L_w]. \tag{5.3}$$

Among the various types of subsystems, we will only study the easiest types: stripes with either x or (inequivalently) w restricted to a smaller length.

5.1 Entanglement entropy for subsystem along x-direction

Let's first consider the subsystem A cut out along the x-direction:

$$x \in [0, \ell_x < L_x], \qquad y \in [0, L_y], \qquad w \in [0, L_w]. \tag{5.4}$$

[7] Here "universal" means the independence from the choice of different delineations of subsystems.

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The three-dimensional minimal surface bordering on ∂A is given by the embedding function $z = z(x)$:

$$\begin{aligned}\text{Area}_x^{\min} &= \int_0^{\ell_x} dx \int_0^{L_y} dy \int_0^{L_w} dw \frac{\tilde{R}}{z} \cdot \frac{\tilde{R}}{z^{2/3}} \cdot \sqrt{\left(\frac{\tilde{R}}{z}\right)^2 + \left(\frac{\tilde{R}}{z} z'\right)^2} \\ &= \tilde{R}^3 L_y L_w \int_0^{\ell_x} dx \frac{1}{z^d} \sqrt{1 + z'^2}, \end{aligned} \tag{5.5}$$

where $d = 1 + 1 + \frac{2}{3} = \frac{8}{3}$ is the total scaling of the boundary system.

This is a Lagrangian system with $\mathcal{L} = \tilde{R}^3 L_y L_w \frac{1}{z^d} \sqrt{1 + z'^2}$. The $z(x)$ that minimizes the surface area is then given by the equation of motion

$$z' = \pm \sqrt{\left(\frac{z_*}{z}\right)^{2d} - 1}, \tag{5.6}$$

where z_* is the peak of z on the minimal surface, across which z' changes sign. It can be solved from $\ell_x = 2 \int_0^{z_*} \frac{dz}{z'}$, which gives $z_* = \frac{\ell_x}{2} \frac{\Gamma(\frac{1}{2d})}{\sqrt{\pi}\Gamma(\frac{1}{2d} + \frac{1}{2})}$. The minimal surface is then

$$\text{Area}_x^{\min} = 2\tilde{R}^3 L_y L_w \frac{1}{z_*^{d-1}} \cdot In \qquad \text{with} \qquad In \equiv \int_0^1 du \frac{1}{u^d} \frac{1}{\sqrt{1 - u^{2d}}}. \tag{5.7}$$

In has a UV divergence at $z \to 0$. Imposing the UV cutoff by choosing the lattice spacing a for the boundary system, we get

$$In = \frac{1}{d-1} \left(\frac{1}{(a/z_*)^{d-1}} - \frac{\sqrt{\pi}\Gamma(\frac{1}{2d} + \frac{1}{2})}{\Gamma(\frac{1}{2d})} \right). \tag{5.8}$$

Then plugging the value of z_* and the five-dimensional Newton's constant $G_5 = \frac{G_{10}}{V_{X_5}}$ with $G_{10} = 8\pi^6 \ell_s^8$, we finally obtain the holographic entanglement entropy for the subsystem divided out along the x-direction:

$$\begin{aligned} S_{EE-x} &= \left(\frac{11}{12}\right)^3 \frac{\pi^2}{\text{Vol}(X_5)} \cdot N^2 L_y L_w \frac{1}{d-1} \left[\frac{1}{a^{d-1}} - \left(\frac{2}{\ell_x}\right)^{d-1} \left(\frac{\sqrt{\pi}\Gamma(\frac{1+d}{2d})}{\Gamma(\frac{1}{2d})} \right)^d \right] \\ &= N^2 L_y L_w \left[\frac{\gamma_1}{a^{5/3}} - \frac{\gamma_2}{(\ell_x)^{5/3}} \right], \end{aligned} \tag{5.9}$$

with $d = \frac{8}{3}$. γ_1 and γ_2 are numerical constants.

Now let's interpret the result. First, the holographic entanglement entropy is proportional to the area of the boundary of the subsystem $\partial A = L_y L_w$ — as expected from the "area law" [29] for the entanglement entropy from direct field theory computations. Second, the first term of the holographic entanglement entropy diverges and is cutoff-dependent. The scaling of a is given by the total scalings of the y and w directions relative to that of the t-direction: $(1 + \frac{2}{3})/1 = \frac{5}{3}$.

The second, finite term of the holographic entanglement entropy is more interesting: it is cutoff-independent therefore can be compared with the field theoretic computation of

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the entanglement entropy. Its coefficient γ_2 gives a measure of the total degrees of freedom. The scaling of ℓ_x is simply the total scalings of the y and w directions relative to that of the x-direction: $(1+\frac{2}{3})/1 = \frac{5}{3}$. Since the scaling of the x-direction is the same as that of the t-direction, the exponents of a and ℓ_x are the same.

5.2 Entanglement entropy for subsystem along w-direction

The next easiest subsystem we can consider is to divide along the w-direction:

$$w \in [0, \ell_w < L_w]. \tag{5.10}$$

The three-dimensional minimal surface bordering on ∂A is given by the embedding function $z = z(w)$:

$$\text{Area}_w^{\min} = \tilde{R}^3 L_x L_y \int_0^{\ell_w} dw \frac{1}{z^3}\sqrt{z^{2/3} + z'^2}. \tag{5.11}$$

Then we could follow the line of the previous subsystem along the x-direction. The computation is straightforward but more complicated so we present here instead a simpler route which utilizes the result for the x-direction subsystem.

The coordinate transformation

$$z = \tilde{z}^{\frac{3}{2}}, \qquad (t, x, y, w) = \frac{3}{2}(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{w}), \tag{5.12}$$

results in the metric

$$ds^2 = \left(\frac{3}{2}\tilde{R}\right)^2 \left(\frac{-d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2}{\tilde{z}^3} + \frac{d\tilde{z}^2}{\tilde{z}^2} + \frac{d\tilde{w}^2}{\tilde{z}^2}\right) + R^2 ds^2_{X_5}. \tag{5.13}$$

Thus we can simply use the result from the x-direction case, with $d = \frac{8}{3}$ replaced by $d_w = 4$. First, we write down the dictionary between values in the original coordinates and the new one.

1. In the new coordinates, the full boundary system is

$$\tilde{x} \in \left[0, \frac{2}{3}L_x\right], \qquad \tilde{y} \in \left[0, \frac{2}{3}L_y\right], \qquad \tilde{w} \in \left[0, \frac{2}{3}L_w\right], \tag{5.14}$$

   while the subsystem A is

$$\tilde{x} \in \left[0, \frac{2}{3}L_x\right], \qquad \tilde{y} \in \left[0, \frac{2}{3}L_y\right], \qquad \tilde{w} \in \left[0, \frac{2}{3}\ell_w < \frac{2}{3}L_w\right]. \tag{5.15}$$

2. The lattice spacing in the new coordinates is related to that in the old coordinates by

$$\tilde{a} = a^{\frac{2}{3}}. \tag{5.16}$$

3. The turning point of z is $\tilde{z}_* = \frac{2}{3}z_*$.

Using the result from the x-direction case, we find the minimal surface in the new coordinates to be

$$Area^{\min} = 3\tilde{R}^3 L_x L_y \frac{1}{d_w - 1}\left[\frac{1}{\tilde{a}^{d_w-1}} - \frac{1}{\tilde{z}_*^{d_w-1}}\frac{\sqrt{\pi}\Gamma(\frac{1}{2d_w}+\frac{1}{2})}{\Gamma(\frac{1}{2d_w})}\right]. \tag{5.17}$$

After translated back into the original coordinates, it gives the entanglement entropy of subsystem along the w-direction

$$\begin{aligned} S_{EE-w} &= \left(\frac{11}{12}\right)^3 \frac{\pi^2}{\mathrm{Vol}(X_5)} \cdot \\ &\quad \cdot N^2 L_x L_y \frac{1}{D-1}\left[\frac{1}{a^{D-1}} - \left(\frac{3}{2}\right)^{d_w-1}\left(\frac{2}{\ell_w}\right)^{d_w-1}\left(\frac{\sqrt{\pi}\Gamma(\frac{1}{2d_w}+\frac{1}{2})}{\Gamma(\frac{1}{2d_w})}\right)^{d_w}\right] \\ &= N^2 L_x L_y\left[\frac{\gamma_1'}{a^2} - \frac{\gamma_2'}{(\ell_w)^3}\right], \end{aligned} \tag{5.18}$$

where $D-1 \equiv \frac{2(d_w-1)}{3} = 2$. The negative finite part has the same form as the result for the subsystem divided along the x-direction with $d = \frac{8}{3}$ replaced by $d_w = 4$.[8]

Now let's compare this result with the one for the subsystem along the x-direction. The essential features are the same. It is proportional to the boundary area $L_x L_y$. There are one cut-off dependent, divergent term and one cut-off independent, finite term. The scaling of the cutoff a is given by the total scalings of the x and y directions relative to that of the t-direction: $(1+1)/1 = 2$. The scaling of the ℓ_w is given by the total scalings of the x and y directions relative to that of the w-direction: $(1+1)/(2/3) = 3$. Unlike the case for the subsystem along the x-direction, since the scaling of the w-direction is different from that of the t-direction, the exponents of a and ℓ_w are different.

6 Perturbative analysis

In order to know the details of the holographic dual field theories, a basic thing to do is to analyze the perturbative spectra around their supergravity solutions. This offers us the information on scale dimensions [2, 3]. For example, scalar perturbations are dual to scalar operators $\mathcal{O}_i$ in the dual field theory. These perturbative modes in supergravity are described by Klein-Gordon equations with various masses in the curved spacetime. Since we have the nontrivial dilaton in our D3-D7 scaling solutions (2.18), it is not clear a priori whether the Klein-Gordon equation should be obtained from the string frame metric or the Einstein frame metric. Actually our scaling property (2.20) of the gravity solutions is only available in the Einstein frame. Also the study of the perturbative spectrum is necessary to determine the stability of the background. Motivated by these, below we will examine the perturbations around our scaling backgrounds (2.18).

[8] This might be understood as follows: when all scalings are normalized with respect to the w-direction, then the total scaling of the (x, y, w)-space is

$$\lambda_x + \lambda_y + \lambda_w = \frac{3}{2} + \frac{3}{2} + 1 = 4. \tag{5.19}$$

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6.1 Description of perturbations

Let us analyze the perturbations of bosonic fields around the D3-D7 scaling solutions defined by (2.18), (2.17), (2.10) and (2.11) in the Einstein frame . We will closely follow the analysis of $AdS_5 \times S^5$ in [32, 33]. We will denote the total ten-dimensional coordinates by $M, N, \ldots = 0, 1, 2, \cdots, 9$. The five-dimensional Lorentzian spacetime (called M_5) is described by the coordinate $\mu, \nu, \cdots = 0, 1, 2, 3, 4$ and the five-dimensional compact manifold X_5 by $\alpha, \beta, \cdots = 5, 6, 7, 8, 9$.

As is clear from the IIB supergravity action in our background, the 3-form fluxes F_3 and H_3 are decoupled from the other fields (i.e. the metric, the dilaton, F_5, and F_1) thus we can concentrate on the latter ones. Then the Lagrangian in the Einstein frame is written as follows

$$\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{2}e^{2\phi}\partial_M\chi\partial^M\chi - \frac{1}{2}\partial_M\phi\partial^M\phi - \frac{1}{4\cdot 5!}F_{MNPQR}F^{MNPQR}\right). \tag{6.1}$$

To make analysis more tractable, let us assume $X_5 = S^5$ here. Then we can define the scalar, vector, traceless symmetric, and antisymmetric spherical harmonics on S^5 by Y^I, Y^I_α, $Y^I_{(\alpha\beta)}$, and $Y^I_{[\alpha\beta]}$, respectively [32]. Using these spherical harmonics, the metric perturbations $\delta g_{MN} = h_{MN}$ can be decomposed as follows [32]

$$h_{(\mu\nu)} = h^I_{(\mu\nu)}Y^I, \quad h^\mu_\mu = h^I Y^I, \quad h_{\mu\alpha} = B^I_\mu Y^I_\alpha, \quad h_{(\alpha\beta)} = \phi^I Y^I_{(\alpha\beta)}, \quad h^\alpha_\alpha = \pi^I Y^I, \tag{6.2}$$

where $(\alpha\beta)$ denotes the traceless symmetric part. We also denoted all indices of the spherical harmonics simply by I. We fix the gauge by requiring

$$\nabla^\alpha h_{(\alpha\beta)} = \nabla^\alpha h_{\mu\alpha} = 0. \tag{6.3}$$

The perturbations of the dilaton and axion are defined as follows:

$$\delta\phi = \varphi^I Y^I, \quad \delta\chi = \eta^I Y^I. \tag{6.4}$$

Finally, the perturbation of the 5-form flux $F_5 = dC_4$ can be express as follows [32]:

$$\begin{aligned}
&C_{\mu_1\mu_2\mu_3\mu_4} = b^I_{\mu_1\mu_2\mu_3\mu_4}Y^I, && C_{\mu_1\mu_2\mu_3\alpha} = b^I_{\mu_1\mu_2\mu_3}Y^I_\alpha,\\
&C_{\mu_1\mu_2\alpha_1\alpha_2} = b^I_{\mu_1\mu_2}Y^I_{[\alpha_1\alpha_2]},\\
&C_{\mu\alpha_1\alpha_2\alpha_3} = b^I_\mu \epsilon_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2}\nabla_{\beta_1}Y^I_{\beta_2}, && C_{\alpha_1\alpha_2\alpha_3\alpha_4} = b^I \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}{}^{\beta}\nabla_\beta Y^I.
\end{aligned} \tag{6.5}$$

They satisfy the gauge fixing condition $\nabla^\alpha C_{\alpha\cdots} = 0$, and the self-duality of F_5 allows us to eliminate $b^I_{\mu_1\mu_2\mu_3\mu_4}$ and $b^I_{\mu_1\mu_2\mu_3}$.

Next we substitute (6.2), (6.4), and (6.5) into equations of motion of (6.1) and derive the perturbative differential equations. We omit the details of analysis here and put them in appendix A as many parts of the calculations are essentially the same as those in [32].

In the end, we find that the following modes

$$\begin{aligned}
&\text{Scalar modes} : \phi^I,\ (h^I, \pi^I, b^I),\\
&\text{Vector modes} : (B^I_\mu, b^I_\mu),\\
&\text{Tensor modes} : b^I_{\mu\nu},
\end{aligned} \tag{6.6}$$

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satisfy free massive field equations which are precisely the same expressions as those in the $AdS_5 \times S^5$ background.[9] In the above, the fields in the same parenthesis mix with each other. Therefore, these perturbations (6.6) obey free field equations of motion constructed from the Einstein frame metric (2.18) instead of the string frame metric.

As an example, let us concentrate on the scalar mode ϕ^I. Its equation of motion is written as

$$\left(\Box_x + \Box_y - \frac{2}{R^2}\right) \phi^I Y^I_{(\alpha\beta)} = 0, \tag{6.7}$$

where $\Box_x$ and $\Box_y$ are the Laplacians of the Lorentzian part M_5 and the sphere part S^5, respectively. Using the eigenvalues of $Y^I_{(\alpha\beta)}$, we eventually obtain[10]

$$\left(\Box_x - \frac{k(k+4)}{R^2}\right) \phi^I = 0, \quad (k = 2, 3, 4, \cdots). \tag{6.8}$$

On the other hands, the other modes φ^I, η^I, and $h_{\mu\nu}$ mix with each other and obey equations of motion more complicated than those in the $AdS_5 \times S^5$ case (see appendix A).

6.2 Scaling dimensions and stability

We have observed that a large class of supergravity modes (6.6), though not all of them, satisfy the conventional free field equations with various masses via the Kaluza-Klein compactification on S^5. The scalar modes in (6.6) satisfy the equations of motion of the form (we denote such a scalar field by Φ here)

$$\left(\Box_x - m^2\right) \Phi = 0, \tag{6.9}$$

where the Laplacian $\Box_x = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is constructed from the Einstein frame metric (2.18).

Now let us consider a scalar field Φ on a slightly generalized scaling background

$$ds^2 = \tilde{R}^2 \left(\frac{dz^2}{z^2} + \frac{-dt^2 + dx^2 + dy^2}{z^2} + \frac{dw^2}{z^{2\nu}}\right), \tag{6.10}$$

where the scaling exponent ν is related to the scaling exponent z in (1.4) by $\nu = \frac{1}{z}$. Especially, our scaling background (2.18) corresponds to $\nu = \frac{2}{3}$. The equation of motion (6.9) is written as follows:

$$-\Phi'' + \frac{\nu+2}{z}\Phi' + \left(\frac{m^2\tilde{R}^2}{z^2} + p^2 - \omega^2 + p_w^2\ z^{2(\nu-1)}\right)\Phi = 0. \tag{6.11}$$

Here ω, p, and p_w are the frequency, the momenta in (x, y)- and w-direction, respectively.

After redefining the wave function by $\Phi(z) = z^{\frac{\nu+2}{2}}\Psi(z)$, we obtain the Schrodinger-like equation

$$-\Psi'' + V(z)\Psi = \omega^2\Psi, \tag{6.12}$$

[9] In other words, for these modes the differences from $AdS_5 \times S^5$ only come from the background metric which is employed to write down the free field equations.

[10] Please distinguish the total angular momentum k of the spherical harmonics of S^5 from the number k of D7-branes.

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where

$$V(z) = \frac{m^2\tilde{R}^2 + \frac{(3+\nu)^2-1}{4}}{z^2} + p^2 + p_w^2\ z^{2(\nu-1)}. \tag{6.13}$$

When z is small, the third term in (6.13) is small compared to the first term, assuming $\nu > 0$. Thus, as in the AdS/CFT case (i.e. $\nu = 1$), we can expect[11] that the stability against the (normalizable) perturbations is the same as that of Schrodinger problem with the potential $V(z) = \frac{\lambda-\frac{1}{4}}{z^2}$. It is well-known that the latter system is stable iff $\lambda \geq 0$.

In this way, we speculate that in the background (6.10), the stability condition requires

$$m^2\tilde{R}^2 \geq -\frac{(3+\nu)^2}{4}. \tag{6.14}$$

Notice that if we set $\nu = 1$ in (6.14), we obtain $m^2\tilde{R}^2 \geq -4$, which is the well-known Breitenlohner-Freedman (BF) bound of AdS_5.

This condition can equally be implied from the behavior of the scalar field near the boundary $z \to 0$

$$\phi(z) \sim Az^{\Delta_+} + Bz^{\Delta_-} + \cdots, \tag{6.15}$$

where

$$\Delta_\pm = \frac{\nu+3}{2} \pm \sqrt{\frac{(\nu+3)^2}{4} + m^2\tilde{R}^2}. \tag{6.16}$$

The quantity $\Delta_\pm$ is holographically interpreted as the scaling dimension in the dual anisotropic scale invariant theory. Notice that the condition (6.14) requires that the scaling dimension should be real-valued.

Now let us go back to our D3-D7 scaling solutions (2.18). In this case we obtain the stability condition by setting $\nu = 2/3$ and $R^2 = \frac{11}{12}\tilde{R}^2$ as follows

$$m^2R^2 \geq -\frac{11}{3}. \tag{6.17}$$

We can apply this condition to the scalar modes in (6.6). As is clear from (6.8), all the scalar modes ϕ^I satisfy this condition. However, we find that one of the infinitely many mixed modes of (h^I, π^I, b^I) actually has the largest tachyonic mass $m^2R^2 = -4$, which saturates the BF bound of AdS_5. This occurs only for the second spherical harmonics $k = 2$. Even though this tachyonic mode is stable in the AdS_5 spacetime, it seems to become an unstable mode in our D3-D7 scaling background as (6.17) is violated.[12]

Nevertheless, we can replace S^5 with an arbitrary Einstein manifolds X_5 with the same Ricci curvature, keeping the same scaling solution (2.18). Define the eigenvalues Λ of Laplacian of a scalar function Y such that $-R^2\Box_y Y = \Lambda Y$. In this case, if

$$\frac{\Lambda}{16} + 1 - \sqrt{\frac{\Lambda}{4} + 1} \geq -\frac{11}{48}, \tag{6.18}$$

[11] We are very grateful to Andreas Karch for illuminating explanations on the stability analysis in Poincaré AdS spaces.

[12] A perturbative instability has also been noticed in [21] for type IIB backgrounds dual to the non-relativistic CFT when the 3-form fluxes are vanishing.

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is always satisfied, then the above lowest mass mode becomes stable. In other words, if there is no eigenvalue between

$$\frac{37-8\sqrt{3}}{3}(\simeq 7.71) < \Lambda < \frac{37+8\sqrt{3}}{3}(\simeq 16.95), \tag{6.19}$$

then the background can be perturbatively stable. Notice that $\Lambda = 12$ saturates the BF bound and it is the unstable mode when $X_5 = S^5$. It is intriguing whether there exists such a stable (Sasaki-)Einstein manifold.

7 D4-D6 Scaling Solutions

In type IIA string, the closest analogue to the previous D3-D7 system is the following D4-D6 system:

$\mathcal{M}_4 \times T^2 \times X_4$:	t	x	y	r	w_1	w_2	s_1	s_2	s_3	s_4
N D4:	×	×	×		×	×				
k D6:	×	×	×				×	×	×	×

Here (w_1, w_2) span a two-manifold that supports the D6 flux (we will choose it to be T^2 for simplicity) and (s_1, s_2, s_3, s_4) span a four-dimensional Einstein manifold X_4 with the same Ricci curvature as that of S^4. However, as we will see, this system does not support a scaling-invariant solutions. In this section, we will present a gravity solution that is closest to a scaling solution: under $(t, x, y, r, w_1, w_2) \to (\lambda t, \lambda x, \lambda y, \frac{r}{\lambda}, \lambda^{\frac{2}{3}} w_1, \lambda^{\frac{2}{3}} w_2)$, the line element $ds^2 \to \lambda^{-\frac{1}{3}} ds^2$. We will also give its black brane generalization.

The fluxes given by these N D4-branes and k D6-branes are:

$$F_2 = \frac{(2\pi)k}{L^2} dw_1 \wedge dw_2, \qquad F_4 = (2\pi)^3 N \frac{1}{\mathrm{Vol}(X_4)} \Omega_{X_4}, \qquad B_2 = 0, \tag{7.1}$$

where X_4 is a unit-radius Einstein 4-manifold (whose Ricci tensor satisfies $R_{ij} = 3g_{ij}$) and Ω_{X_4} is its volume-form; L is the periodicity of w_i. This flux profile satisfies the flux equations of motion. The corresponding (string frame) metric ansatz is:

$$ds_s^2 = e^{2B(r)}(-dt^2 + dx^2 + dy^2) + e^{-2A(r)} dr^2 + e^{2T(r)}(dw_1^2 + dw_2^2) + e^{2Z(r)} ds_{X_4}^2, \tag{7.2}$$

with a possibly non-constant dilaton $\phi(r)$.

For a scaling solution, $\{A, B, T, Z, \phi\}$ are

$$\begin{aligned} A(r) &= a_1 \log r + a_0, & B(r) &= b_1 \log r + b_0, & T &= t_1 \log r + t_0, \\ Z &= z_1 \log r + z_0, & \phi(r) &= \eta_s \log r + \phi_0. \end{aligned} \tag{7.3}$$

There are one equation of motion from the dilaton and five from the gravity part. For the scaling ansatz, they all reduce to algebraic equations and the solution is easily found to be:

$$\begin{aligned}
A(r) &= \left(1-\frac{1}{3}\eta_s\right)\log(r) - \frac{1}{3}\log\left(\frac{8\pi^3}{3\mathrm{Vol}(X_4)}e^{\phi_0}N\right) - \frac{1}{2}\log\left(\frac{17}{18}\right) - \log\eta_s,\\
B(r) &= \frac{5}{6}\eta_s\log r + b_0,\\
T(r) &= \frac{2}{3}\eta_s\log r + \frac{1}{6}\log\left(\frac{8\pi^3}{3\mathrm{Vol}(X_4)}e^{\phi_0}N\right) + \frac{1}{2}\log\left(\frac{2\pi}{L^2}e^{\phi_0}k\right),\\
Z(r) &= \frac{1}{3}\eta_s\log(r) + \frac{1}{3}\log\left(\frac{8\pi^3}{3\mathrm{Vol}(X_4)}e^{\phi_0}N\right),\\
\phi(r) &= \eta_s\log r + \phi_0.
\end{aligned} \tag{7.4}$$

(η_s, b_0, ϕ_0) are three gauge parameters. η_s corresponds to the gauge freedom of $r \to r^\alpha$, b_0 corresponds to that of rescaling the (t, x, y) directions, and ϕ_0 gives the string coupling at $r = 1$ thus corresponds to rescalings of the r and T^2 directions.

Without loss of generality, we choose

$$\eta_s = 2, \quad b_0 = \frac{1}{3}\log\left(\frac{8\pi^3}{3\mathrm{Vol}(X_4)}e^{\phi_0}N\right) + \frac{1}{2}\log\frac{34}{9}, \quad \phi_0 = \frac{1}{2}\log\left(\frac{L^6N}{3\mathrm{Vol}(X_4)k^3}\right) + \frac{3}{2}\log\frac{34}{9}. \tag{7.5}$$

The solution in the string frame is

$$ds_s^2 = \tilde{R}_s^2\left[r^{\frac{10}{3}}(-dt^2+dx^2+dy^2) + \frac{dr^2}{r^{\frac{2}{3}}} + r^{\frac{8}{3}}(dw_1^2+dw_2^2)\right] + R_s^2 r^{\frac{4}{3}} ds_{X_4}^2, \tag{7.6}$$

with $R_s^2 = \frac{9}{34}\tilde{R}_s^2 = \frac{8\pi^2}{3\mathrm{Vol}(X_4)} \cdot \frac{17}{9} \cdot N\frac{L^2}{k}$. In the Einstein frame, it is

$$ds_E^2 = \tilde{R}^2\left[r^{\frac{7}{3}}(-dt^2+dx^2+dy^2) + \frac{dr^2}{r^{\frac{5}{3}}} + r^{\frac{5}{3}}(dw_1^2+dw_2^2)\right] + R^2 r^{\frac{1}{3}} ds_{X_4}^2, \tag{7.7}$$

with $R^2 = \frac{9}{34}\tilde{R}^2 = (\frac{8\pi^2}{3\mathrm{Vol}(X_4)})^{\frac{3}{4}}(\frac{17\pi^2}{9} \cdot N^3\frac{L^2}{k})^{1/4}$.

This background is no longer scaling invariant. Under the scaling transformation

$$(t, x, y, r, w_1, w_2) \to \left(\lambda t, \lambda x, \lambda y, \frac{r}{\lambda}, \lambda^{\frac{2}{3}}w_1, \lambda^{\frac{2}{3}}w_2\right), \tag{7.8}$$

the metric scales as

$$ds^2 \to \lambda^{-\frac{1}{3}} ds^2, \tag{7.9}$$

instead of staying invariant. This is not surprising since the D4-brane theory is not conformal in the first place.

Generalizing to finite temperature, the corresponding black brane solution (in the Einstein frame) is

$$ds_E^2 = \tilde{R}^2\left[r^{\frac{7}{3}}(-F(r)dt^2+dx^2+dy^2) + \frac{dr^2}{F(r)r^{\frac{5}{3}}} + r^{\frac{5}{3}}(dw_1^2+dw_2^2)\right] + R^2 r^{\frac{1}{3}} ds_{X_4}^2, \tag{7.10}$$

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with

$$F(r) = 1 - \frac{\mu}{r^{\frac{17}{3}}}, \tag{7.11}$$

where μ is the mass parameter of the black hole. Its Hawking temperature is

$$T_H = \frac{17}{12\pi}\mu^{\frac{3}{17}}. \tag{7.12}$$

The Bekenstein-Hawing entropy is

$$S = \gamma \cdot \left(\frac{8\pi^2}{3\mathrm{Vol}(X_4)}\right)^2 \cdot T_H^{\frac{14}{3}} \cdot N^3 \cdot V_2 \cdot \frac{L^4}{k}, \tag{7.13}$$

with $\gamma = 2^{\frac{28}{3}} \cdot 3^{-\frac{7}{3}} \cdot 17^{-\frac{5}{3}} \cdot \pi^{\frac{8}{3}}$ and V_2 is the area in the (x, y) directions.

8 Conclusions and discussions

In this paper, we presented a class of gravity duals of Lifshitz-like fixed points in type IIB supergravity. They represent backgrounds with intersecting D3 and D7-branes and their Einstein frame metrics (2.18) enjoy anisotropic scale invariance. We also extended them to black brane solutions dual to finite temperature theories. Moreover, we showed the existence of solutions which interpolate between our anisotropic solutions in the IR and the familiar $AdS_5 \times X_5$ solutions in the UV. Then the holography asserts that our Lifshitz-like fixed points can be obtained from various four-dimensional CFTs including $\mathcal{N} = 4$ super Yang-Mills via RG flows. These flows are triggered by the relevant and anisotropic perturbation which makes the θ-angle (the coefficient in front of the topological Yang-Mills coupling $F \wedge F$) linearly dependent on one of the space-like coordinates i.e. $\theta \propto w$. When w is compactified, the perturbation induces the Chern-Simons coupling $\int A \wedge F + \frac{2}{3}A^3$. This theory itself seems an intriguing model worth pursuing in a future work, as the equation of motion becomes local in spite of the violation of the Lorentz invariance.

Employing our supergravity solutions we studied the thermal entropy and the entanglement entropy to measure the degrees of freedom of the holographic dual theories. We found characteristic scaling properties in both quantities. We also holographically computed the shear and bulk viscosities. A more general analysis of hydrodynamics with the momentum in the w-direction taken into account was left as an interesting future problem.

Moreover, we performed a perturbative analysis around our solutions and found that a large class of scalar modes obey the Klein-Gordon equation in the curved spacetime which has the expected scaling property. Also we found an unstable scalar mode when the compact manifold X_5 is S^5. Since this unstable mode occurs only for a 'd-wave' spherical harmonics, S^5 might decay into a less symmetric Einstein manifold and be stabilized. Thus we have reason to hope that there exists a class of (Sasaki-)Einstein manifolds with which our scaling solutions become stable. Even the background with $X_5 = S^5$ is still useful at least in capturing qualitative properties of gravity duals of Lifshitz-like fixed points, with unstable modes simply neglected. The construction of manifestly stable and non-dilatonic gravity duals of Lifshitz-like fixed points still remains as a very interesting future problem.

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The three-form fluxes which we assume to be vanishing in our solution might play an important role.

It is also intriguing to generalize our solutions to other values of p and d in (1.2). For example, the simplest case $p = 0$ deserves particular attention. It can be formally obtained from our solution (2.18) by the double Wick rotation $t \to iw$ and $w \to it$. However, the axion field χ and its flux become imaginary-valued therefore the solution is not physical in the ordinary type IIB supergravity. A slightly better example which realizes the case $p = 0$ is a background based on D3-D5 systems, where D5-branes are regarded as the baryons [34]. We cannot get any consistent solution if we restrict to the ordinary type IIB supergravity because the tadpole for the H-flux is generated by F-stings which attach to D5-branes and stretch into the boundary [34]. To construct a solution in this background we need to add the F-string action as an extra term to the type IIB supergravity. Under this slightly unusual assumption, we can indeed find the following black brane solution in the Einstein frame [35]

$$\begin{aligned} ds_E^2 &= -A(\rho)\rho^{14}dt^2 + \rho^2(dx^2+dy^2+dz^2) + \tilde{L}^2\frac{d\rho^2}{\rho^2 A(\rho)} + L^2 d\Omega^2, \\ e^{\phi(\rho)} &= e^{\phi_0}\rho^6, \qquad A(\rho) = 1 - \frac{\mu}{\rho^{10}}, \qquad \tilde{L}^2 = 10L^2, \end{aligned} \tag{8.1}$$

with constant 3-form fluxes H_3, F_3 and the RR 5-form F_5. A derivation of this solution is briefly reviewed in appendix C. At zero temperature, this corresponds to the metric (1.2) with $p = 0$, $d = 3$ and $z = 7$.

Finally, it is also intriguing to apply our backgrounds to realistic condensed matter systems. Our D3-D7 model was originally introduced to model the holographic dual of fractional quantum Hall effects in string theory [22] (refer to [36–38] for other holographic realizations of quantum Hall effects). Therefore one of the future problems is to calculate physical quantities such as finite temperature corrections to the Hall and longitudinal conductivities in this theory. The anisotropic critical points we found in this paper may also be useful to analyze the systems like liquid crystals and some anisotropic spin systems.

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A Perturbative analysis

In this appendix, we consider perturbative fluctuations around the D3-D7 background (2.10), (2.11), (2.17), and (2.18) in the Einstein frame with $X_5 = S^5$, and analyze the stability of it by using the linearized equations of motion. Our analysis below mostly follows [32, 33]. Here we notice that RR 3-form flux F_3 and NSNS 3-form flux H_3 vanish on this background, and their fluctuations do not mix with that of the other fields in the linear order. Thus we start with the action (6.1). The fluctuation of the metric, dilaton, axion, and RR 5-form flux $dC_4 = F_5$ along with their decompositions in terms of spherical harmonics on S^5 are summarized in (6.2), (6.4), and (6.5). Here we consider the fluctuations satisfying the gauge fixing conditions $\nabla^\alpha h_{(\alpha\beta)} = \nabla^\alpha h_{\mu\alpha} = 0$ for the metric and $\nabla^\alpha C_{\alpha IJKL} = 0$ for the RR 5-form flux $F_5 = dC_4$. For simplicity, we denote the background metric, dilaton, axion, and 5-form flux by g_{MN}, ϕ, χ, and F_5, respectively. In this section we define $\tilde{\alpha} = \alpha/R^5 = 4/R$.

A.1 Some conventions

Before writing down linearized equations of motion, we summarize some of our conventions.

We normalized the ϵ tensor on the five-dimensional Lorentzian part M_5 and the S^5 part in (2.18) by

$$\begin{aligned} \epsilon_{01234} &= \sqrt{-g_{M_5}}, & \epsilon^{01234} &= -\frac{1}{\sqrt{-g_{M_5}}}, \\ \epsilon_{56789} &= \sqrt{g_{S^5}}, & \epsilon^{56789} &= \frac{1}{\sqrt{g_{S^5}}}. \end{aligned} \tag{A.1}$$

In this convention,

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5} = -5!\ , \qquad \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = 5!\ . \tag{A.2}$$

We also define the ten-dimensional ϵ tensor by

$$\epsilon_{0123456789} = \sqrt{-g_{10}} = \epsilon_{01234} \cdot \epsilon_{56789}. \tag{A.3}$$

It is also useful to define the Laplacian for M_5 and S^5 by

$$\Box_x = g^{\mu\nu}\nabla_\mu\nabla_\nu, \qquad \Box_y = g^{\alpha\beta}\nabla_\alpha\nabla_\beta. \tag{A.4}$$

A.2 Spherical harmonics on S^5

In this appendix, we decompose the linearized equations of motion for the fluctuations by using the spherical harmonics on S^5. Thus we define Y^I, Y^I_α, $Y^I_{(\alpha\beta)}$, and $Y^I_{[\alpha\beta]}$, which represent scalar, vector,traceless symmetric, and antisymmetric spherical harmonics, respectively. They satisfy the transverse conditions

$$\nabla^\alpha Y^I_\alpha = \nabla^\alpha Y^I_{(\alpha\beta)} = \nabla^\alpha Y^I_{[\alpha\beta]} = 0. \tag{A.5}$$

For the vector spherical harmonics, we define the Hodge-de Rham operator Δ_y by $\Delta_y Y^I_\beta = \Box_y Y^I_\beta - R^\alpha_\beta Y^I_\alpha$. We can define the Hodge-de Rham operators for the other spherical

harmonics in the similar manner and the eigenvalues of Δ_y on the S^5 with the radius R are given by[13] [32]

$$\Delta_y Y^I = \Box_y Y^I = -\frac{k(k+4)}{R^2} Y^I, \qquad (k = 0, 1, 2, \cdots) \tag{A.6}$$

$$\Delta_y Y^I_\alpha = \left(\Box_y - \frac{4}{R^2}\right) Y^I_\alpha = -\frac{(k+1)(k+3)}{R^2} Y^I_\alpha, \qquad (k = 1, 2, \cdots) \tag{A.7}$$

$$\Delta_y Y^I_{(\alpha\beta)} = \left(\Box_y - \frac{10}{R^2}\right) Y^I_{(\alpha\beta)} = -\frac{k^2+4k+8}{R^2} Y^I_{(\alpha\beta)}, \qquad (k = 2, 3, 4, \cdots) \tag{A.8}$$

$$\Delta_y Y^I_{[\alpha\beta]} = \left(\Box_y - \frac{6}{R^2}\right) Y^I_{[\alpha\beta]} = -\frac{(k+2)^2}{R^2} Y^I, \qquad (k = 1, 2, \cdots). \tag{A.9}$$

A.3 Five-form flux equation

Let us first consider the perturbation for the RR 5-form flux F_5. It satisfies the self-duality constraint

$$F_{PQRST} = \frac{1}{5!} \epsilon_{PQRST}{}^{ABCDE} F_{ABCDE}. \tag{A.10}$$

from which the equation of motion $d * F_5 = 0$ follows automatically. By denoting the fluctuation of the 5-form around the background as $\delta F_5 = f_5$, (A.10) in the linear order is written as

$$\begin{aligned} f_{PQRST} = {} & \frac{1}{5!} \epsilon_{PQRST}{}^{ABCDE} f_{ABCDE} + \frac{1}{2 \cdot 5!} h_M{}^M \epsilon_{PQRST}{}^{ABCDE} F_{ABCDE} \\ & - \frac{1}{4!} \epsilon_{PQRSTA_1}{}^{BCDE} F_{A_2BCDE} h^{A_1A_2}. \end{aligned} \tag{A.11}$$

By substituting (6.2) and (6.5) into this equation and then decomposing it by the spherical harmonics on S^5, we obtain five equations:

$$\left[5\nabla_{\mu_1} b^I_{\mu_2\mu_3\mu_4\mu_5} - \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5} \left(\frac{\tilde{\alpha}}{2} h^I - \frac{\tilde{\alpha}}{2} \pi^I + b^I \Box_y\right)\right] Y^I = 0, \tag{A.12}$$

$$\left[4\nabla_{\mu_1} b^I_{\mu_2\mu_3\mu_4} + \epsilon_{\mu_1\mu_2\mu_3\mu_4}{}^{\nu} (b^I_\nu \Box_y - \tilde{\alpha} B^I_\nu)\right] Y^I_\alpha = 0, \tag{A.13}$$

$$\left[b^I_{\mu_1\mu_2\mu_3\mu_4} + \epsilon_{\mu_1\mu_2\mu_3\mu_4}{}^{\nu} \nabla_\nu b^I\right] \nabla_\alpha Y^I = 0, \tag{A.14}$$

$$\nabla_{\mu_1} b^I_{\mu_2\mu_3} Y^I_{[\alpha\beta]} - \frac{1}{12} \epsilon_{\mu_1\mu_2\mu_3}{}^{\nu_1\nu_2} b^I_{\nu_1\nu_2} \epsilon_{\alpha\beta}{}^{\gamma_1\gamma_2\gamma_3} \nabla_{\gamma_1} Y^I_{[\gamma_2\gamma_3]} = 0, \tag{A.15}$$

$$\left[b^I_{\mu_1\mu_2\mu_3} + \epsilon_{\mu_1\mu_2\mu_3}{}^{\nu_1\nu_2} \nabla_{\nu_1} b^I_{\nu_2}\right] \nabla_{[\alpha} Y^I_{\beta]} = 0. \tag{A.16}$$

Here $\tilde{\alpha} = \alpha / R^5 = 4/R$. Now, we can simply solve (A.14) and (A.16) algebraically, assuming $k \geq 1$. Then, we obtain the following three equations from (A.12) and (A.14), (A.13) and (A.16), and (A.15) respectively:

$$\left[(\Box_x + \Box_y) b^I + \frac{\tilde{\alpha}}{2} h^I - \frac{\tilde{\alpha}}{2} \pi^I\right] Y^I = 0, \tag{A.17}$$

$$\left[\Box_x b^I_\mu - \nabla^\nu \nabla_\mu b^I_\nu + \Delta_y b^I_\mu - \tilde{\alpha} B^I_\mu\right] Y^I_\alpha = 0, \tag{A.18}$$

$$\left[3\nabla_{\mu_1} b^I_{\mu_2\mu_3} \mp \frac{i}{2} \epsilon_{\mu_1\mu_2\mu_3}{}^{\nu_1\nu_2} b^I_{\nu_1\nu_2} \sqrt{-\Delta_y}\right] Y^\pm_{[\alpha\beta]} = 0. \tag{A.19}$$

[13] Please distinguish the total angular momentum k of the spherical harmonics of S^5 from the number k of D7-branes.

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A.4 Einstein equations

Let us next consider the perturbation for the Einstein equation. The Einstein equation itself can be obtained from the action (6.1) as

$$R_{MN} - \frac{1}{2}g_{MN}\left(R - \frac{1}{2}e^{2\phi}\partial_P\chi\partial^P\chi - \frac{1}{2}\partial_P\phi\partial^P\phi - \frac{1}{4\cdot 5!}F_{PQRST}F^{PQRST}\right) \tag{A.20}$$
$$-\frac{1}{2}e^{2\phi}\partial_M\chi\partial_N\chi - \frac{1}{2}\partial_M\phi\partial_N\phi - \frac{1}{4\cdot 4!}F_{MPQRS}F_N{}^{PQRS} = 0.$$

By using

$$R - \frac{1}{2}e^{2\phi}\partial_P\chi\partial^P\chi - \frac{1}{2}\partial_P\phi\partial^P\phi = 0. \tag{A.21}$$

and

$$F_{PQRST}F^{PQRST} = 0, \tag{A.22}$$

derived from the trace part of (A.20) and the self-duality condition (A.10) respectively, we can reduce the Einstein equation (A.20) to a simpler form:

$$R_{MN} - \frac{1}{2}e^{2\phi}\partial_M\chi\partial_N\chi - \frac{1}{2}\partial_M\phi\partial_N\phi - \frac{1}{4\cdot 4!}F_{MPQRS}F_N{}^{PQRS} = 0. \tag{A.23}$$

By linearly perturbing (A.23) and decomposing it in terms of the spherical harmonics on S^5, we obtain some equations for the fluctuations around the background. We summarize the resulting equations below.

A.4.1 $\alpha\beta$ components

From (α,β)-components of (A.23), we obtain the following four equations:

$$\left[\Box_x + \Box_y - \frac{2}{R^2}\right]\phi^I Y^I_{(\alpha\beta)} = 0, \tag{A.24}$$
$$\nabla^\mu B^I_\mu \nabla_{(\alpha}Y^I_{\beta)} = 0, \tag{A.25}$$
$$\left[h^I + \frac{3}{5}\pi^I\right]\nabla_{(\alpha}\nabla_{\beta)}Y^I = 0, \tag{A.26}$$
$$\left[\frac{1}{10}\Box_x\pi^I + \frac{4}{25}\pi^I\Box_y + \frac{1}{10}h^I\Box_y + \frac{\tilde\alpha}{2}b_I\Box_y - \frac{\tilde\alpha^2}{5}\pi^I\right]Y^I = 0. \tag{A.27}$$

A.4.2 $\mu\alpha$ components

In a similar way, from (μ,α)-component of (A.23), we obtain the following equations:

$$\left[\frac{1}{2}(\Box_x B^I_\mu - \nabla^\nu\nabla_\mu B^I_\nu) + \frac{1}{2}B^I_\mu\Delta_y + \frac{\tilde\alpha}{4}b^I_\mu\Delta_y + \frac{\tilde\alpha}{4!}\epsilon_\mu{}^{\rho_1\rho_2\rho_3\rho_4}\nabla_{\rho_1}b^I_{\rho_2\rho_3\rho_4}\right]Y^I_\alpha = 0. \tag{A.28}$$
$$\left[-\frac{1}{2}\nabla^\nu h^I_{\nu\mu} + \frac{1}{2}\nabla_\mu h^I + \frac{2}{5}\nabla_\mu\pi^I + \frac{\tilde\alpha}{4}\nabla_\mu b^I + \frac{\tilde\alpha}{96}\epsilon_\mu{}^{\rho_1\rho_2\rho_3\rho_4}b^I_{\rho_1\rho_2\rho_3\rho_4}\right.$$
$$\left. + \delta_{\mu,w}\frac{1}{2}e^{2\phi}\partial_w\chi\,\eta^I + \delta_{\mu,r}\frac{1}{2}\partial_r\phi\,\varphi^I\right]\nabla_\alpha Y^I = 0. \tag{A.29}$$

A.4.3 $\mu\nu$ components

Finally, from (μ,ν)-component of (A.23), we obtain the following equations:

$$\begin{aligned}0 = &-\frac{1}{2}(\Box_x+\Box_y)h_{\mu\nu}-\frac{1}{2}\nabla_\mu\nabla_\nu(h+\pi)+\frac{1}{2}(\nabla_\mu\nabla^\rho h_{\rho\nu}+\nabla_\nu\nabla^\rho h_{\rho\mu})+R_{\mu\rho\sigma\nu}h^{\rho\sigma}\\&+\frac{1}{2}(R^\rho_\mu h_{\nu\rho}+R^\rho_\nu h_{\mu\rho})-e^{2\phi}(\partial_\mu\chi\partial_\nu\chi)\varphi-\frac{e^{2\phi}}{2}(\partial_\mu\chi\partial_\nu\eta+\partial_\nu\chi\partial_\mu\eta)\\&-\frac{1}{2}\partial_\mu\phi\partial_\nu\varphi-\frac{1}{2}\partial_\nu\phi\partial_\mu\varphi-\frac{\tilde\alpha}{48}g_{\mu\nu}\epsilon^{\rho_1\rho_2\rho_3\rho_4\rho_5}\nabla_{\rho_1}b^I_{\rho_2\rho_3\rho_4\rho_5}Y^I-\frac{\tilde\alpha^2}{4}g_{\mu\nu}h+\frac{\tilde\alpha^2}{4}h_{\mu\nu}.\end{aligned}\tag{A.30}$$

A.5 Dilaton and axion equations

Let us then move to the linear perturbation for the dilaton and axion equations of motion. They are given by

$$\partial_M(\sqrt{-g}g^{MN}\partial_N\phi)-\sqrt{-g}e^{2\phi}g^{MN}\partial_M\chi\partial_N\chi=0,\tag{A.31}$$

$$\partial_M(\sqrt{-g}e^{2\phi}g^{MN}\partial_N\chi)=0.\tag{A.32}$$

respectively. When fluctuating around our background, these equations are rewritten as

$$\begin{aligned}&(\Box_x+\Box_y)\varphi+\frac{1}{2}g^{rr}\partial_r\phi\partial_r(h+\pi)-(\partial_r\phi)(\nabla_\mu h^{\mu r})-(\nabla_I\partial_J\phi)h^{IJ}\\&\quad-2\varphi e^{2\phi}g^{ww}(\partial_w\chi)^2+e^{2\phi}h^{ww}(\partial_w\chi)^2-2e^{2\phi}g^{ww}(\partial_w\chi)(\partial_w\eta)=0,\end{aligned}\tag{A.33}$$

$$\begin{aligned}&(\Box_x+\Box_y)\eta+2(\partial_r\phi)(\partial_r\eta)g^{rr}+2g^{ww}(\partial_w\varphi)(\partial_w\chi)+\frac{1}{2}g^{ww}\partial_w(h+\pi)\partial_w\chi\\&\quad-(\nabla_\mu h^{\mu w})(\partial_w\chi)-h^{IJ}(\nabla_I\partial_J\chi)-2h^{rw}(\partial_r\phi)(\partial_w\chi)=0.\end{aligned}\tag{A.34}$$

A.6 Spectrum for decoupled modes

As we derived the linearized equations of motion for the fluctuations in the appendix A.3, A.4, and A.5, we then analyze the spectrum for them. We start with the one for those modes which do not mix with other modes in a complicated way. The analysis turns out to be essentially the same as the case of $AdS_5\times S^5$ [32].

A.6.1 ϕ^I mode from $h_{(\alpha\beta)}$

First we consider the scalar mode $h_{(\alpha\beta)}=\phi^I Y^I_{(\alpha\beta)}$ from the fluctuation of the metric. It obeys the equation of motion (A.24) and, by using (A.8), we obtain

$$\left(\Box_x-\frac{k(k+4)}{R^2}\right)\phi^I=0.\quad(k=2,3,4,\cdots)\tag{A.35}$$

Thus we find its mass $m^2=\frac{k(k+4)}{R^2}$, which obviously satisfies the stability condition.

A.6.2 $b^I_{\mu\nu}$ mode from $C_{\mu\nu\alpha\beta}$

For $b^I_{\mu\nu}$ from the fluctuation of $C_{\mu\nu\alpha\beta}$, (A.19) leads to

$$(\mathrm{Max}_x+\Delta_y)b^I_{\mu\nu}Y^I_{[\alpha\beta]}=0.\tag{A.36}$$

The Maxwell operator Max_x on M_5 is defined by $\mathrm{Max}_x b^I_\mu=\Box_x b_{\mu_1}-\nabla^\nu\nabla_\mu b_\nu$ for the vector b^I_μ and we can generalize the definition for teonsors. Since the mass for this mode is given by $m^2=-\Delta_y=\frac{(k+2)^2}{R^2}$ $(k=1,2,\cdots)$, this mode turns out to be stable.

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A.6.3 b^I_μ and B^I_μ modes from $g_{\mu\alpha}$ and $C_{\mu\alpha\beta\gamma}$

Let us next consider b^I_μ and B^I_μ from the fluctuation of $g_{\mu\alpha}$ and $C_{\mu\alpha\beta\gamma}$ respectively. From (A.18), (A.28), and (A.16), we obtain the equations for these modes as

$$((\mathrm{Max}_x + \Delta_y) b^I_\mu - \tilde{\alpha} B^I_\mu) Y^I_\alpha = 0, \tag{A.37}$$

$$((\mathrm{Max}_x + \Delta_y) B^I_\mu - \frac{\tilde{\alpha}}{2}(Max - \Delta_y) b^I_\mu) Y^I_\alpha = 0. \tag{A.38}$$

or, by denoting in a different expression, as

$$\mathrm{Max}_x \cdot \begin{pmatrix} b^I_\mu \\ B^I_\mu \end{pmatrix} + \begin{pmatrix} \Delta_y & -\tilde{\alpha} \\ \tilde{\alpha}\Delta_y & \Delta_y - \frac{\tilde{\alpha}^2}{2} \end{pmatrix} \cdot \begin{pmatrix} b^I_\mu \\ B^I_\mu \end{pmatrix} = 0. \tag{A.39}$$

The eigenvalues of 2×2 matrix in the second term are

$$-m^2_\pm = \Delta_y - \frac{\tilde{\alpha}^2}{4} \pm \sqrt{\frac{\tilde{\alpha}^4}{16} - \tilde{\alpha}^2 \Delta_y}\ . \tag{A.40}$$

More explicitly, by using (A.7), we get the masses for these vector fields

$$m^2_+ = \frac{k^2 - 1}{R^2}, \qquad m^2_- = \frac{(k+3)(k+5)}{R^2}, \qquad (k = 1, 2, 3, \cdots). \tag{A.41}$$

Therefore we can find that these modes are stable, too.

A.7 Spectrum for mixed modes: h^I, π^I, and b^I

Let us determine the spectrum for the scalar perturbation h^I, π^I, and b^I here. We first assume $k \geq 2$ and then find $h^I = -\frac{3}{5}\pi^I$ from (A.26). Thus we can rewrite (A.17) and (A.27) as follows:

$$\Box_x \cdot \begin{pmatrix} b^I \\ \pi^I \end{pmatrix} + \begin{pmatrix} \Box_y & -\frac{4}{5}\tilde{\alpha} \\ 5\tilde{\alpha}\Box_y & \Box_y - 2\tilde{\alpha}^2 \end{pmatrix} \cdot \begin{pmatrix} b^I \\ \pi^I \end{pmatrix} = 0. \tag{A.42}$$

The matrix in the second term is diagonalized and the eigenvalues are given by

$$-m^2_\pm = \Box_y - \tilde{\alpha}^2 \pm \sqrt{\tilde{\alpha}^4 - 4\tilde{\alpha}^2 \Box_y}, \tag{A.43}$$

or, more explicitly, by

$$m^2_+ = \frac{k^2 - 4k}{R^2}, \tag{A.44}$$

$$m^2_- = \frac{(k+4)(k+8)}{R^2}. \quad . \tag{A.45}$$

Even for $k = 0, 1$, we can see that the expression (A.45) is correct. Thus we find the lowest mass in this mode is $m^2_+ = -\frac{4}{R^2}$ when $k = 2$. This violates the stability bound.

A.8 Spectrum for other modes: $h_{\mu\nu}$, φ, and η

Now, the remaining modes are $h_{\mu\nu}$, φ, and η. Since the analysis of massive modes looks highly complicated, below we would like to consider only zero modes on S^5 of $h_{\mu\nu}$ and φ, η. They are useful to the calculations of the viscosity in section 4. For this reason, we generalize the background to the black brane metric (2.27) and write down the linearized equations for the fluctuation around it. By taking $\mu \to 0$, we can reproduce those for the background metric (2.18).

A.8.1 Zero modes on S^5

We concentrate on the zero modes on S^5 i.e. $k = 0$ modes of the spherical harmonics. Notice that in this case, we have $b = 0$ and $b_{\mu_1\mu_2\mu_3\mu_4}$ is expressed in terms of h and π as

$$5\nabla_{\mu_1} b_{\mu_2\mu_3\mu_4\mu_5} = \frac{2}{R}\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}(h - \pi), \tag{A.46}$$

from (A.10). From (A.27), the mode π satisfies

$$\left(\Box_x - \frac{32}{R^2}\right)\pi = 0. \tag{A.47}$$

It is also useful to define the Weyl shifted metric $H_{\mu\nu}$ by

$$H_{\mu\nu} = h_{\mu\nu} + \frac{1}{3}g_{\mu\nu}\pi, \tag{A.48}$$

which corresponds to the metric perturbation around the five-dimensional background obtained by the reduction of S^5. Then the Einstein equation (A.30) can be rewritten as follows:

$$\delta R^{(5)}_{\mu\nu} + \frac{4}{R^2}H_{\mu\nu} - e^{2\phi}(\partial_\mu\chi\partial_\nu\chi)\varphi - \frac{e^{2\phi}}{2}(\partial_\mu\chi\partial_\nu\eta + \partial_\nu\chi\partial_\mu\eta) - \frac{1}{2}(\partial_\mu\phi\partial_\nu\varphi + \partial_\nu\phi\partial_\mu\varphi) = 0. \tag{A.49}$$

Here $\delta R^{(5)}_{\mu\nu}$ is the perturbation of the purely five dimensional Ricci tensor (neglecting the S^5 contributions) due to the metric perturbation $H_{\mu\nu}$. Notice that when $\mu, \nu \neq w, r$, we obtain the simple Einstein equation $\delta R^{(5)}_{\mu\nu} + \frac{4}{R^2}H_{\mu\nu} = 0$. This simplification is applicable, for example, to the shear modes H_{tx} and H_{xy}.

For the dilaton equation of motion (A.33) with $k = 0$, by using the results

$$\begin{aligned}
\nabla_t\partial_t\phi &= -\frac{2}{3}r^2\left(1 - \frac{\mu}{r^{11/3}}\right)\left(1 + \frac{5\mu}{6r^{11/3}}\right) \equiv f_t, \\
\nabla_x\partial_x\phi &= \nabla_y\partial_y\phi = \frac{2}{3}r^2\left(1 - \frac{\mu}{r^{11/3}}\right) \equiv f_x, \\
\nabla_w\partial_w\phi &= \frac{4}{9}r^{\frac{4}{3}}\left(1 - \frac{\mu}{r^{11/3}}\right) \equiv f_w, \\
\nabla_r\partial_r\phi &= \frac{11\mu}{9r^{17/3}}\left(1 - \frac{\mu}{r^{11/3}}\right)^{-1} \equiv f_w,
\end{aligned} \tag{A.50}$$

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and the Weyl shifted metric $H_{\mu\nu}$, we can rewrite it as follows:

$$\begin{aligned}\Box_x\varphi+\frac{1}{3r}g^{rr}(\partial_r H)-f_t H^{tt}-f_x(H^{xx}+H^{yy})-f_w H^{ww}-f_r H^{rr}\\ +\frac{1}{3}\pi(g^{tt}f_t+2g^{xx}f_x+g^{ww}f_w+g^{rr}f_r)-\frac{2}{3r}\nabla_\mu H^{\mu r}\\ +\beta^2 e^{2\phi}(H^{ww}-2g^{ww}\varphi-\frac{1}{3}\pi g^{ww})-2\beta e^{2\phi}g^{ww}\partial_w\eta=0,\end{aligned}\tag{A.51}$$

where $H\equiv H^\mu_\mu=h+\frac{5}{3}\pi$.

On the other hand, for the axion equation of motion (A.34) with $k=0$, by using the values

$$\nabla_r\partial_w\chi=\nabla_w\partial_r\chi=-\frac{2k}{3Lr},\tag{A.52}$$

it can be rewritten as follows

$$\Box_x\eta+\frac{4}{3r}g^{rr}\partial_r\eta+\beta g^{ww}\partial_w\left(2\varphi+\frac{H}{2}\right)-\beta\nabla_\mu H^{\mu w}=0.\tag{A.53}$$

We can take the radial gauge $H_{\mu r}=0$ and have 5 physical modes for $H_{\mu\nu}$. Then the dilaton and axion equations of motion (A.51) and (A.53) become

$$\begin{aligned}\Box_x\varphi+\frac{1}{3r}g^{rr}(\partial_r H)-f_t H^{tt}-f_x(H^{xx}+H^{yy})\\ -f_w H^{ww}+\frac{1}{3}\pi(g^{tt}f_t+2g^{xx}f_x+g^{ww}f_w+g^{rr}f_r)\\ +\beta^2 e^{2\phi}(H^{ww}-2g^{ww}\varphi-\frac{1}{3}\pi g^{ww})-2\beta e^{2\phi}g^{ww}\partial_w\eta=0,\end{aligned}\tag{A.54}$$

and

$$\Box_x\eta+\frac{4}{3r}g^{rr}\partial_r\eta+\beta g^{ww}\partial_w\left(2\varphi+\frac{H}{2}\right)-\beta\nabla_\mu H^{\mu w}=0.\tag{A.55}$$

In summary, we need to solve the zero mode equations of motion (A.46), (A.47), (A.49), (A.54), and (A.55) to find the variables $b_{\mu\nu\rho\sigma}$, π, φ, η, and $H_{\mu\nu}$.

B Linearized equations for the shear and sound channel

In this appendix we summarize the linearized equations for the shear and sound channel. They are useful to derive the differential equations for the gauge invariant combinations in section 4.

For this purpose, we consider the equations (A.46), (A.47), (A.49), (A.54), and (A.55) derived in the appendix A.8.1. Then we substitute the fluctuation corresponding to the shear channel (4.1) or sound channel (4.3) into them and then derive explicit expressions for the linearized equations of motion. We summarize the resulting equations below.

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B.1 Shear channel

By considering the fluctuations H_{tx} and H_{xy} which correspond to the shear channel in the radial gauge $H_{\mu r} = 0$, we obtain the equations for them

$$H''_{tx} + \ln'(c_x^{14/3})H'_{tx} - \frac{q}{c_x^2 c_t^2}(qH_{tx} + \omega H_{xy}) = 0, \tag{B.1}$$

$$H''_{xy} + \ln'(c_t^2 c_x^{8/3})H'_{xy} + \frac{\omega}{c_t^4}(qH_{tx} + \omega H_{xy}) = 0, \tag{B.2}$$

$$qH'_{xy} + \omega\frac{c_x^2}{c_t^2}H'_{tx} = 0, \tag{B.3}$$

from (t,x)-, (x,y)-, and (x,r)-component of (A.49), respectively. We can directly show that two of the three equations are independent.

B.2 Sound channel

Let us next consider the sound mode. The corresponding fluctuations are $H_{tt}, H_{ty}, H_{xx}, H_{yy}, H_{ww}, \varphi, \pi$ and $f_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ in the radial gauge. In this case the linearized equations (A.47), (A.49), and (A.54) are summarized as

$$H''_{tt} + \ln'\left(c_t^3 c_x^{8/3}\right)H'_{tt} - \ln'(c_t)\left(H'_{yy} + H'_{ii}\right) - \frac{1}{c_t^2}\left(\frac{\omega^2}{c_t^2}(H_{yy} + H_{ii}) + \frac{2q\omega}{c_t^2}H_{ty} + \frac{q^2}{c_x^2}H_{tt}\right) = 0, \tag{B.4}$$

$$H''_{xx} + \ln'\left(c_t^2 c_x^{8/3}\right)H'_{xx} - \ln'(c_x)\left(H'_{tt} - H'_{yy} - H'_{ii}\right) + \frac{1}{c_t^2}\left(\frac{\omega^2}{c_t^2} - \frac{q^2}{c_x^2}\right)H_{xx} = 0, \tag{B.5}$$

$$H''_{yy} + \ln'(c_t^2 c_x^{8/3})H'_{yy} - \ln'(c_x)(H'_{tt} - H'_{yy} - H'_{ii}) + \frac{1}{c_t^2}\left(\frac{\omega^2}{c_t^2}H_{yy} + \frac{2q\omega}{c_t^2}H_{ty} + \frac{q^2}{c_x^2}(H_{tt} - H_{ii})\right) = 0, \tag{B.6}$$

$$H''_{ww} + \ln'(c_t^2 c_x^{8/3})H'_{ww} - \ln'(c_x^{2/3})(H'_{tt} - H'_{yy} - H'_{ii}) + \frac{1}{c_t^2}\left(\frac{\omega^2}{c_t^2} - \frac{q^2}{c_x^2}\right)H_{ww} + \frac{44}{9c_t^2}\left(\varphi - \frac{1}{2}H_{ww}\right) = 0, \tag{B.7}$$

$$(H''_{tt} - H''_{yy} - H''_{ii}) + \ln'(c_t c_x^2)(H'_{tt} - H'_{yy} - H'_{ii}) + \ln'(c_t^2 c_x^{-2})H'_{tt} - \ln' c_x^{4/3}\left(\varphi' - \frac{1}{2}H'_{ww}\right) = 0, \tag{B.8}$$

$$H''_{ty} + \ln'(c_x^{14/3})H'_{ty} + \frac{q\omega}{c_t^2 c_x^2}H_{ii} = 0, \tag{B.9}$$

$$(H'_{yy}+H'_{ii})+\frac{q}{\omega}H'_{tz}+\ln'(c_x c_t^{-1})(H_{yy}+H_{ii})$$
$$+\frac{q}{\omega}\ln'(c_x^2 c_t^{-2})H_{ty}+\ln(c_x^{2/3})\left(\varphi-\frac{1}{2}H_{ww}\right)=0, \quad \text{(B.10)}$$

$$(H'_{tt}-H'_{ii})+\frac{\omega}{q}\frac{c_x^2}{c_t^2}H'_{tz}-\ln'(c_x c_t^{-1})H_{tt}-\ln'(c_x^{2/3})\left(\varphi-\frac{1}{2}H_{ww}\right)=0, \quad \text{(B.11)}$$

$$\varphi''+\ln'(c_t^2 c_x^{8/3})\varphi'-\ln'(c_x^{1/3})(H'_{tt}-H'_{yy}-H'_{ii})+\frac{1}{c_t^2}\left(\frac{\omega^2}{c_t^2}-\frac{q^2}{c_x^2}\right)\varphi$$
$$-\frac{44}{9c_t^2}\left(\varphi-\frac{1}{2}H_{ww}\right)+\frac{11(\mu^3-2\mu^2 r^{11/3}+2r^{11})}{27r^{28/3}(r^{11/3}-\mu)}\pi=0,$$
$$\pi''+\ln'(c_t^2 c_x^{8/3})\pi'+\frac{1}{c_t^2}\left(\frac{\omega^2}{c_t^2}-\frac{q^2}{c_x^2}\right)\pi-\frac{32}{c_t^2}\pi=0, \quad \text{(B.12)}$$

where $H_{ii}=H_{xx}+H_{ww}$.

C D3-D5 scaling solution with F-string sources

Here we briefly review the D3-D5 scaling solution in type IIB supergravity with F-string sources [35]. This solution can be regarded as a back-reacted solution dual to a homogeneous baryon condensation in $\mathcal{N}=4$ super Yang-Mills. Each baryon (= D5-brane) situated near the horizon $r=0$ carries N F-strings which extend into the boundary $r=\infty$ due to the string creation [34]. Below, we work in the Einstein frame and the supergravity action is obtained by rewriting the one in the string frame (2.1) by the Einstein frame metric $G^E_{MN}=e^{-\phi/2}G^{\text{string}}_{MN}$.

For this brane setup, it is appropriate to use the following ansatz for the metric

$$ds^2=-e^{u(r)}A(r)dt^2+e^{b(r)}\left(\sum_{i=1}^{3}(dx_i)^2\right)+e^{c(r)}(A^{-1}(r)dr^2+r^2ds^2_{X_5}), \quad \text{(C.1)}$$

where X_5 represents a Einstein manifold with the same Ricci curvature as the unit radius S^5. For fluxes, the ansatz is

$$F_3=e^{\frac{3}{2}b(r)}\cdot h(r)\cdot dx_1\wedge dx_2\wedge dx_3,$$
$$F_5=r^5 e^{\frac{5}{2}c(r)}f(r)(\Omega_{X_5}+*\Omega_{X_5}), \quad \text{(C.2)}$$

where Ω_{X_5} is the volume form on X_5. We also assume that the dilaton is dependent on the radial coordinate r only. If we treat this solution within supergravity, we will have a tadpole of the NSNS 3-form flux H_3 as is clear from the equation of motion in the presence of the Chern-Simons term

$$d(e^{-\phi}*H_3)=F_3\wedge F_5, \quad \text{(C.3)}$$

where F_3 and F_5 are sourced by the D5 and D3-branes, respectively. Thus we cannot construct consistent supergravity solutions under the assumption of symmetry of spacetime implied by (C.1).

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To resolve this problem, we notice that the effective F-string charges are generated from the D3-brane and D5-brane charges via the Chern-Simons term

$$S_{\rm CS} = -\frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3. \tag{C.4}$$

Therefore, to resolve this problem, we treat the created F-strings as the probe action and add it to the supergravity action. The probe action for a single F-string in the Einstein frame is given by

$$S_{\rm string} = -\frac{1}{2\pi}\int d\tau d\sigma e^{\frac{\phi}{2}}\sqrt{-G_E} + \frac{1}{2\pi}\int d\tau d\sigma B_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu. \tag{C.5}$$

By identifying $\tau = t$ and $\sigma = r$ and taking the sum over infinitely many F-strings, we obtain the probe action for the created F-strings

$$\sum_i S_{{\rm string}(i)} = \int dx_1 dx_2 dx_3 \Omega_{X_5}\ \rho\ S_{\rm string}, \tag{C.6}$$

where i labels i-th F-string and the density of F-strings ρ is assumed to be constant. In the following, we solve the equations of motion derived from the supergravity action with this probe action added.

C.1 Equation of motions

The Bianchi identity for F_3 and the equation of motion for F_5 are written as

$$\partial_r(r^5 e^{\frac{5}{2}c(r)} f(r)) = 0, \qquad \partial_r(e^{\frac{3}{2}b(r)}h(r)) = 0. \tag{C.7}$$

Here we notice that the equation of motion for F_3 is automatically satisfied for the ansatz introduced above. From these equations, we can define the constants F and H as follows:

$$f(r) = F\cdot r^{-5}e^{-\frac{5}{2}c(r)}, \qquad h(r) = H\cdot e^{-\frac{3}{2}b(r)}. \tag{C.8}$$

Then, from (C.3), the density of F-strings turns out to be

$$\rho = \frac{\pi F H}{2\kappa_{10}^2}. \tag{C.9}$$

To derive the dilaton equation of motion, we notice that the F-string action now looks like

$$S_{F-string} = -\frac{1}{2\pi}\int dt dr e^{\frac{\phi}{2}}\sqrt{-G_E} = -\frac{1}{2\pi}\int dt dr e^{\frac{\phi}{2}+\frac{c}{2}+\frac{u}{2}}. \tag{C.10}$$

Thus the equation of motion becomes

$$\partial_r(\phi'(r)A(r)e^{2c(r)+\frac{3}{2}b(r)+\frac{u(r)}{2}}r^5) = \frac{FH}{4}e^{\frac{\phi(r)}{2}+\frac{c(r)}{2}+\frac{u(r)}{2}} + \frac{1}{2}r^5 e^{\phi(r)+\frac{3}{2}b(r)+3c(r)+\frac{u(r)}{2}}h(r)^2. \tag{C.11}$$

We can also derive the Einstein equations for the type IIB supergravity with the F-string action. Combined with (C.11), we can summarize the equations of motion as follows:

$$[A(\log A+u)'r^5e^{\frac{3}{2}b+2c+\frac{1}{2}u}]' = \frac{3}{8}FHe^{\frac{1}{2}\phi+\frac{1}{2}c+\frac{1}{2}u}+\frac{1}{2}F^2r^{-5}e^{-2c+\frac{3}{2}b+\frac{1}{2}u} + \frac{1}{4}H^2r^5e^{\phi+3c-\frac{3}{2}b+\frac{1}{2}u}, \tag{C.12}$$

$$[b'Ar^5e^{\frac{3}{2}b+2c+\frac{1}{2}u}]' = -\frac{1}{8}FHe^{\frac{1}{2}\phi+\frac{1}{2}c+\frac{1}{2}u}+\frac{1}{2}F^2r^{-5}e^{-2c+\frac{3}{2}b+\frac{1}{2}u} - \frac{3}{4}H^2r^5e^{\phi+3c-\frac{3}{2}b+\frac{1}{2}u}, \tag{C.13}$$

$$[(c+2\log r)'Ar^5e^{\frac{3}{2}b+2c+\frac{1}{2}u}]' = -\frac{1}{8}FHe^{\frac{1}{2}\phi+\frac{1}{2}c+\frac{1}{2}u}-\frac{1}{2}F^2r^{-5}e^{-2c+\frac{3}{2}b+\frac{1}{2}u} + \frac{1}{4}H^2r^5e^{\phi+3c-\frac{3}{2}b+\frac{1}{2}u}+8r^3e^{2c+\frac{3}{2}b+\frac{1}{2}u}, \tag{C.14}$$

$$[(\log A+u-b-2\phi)'Ar^5e^{\frac{3}{2}b+2c+\frac{1}{2}u}]' = 0, \tag{C.15}$$

$$2(\phi')^2+3(b')^2+10\frac{c'}{r}-3b'c'+6b''+10c''-3b'u'-5c'u'-10\frac{u'}{r}=0. \tag{C.16}$$

It is also useful to derive the following equation from a linear combination of (C.12), (C.13), and (C.14)

$$[(\log A+u+b+2c+4\log r)'Ar^5e^{\frac{3}{2}b+2c+\frac{1}{2}u}]' = 16r^3e^{2c+\frac{3}{2}b+\frac{1}{2}u}. \tag{C.17}$$

In the above discussion, we have derived five equation of motion for five variables $A(r)$, $u(r)$, $b(r)$, $c(r)$, $\phi(r)$. However, we can eliminate one of them, say $u(r)$, by the diffeomorphism $r\to\tilde{r}=\tilde{r}(r)$. In this sense, the independent degree of freedom under the symmetry ansatz is the four variables. Thus we should show the five equations of motion are not over constrained. Indeed we can show the following identity from the four equations of motion (C.12), (C.13), (C.14), and (C.15)

$$\left[r^{10}A(r)^2e^{3b(r)+4c(r)+u(r)}\cdot E_{\text{constraint}}(r)\right]'=0, \tag{C.18}$$

where $E_{\text{constraint}}(r)$ is the left-hand side of (C.16). This guarantees that the fifth equation of motion $E_{\text{constraint}}(r)=0$ is satisfied if it is vanishing at any particular value of r.

C.2 Scaling solutions

Let us assume the following simple scaling profile for the unknown functions:

$$u(r)=u_1\log r+u_0,\quad b(r)=b_1\log r+b_0,\quad c(r)=c_1\log r+c_0,\quad \phi(r)=\phi_1\log r+\phi_0. \tag{C.19}$$

At first we also assume $A(r)=1$, which corresponds to the extremal case.

Using the three equations of motion (C.14), (C.15), and (C.16), we find that the following coefficients u_1, b_1, c_1 and ϕ_1 satisfy equations of motion

$$u_1=7\sqrt{\frac{2}{5}},\quad b_1=\sqrt{\frac{2}{5}},\quad c_1=-2,\quad \phi_1=3\sqrt{\frac{2}{5}}. \tag{C.20}$$

By substituting (C.19) with (C.20) into (C.12), (C.13) and (C.14), we can find four solutions for the pair $(F,\ H)$ in terms of u_0, b_0, c_0 and ϕ_0 as follows:

$$F = \pm e^{2c_0} \cdot \frac{\sqrt{83 - \sqrt{1081}}}{\sqrt{3}},$$
$$H = \frac{\sqrt{1081} - 5}{44} \cdot e^{\frac{3}{2}b_0 - \frac{5}{2}c_0 - \frac{1}{2}\phi_0} \cdot F, \tag{C.21}$$

and

$$F = \pm e^{2c_0} \cdot \frac{\sqrt{83 + \sqrt{1081}}}{\sqrt{3}},$$
$$H = -\frac{\sqrt{1081} + 5}{44} \cdot e^{\frac{3}{2}b_0 - \frac{5}{2}c_0 - \frac{1}{2}\phi_0} \cdot F. \tag{C.22}$$

Thus, in order to realize the fluxes with arbitrary values and sign, we have only to choose one of the four solutions and tune c_0 and ϕ_0 appropriately. We set $u_0 = b_0 = 0$ below.

As we constructed the extremal solution, generalization to a black brane solution is straightforward by considering more general functions $A(r)$. Let us again assume the ansatz (C.19) with the values (C.20). Then it immediately follows that all equation of motion are satisfied only if

$$A(r) = 1 - \frac{M}{r^{\sqrt{10}}}, \tag{C.23}$$

where M is an arbitrary constant related to the ADM mass of the black brane. We also notice that the other profiles are the same as the extremal solutions.

In summary we obtain the following metric in the Einstein frame and the dilaton

$$ds^2 = -r^{u_1} A(r) dt^2 + r^{b_1} d\vec{x}^2 + e^{c_0} \left(\frac{dr^2}{r^2 A(r)} + (d\Omega_5)^2 \right), \qquad e^{\phi(r) - \phi_0} = r^{\phi_1}. \tag{C.24}$$

and, after the redefinition of radial coordinate $\rho = r^{\frac{1}{\sqrt{10}}}$, we reach the expression (8.1).

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