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Non-BPS black rings and black holes in Taub-NUT

Iosif Bena,[a] Gianguido Dall'Agata,[b] Stefano Giusto,[a] Clément Ruef[a] and Nicholas P. Warner[c]

[a]*Institut de Physique Théorique, CEA Saclay, 91191 Gif sur Yvette, France*
[b]*Dipartimento di Fisica "Galileo Galilei" & INFN, Sezione di Padova, Università di Padova, Via Marzolo 8, 35131 Padova, Italy*
[c]*Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089, U.S.A.*
E-mail: iosif.bena@cea.fr, gianguido.dallagata@pd.infn.it, stefano.giusto@cea.fr, clement.ruef@cea.fr, warner@usc.edu

Abstract: We solve the recently-proposed equations describing non-BPS extremal multi-center configurations, and construct explicit solutions describing non-supersymmetric extremal black rings in Taub-NUT, as well as the seed solution for the most general extremal non-BPS under-rotating black hole in four dimensions. We also find solutions that contain both a black hole and a black ring, which descend to four-dimensional extremal non-BPS two-center black holes with generic charges.

Keywords: Black Holes in String Theory, Black Holes, String Duality

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1 Introduction

Supersymmetric solutions that preserve the same supersymmetries as three-charge black holes or black rings in five dimensions are well understood and can be written in terms of three self-dual two-forms describing magnetic fluxes on a hyper-Kähler four-dimensional base, three warp factors, sourced either by the two-forms or by singular sources, and an angular momentum one-form [1]. These solutions to M-theory, or type II string theory, can be recast in terms of BPS solutions of five-dimensional $U(1)^3$ ungauged supergravity and can also be easily generalized to $U(1)^N$ supergravities [2]. If the four-dimensional hyper-Kähler base space is Gibbons-Hawking (or Taub-NUT), the two-forms, the warp factors and the angular momentum can be determined entirely in terms of eight $(2N + 2)$ harmonic functions [3–5], and descend to four-dimensional BPS multi-centered black hole configurations [6].

Implicit in the construction of the supersymmetric solutions is the choice of an orientation for the hyper-Kähler four-dimensional base: The curvature tensor can be arranged to be either self-dual or anti-self dual. For supersymmetry it is crucial that the Riemann curvature of this base has the same duality as the three magnetic two-forms: They must all be self-dual or anti-self-dual. The difference in choice merely amounts to an overall reversal of orientation and is usually neglected. However, there has been a very nice recent observation [7] that one can obtain extremal non-supersymmetric solutions of the supergravity equations of motion by flipping the relative dualities of the hyper-Kähler base and the magnetic two-forms.[1] This means that supersymmetries are "locally preserved" by the sources but globally broken by the incompatible holonomy of the background metric on the base.

A simple example of this, and a very useful tool in our analysis, is to start by noting that there are two ways of writing the flat metric on $\mathbb{R}^4$ in Gibbons-Hawking (GH) form: One that looks self-dual and one that looks anti-self-dual. While this distinction is a coordinate artifact for $\mathbb{R}^4$ (because the curvature is trivial), one of the choices will break supersymmetry in more general backgrounds. Indeed, it is fairly straightforward to adapt what appears as an orientation reversal in $\mathbb{R}^4$ to a highly non-trivial, supersymmetry-breaking transformation in Taub-NUT. Thus, given an asymptotically $\mathbb{R}^4$ solution, one can find two ways of extending it to an asymptotically Taub-NUT solution: one that preserves the supersymmetry and one that does not.

The basic technique is also easily understood in terms of the underlying brane construction. For example, an asymptotically five-dimensional black ring solution (with a flat $\mathbb{R}^4$ base) preserves the four supersymmetries respected by its three constituent electric M2 branes. When one replaces the $\mathbb{R}^4$ base by a Taub-NUT space and considers the solution from the IIA perspective, the M2 branes descend to D2 branes while the tip of Taub-NUT descends to a D6 brane. In the BPS embedding, the four Killing spinors preserved by the three sets of D2 branes are the same as those of the D6 brane, and thus the solution is supersymmetric. In the non-BPS embedding the D6 brane has opposite orientation, and hence it does not preserve any of the four Killing spinors of the D2 branes.

An interesting corollary of this D-brane picture is that five-dimensional objects that preserve the same eight Killing spinors as two sets of M2 branes, will still be supersymmetric when embedded in self-dual or anti-self-dual Taub-NUT. Indeed, if only two sets of D2 branes are present, the D6 brane will be mutually BPS with them irrespective of its orientation. Hence, a two-charge supertube embedded in Taub-NUT in the "duality-matched" embedding [8] or in the "duality-flipped" embedding [7] will still be supersymmetric. We will see in section 4 the rather unexpected fashion in which this is realized.

Our purpose in this paper is to give a general algorithm for constructing the most general two-center solution of the "almost BPS equations" presented in [7]. The most obvious solution to look for is a "non-BPS" two-charge supertube in Taub-NUT. However, as we explained above, this solution turns out to be identical to that of the BPS supertube in Taub-NUT. The next obvious solution is the non-BPS three-charge three-dipole charge black ring in Taub-NUT, which we construct in section 3.

Because the new non-BPS black-ring solution becomes identical to the BPS solution both in $\mathbb{R}^4$, and in $\mathbb{R}^3 \times S^1$, it is possible to recycle many of the pieces of the BPS three-

[1] We will consistently fix our hyper-Kähler base to be self-dual (i.e. with self-dual curvature) and so this new prescription amounts to starting with anti-self-dual magnetic two-forms and solving the supersymmetric BPS equations with flipped dualities.

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charge three-dipole charge black ring solutions in $\mathbb{R}^3 \times S^1$ and $\mathbb{R}^4$ [1, 4, 9–11], and the only new ingredient is to solve one non-trivial equation for a piece of the rotation vector. The full solution is again generated from several harmonic functions, determining the M2 charges, M5 dipole charges and angular momentum of the black ring. However, these harmonic functions enter the solution very differently than for BPS black rings in Taub-NUT [5, 12, 13]. Furthermore, in order for the solution to be free of closed timelike curves (CTC's), the harmonic function that determines the angular momentum must have both a $1/r$ source at the tip of Taub-NUT, as well as a "dipole" piece of the form $\cos\theta/r^2$ centered at the black ring location. No such terms appear in the BPS ring solution, and the necessity of their presence is far from obvious without a careful construction of the full solution.

Since these new solutions "locally preserve" supersymmetry but break it globally, one expects that local properties should be the same as those of the BPS counterparts. Indeed, we find that the near-horizon geometry of the non-BPS extremal ring is identical to that of its BPS cousin, and its entropy is given by the $E_{7(7)}$ quartic invariant as a function of its charges [14]. On the other hand, the location, or "radius" of the ring in Taub-NUT is a more global property and is generically different for BPS and non-BPS solutions. For both BPS and non-BPS solutions the location is determined by the requirement that there be no Dirac-Misner strings, but the source terms that can give rise to such strings are very different for BPS and non-BPS solutions. We also show, in section 4, that when black rings are reduced to two-charge supertubes, the BPS and non-BPS solutions coincide, and the two radii become equal.

As observed in [7], the almost BPS equations can be used to re-derive the non-rotating extremal non-BPS four-dimensional single-center black hole obtained in [15, 16]. However their power is much greater, even for single-center solutions: by adding to the angular momentum harmonic function a "dipole" piece of the form $\cos\theta/r^2$ centered at the black hole location, we can give this black hole rotation. The resulting solution is a new rotating extremal non-BPS solution in four dimensions. This solution has five (four-dimensional) quantized charges (corresponding to D6, D0 and three sets of D2 branes) as well as angular momentum.[2]

For particular values of the charges and moduli one can show that this black hole can be related by dualities to the "slowly-rotating" or "ergo-free" extremal limit[3] of the D6-D0 (Rasheed-Larsen) black hole [19] or its D6-D2-D2-D0 dual [20]. However, our solution is much more general, as it can have arbitrary D6-D2-D2-D2-D0 charges. Hence this solution is the seed solution for the most generic extremal under-rotating black hole of the *STU* model and of $\mathcal{N} = 8$ supergravity in four dimensions.

Using our method it also is quite straightforward to find a solution that contains both this generic rotating black hole and a black ring. The presence of the black hole adds an extra source term to the black ring warp factor, and three more terms to the angular momentum vector. It also modifies the black ring radius relation, without changing the near-horizon geometry of either the ring or the hole.

[2]It is also trivial to introduce Wilson lines for the magnetic gauge fields, because they do not affect the rest of the solution in any way (unlike for BPS solutions).

[3]See, for example, [17] or [18] for a discussion of the two extremal limits of this black hole.

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The non-BPS black ring with a black hole in the middle can be compactified to four dimensions, to give a two-center non-BPS solution, with a non-trivial angular momentum. The black hole at one of the centers has five charges (D6-D2-D2-D2-D0), and the black hole at the other center has seven charges (D4-D4-D4-D2-D2-D2-D0). Since we find the solution for arbitrary moduli, this system can be dualized into one where each of the two black holes has D6-D4-D2-D0 charges and can probably be identified to the most generic extremal two-centered solution of the *STU* model.

Before beginning, it is important to note that there exists a rather large body of work on constructing extremal black holes in four-dimensional supergravity, that started from the observation of [21] that the second-order equations underlying these solutions can be factorized as products of easier-to-solve first-order equations.[4] So far, the single-center solutions obtained in this way appear to be captured in the ansatz in [7]. On the other hand there exists a rather complementary body of work on embedding non-extremal five-dimensional solutions in Taub-NUT, that began with [25, 26] and resulted in the recent construction of non-extremal black rings in Taub-NUT [27]. It would be interesting to see if one can construct our extremal non-BPS ring using either of these approaches, and whether, upon extending these approaches to construct our solution, one could access to a larger set of solutions than those contained in the ansatz of [7].

In section 2 we review the ansatz of [7] for finding non-BPS solutions. In section 3 we outline our solution-finding technique by constructing a three-charge three-dipole non-BPS black ring in Taub-NUT. We also analyze its charges, mass and near-horizon limit. In section 4 we construct a non-BPS supertube in Taub-NUT, and show that this is identical to a BPS supertube. In section 5 we construct a five-charge rotating black hole, which is the seed solution for the most general extremal non-BPS under-rotating black hole in four dimensions. We also discuss its relation to the Rasheed-Larsen solution. In section 6 we construct a solution that includes both a rotating black hole at the tip of Taub NUT and a black ring; this solution descends to a two-centered non-BPS black hole solution in four dimensions. We conclude in section 7.

2 "Almost BPS" solutions

BPS solutions of eleven-dimensional supergravity carrying M2 and M5 charges are of the form

$$\begin{aligned} ds^2 &= -(Z_1 Z_2 Z_3)^{-2/3}(dt+k)^2 + (Z_1 Z_2 Z_3)^{1/3} ds_4^2 \\ &\quad + \left(\frac{Z_2 Z_3}{Z_1^2}\right)^{1/3}(dx_1^2+dx_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2}\right)^{1/3}(dx_3^2+dx_4^2) + \left(\frac{Z_1 Z_2}{Z_3^2}\right)^{1/3}(dx_5^2+dx_6^2) \end{aligned} \tag{2.1}$$

$$C^{(3)} = \left(a_1 - \frac{dt+k}{Z_1}\right)\wedge dx_1 \wedge dx_2 + \left(a_2 - \frac{dt+k}{Z_2}\right)\wedge dx_3 \wedge dx_4 + \left(a_3 - \frac{dt+k}{Z_3}\right)\wedge dx_5 \wedge dx_6\,, \tag{2.2}$$

[4]See, for example, [15, 16, 22–24].

where ds_4^2 is a hyper-Kähler four-dimensional metric. Defining the "dipole" field strengths as

$$\Theta_I = da_I \ , \qquad I = 1, 2, 3 \, , \tag{2.3}$$

the equations following from supersymmetry for a self-dual hyper-Kähler base metric are:[5]

$$\begin{aligned} \Theta_I \; &= \; *_4 \Theta_I \, , &(2.4)\\ d *_4 dZ_I \; &= \; \frac{|\epsilon_{IJK}|}{2} \Theta_J \wedge \Theta_K \, , &(2.5)\\ dk + *_4 dk \; &= \; Z_I \Theta_I \, , &(2.6) \end{aligned}$$

where $*_4$ is the Hodge duality operation performed with the metric ds_4^2. The foregoing equations also govern the solutions of arbitrary $\mathrm{U}(1)^N$ ungauged supergravities in five dimensions [2] if one replaces the $|\epsilon_{IJK}|$ by the corresponding triple intersection number C_{IJK}.

It was observed in [7] that a class of extremal solutions of the equations of motion is obtained by reversing the duality of the Θ_I and of k relative to the duality of the curvature of the four-dimensional base. That is, one preserves the metric, ds_4^2, and the duality of its Riemann tensor but flips $*_4 \to -*_4$ in (2.4)–(2.6):

$$\begin{aligned} \Theta_I \; &= \; - *_4 \Theta_I &(2.7)\\ d *_4 dZ_I \; &= \; \frac{C_{IJK}}{2} \Theta_J \wedge \Theta_K &(2.8)\\ dk - *_4 dk \; &= \; Z_I \Theta_I \, . &(2.9) \end{aligned}$$

When the base metric ds_4^2 is flat $\mathbb{R}^4$, the flip of orientation can be re-written as a change of coordinates, and solutions to equations (2.7)–(2.9) are still BPS. When ds_4^2 is not flat, as in Taub-NUT space, equations (2.7)–(2.9) define, in general, non-BPS solutions, which were named "almost BPS" in [7].

2.1 Gibbons-Hawking base

As with the BPS solutions, equations (2.7)–(2.9) are easier to solve if one specializes to Gibbons-Hawking base metrics:

$$ds_4^2 = V^{-1}(d\psi + \vec{A})^2 + V ds_3^2 \, , \qquad *_3 d\vec{A} = dV \, . \tag{2.10}$$

We will also only look for solutions that are invariant under ψ-translations.

The four-dimensional geometry is encoded in the function V, which is harmonic with respect to the flat three-dimensional euclidean metric ds_3^2. The Hodge star operation in $\mathbb{R}^3$ is denoted by $*_3$ and one-forms on $\mathbb{R}^3$ are denoted by a vector superscript. In general, for a GH base one can take $*_3 d\vec{A} = \pm dV$ and this leads to self-dual or anti-self-dual Riemann tensors. The choice in (2.10) means we are choosing a self-dual curvature.

The one-form potentials for the anti-self dual field strengths have the form:

$$a_I = K_I(d\psi + \vec{A}) + \vec{a}_I \, , \qquad *_3 d\vec{a}_I = V dK_I - K_I dV \, , \tag{2.11}$$

[5] If one uses a hyper-Kähler base with an anti-self-dual curvature then the dualities in (2.4)–(2.6) are flipped to the form (2.7)–(2.9).

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where K_I is a harmonic function on $\mathbb{R}^3$. Such a_I's thus provide the general solution to eq. (2.7).

Using this result in eq. (2.8), one finds that the warp factors Z_I must satisfy

$$d *_3 dZ_I = \frac{1}{2} C_{IJK} V d *_3 d(K_J K_K) . \tag{2.12}$$

Unlike the BPS solution, this equation does not, in general, admit a closed form solution written solely in terms of the functions V and K_I. However, in practice, it is still relatively straightforward to obtain exact solutions for Z_I.

Expanding k along the fiber and base of the Gibbons-Hawking space:

$$k \;=\; \mu(d\psi + \vec{A}) + \vec{\omega} \,, \tag{2.13}$$

one can reduce (2.9) to:

$$d(V\mu) \;+\; *_3 \, d\vec{\omega} \;=\; V Z_I dK_I \,. \tag{2.14}$$

Acting with $d*_3$ one obtains the following equation for μ:

$$d *_3 d(V\mu) \;=\; d(V Z_I) *_3 \wedge dK_I \,. \tag{2.15}$$

This equation is the integrability condition for (2.14). Again, one does not seem to be able to find a simple, general solution to this equation, but we will obtain particular solutions in later sections.

3 Non-BPS extremal black ring

In this section we derive one of the main results of this paper: an exact solution representing a non-BPS extremal regular black ring in Taub-NUT space. This space is described by the Gibbons-Hawking potential

$$V = h \;+\; \frac{Q_6}{r} \quad \Rightarrow \quad \vec{A} \;=\; Q_6 \cos\theta d\phi \,. \tag{3.1}$$

We have introduced a generic constant h in V to facilitate comparison with the flat space ($\mathbb{R}^4$) limit, which corresponds to taking $h = 0$. Taking $Q_6 = 0$ corresponds to the infinite radius limit of the black ring, in which the base reduces to $\mathbb{R}^3 \times S^1$. In both of these limits the non-BPS solution must reduce to the known BPS black ring solution.

3.1 Solving the equations

We take the position of the black ring in $\mathbb{R}^3$ to be along the positive z axis at a distance R from the origin of Taub-NUT. We denote polar coordinates centered at the black ring position by (Σ, θ_Σ). Their relation to the polar coordinates (r, θ) centered at the origin is:

$$\Sigma = \sqrt{r^2 + R^2 - 2rR\cos\theta} \,, \qquad \cos\theta_\Sigma = \frac{r\cos\theta - R}{\Sigma} \,. \tag{3.2}$$

The black ring carries dipole charges associated with the harmonic functions[6]

$$K_I \; = \; \frac{d_I}{\Sigma} \, , \qquad I = 1, 2, 3 \, . \tag{3.3}$$

According to eq. (2.11), the corresponding dipole gauge fields are given by:

$$a_I = \frac{d_I}{\Sigma}(d\psi + \vec{A}) + \vec{a}_I \, , \qquad \vec{a}_I = h\, d_I \frac{r\cos\theta - R}{\Sigma} d\phi + Q_6 d_I \frac{r - R\cos\theta}{R\Sigma} d\phi \, . \tag{3.4}$$

The warp factors Z_I are determined by the equation:

$$d *_3 dZ_I = \frac{C_{IJK}}{2} V d *_3 d(K_J K_K) = \frac{C_{IJK}}{2} \left(h + \frac{Q_6}{r} \right) d *_3 d \left(\frac{d_J d_K}{\Sigma^2} \right) . \tag{3.5}$$

The solution Z_I can be written as the linear combination of two terms. The first term satisfies the equation:

$$d *_3 dZ_I^{(1)} = \frac{C_{IJK}}{2} \, h \, d *_3 d \left(\frac{d_J d_K}{\Sigma^2} \right) , \tag{3.6}$$

which is trivially solved by:

$$Z_I^{(1)} = \frac{C_{IJK}}{2} \, h \, \frac{d_J d_K}{\Sigma^2} \, . \tag{3.7}$$

The second term is found by solving

$$d *_3 dZ_I^{(2)} = \frac{C_{IJK}}{2} \frac{Q_6}{r} d *_3 d \left(\frac{d_J d_K}{\Sigma^2} \right) . \tag{3.8}$$

This is the same equation as the one in a flat $\mathbb{R}^4$ base and BPS and "almost BPS" solutions are related by simple change of coordinates (essentially, the exchange of the coordinates ψ and ϕ). One can therefore borrow the known BPS solution and see that the equation above is solved by:

$$Z_I^{(2)} = \frac{C_{IJK}}{2} \frac{Q_6 \, d_J d_K}{R^2} \frac{r}{\Sigma^2} \, . \tag{3.9}$$

Moreover we can add to Z_I a harmonic function L_I, which has a pole at the location of the ring:

$$L_I = l_I + \frac{Q_I}{\Sigma} \, . \tag{3.10}$$

It is not much more difficult to add a pole in L_I at the center of the TN space, which corresponds to placing a black hole inside the black ring. We will construct this more general solution in section 6. The total solution for Z_I is then

$$Z_I = l_I + \frac{Q_I}{\Sigma} + \frac{C_{IJK}}{2} \frac{d_J d_K}{\Sigma^2} \left(h + \frac{Q_6 r}{R^2} \right) . \tag{3.11}$$

The equation for $k = \mu(d\psi + \vec{A}) + \vec{\omega}$ is now:

$$\begin{aligned} d(V\mu) + *_3 d\vec{\omega} \; &= \; V Z_I dK_I \\ &= \left[\left(h + \frac{Q_6}{r} \right) \left(l_I + \frac{Q_I}{\Sigma} \right) + \left(h^2 + \frac{Q_6^2}{R^2} + Q_6 h \left(\frac{1}{r} + \frac{r}{R^2} \right) \right) \frac{C_{IJK}}{2} \frac{d_J d_K}{\Sigma^2} \right] d \left(\frac{d_I}{\Sigma} \right) , \end{aligned} \tag{3.12}$$

[6] As one can see from (2.11), adding a constant κ_I to K_I has the only effect of shifting the dipole potential a_I by the constant one-form $k_I d\psi$. Hence a constant in K_I is physically irrelevant.

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and we then expand the source term on the right-hand side into simpler component pieces. It is then straightforward to find a solution for each piece. We list in the following the solutions for the various terms:

$$d(V\mu_1) + *_3 d\vec{\omega}_1 = \left(h + \frac{Q_6}{r}\right) l_I d\left(\frac{d_I}{\Sigma}\right) \tag{3.13}$$
$$\Rightarrow \mu_1 = \frac{l_I d_I}{2\Sigma}\,, \qquad \vec{\omega}_1 = \frac{h\, l_I d_I}{2}\frac{r\cos\theta - R}{\Sigma}d\phi + \frac{Q_6 l_I d_I}{2}\frac{r - R\cos\theta}{R\Sigma}d\phi\,.$$

$$d(V\mu_2) + *_3 d\vec{\omega}_2 = h\,\frac{Q_I}{\Sigma}d\left(\frac{d_I}{\Sigma}\right) \qquad \Rightarrow \qquad \mu_2 = h\,\frac{Q_I d_I}{2V\,\Sigma^2}\,, \quad \vec{\omega}_2 = 0\,. \tag{3.14}$$

$$d(V\mu_3) + *_3 d\vec{\omega}_3 = \frac{Q_6}{r}\frac{Q_I}{\Sigma}d\left(\frac{d_I}{\Sigma}\right)\,. \tag{3.15}$$

$$d(V\mu_4) + *_3 d\vec{\omega}_4 = \left(h^2 + \frac{Q_6^2}{R^2}\right)\frac{C_{IJK}}{2}\frac{d_J d_K}{\Sigma^2}\,d\left(\frac{d_I}{\Sigma}\right)\,. \tag{3.16}$$

$$d(V\mu_5) + *_3 d\vec{\omega}_5 = Q_6 h\left(\frac{1}{r} + \frac{r}{R^2}\right)\frac{C_{IJK}}{2}\frac{d_J d_K}{\Sigma^2}d\left(\frac{d_I}{\Sigma}\right)\,. \tag{3.17}$$

To find a solution to the third equation it is useful to reinterpret it as the equation for a one-form $\tilde{k} \equiv rV\mu_3(d\psi + \vec{A}) + \vec{\omega}_3$ in a flat $\mathbb{R}^4$ base, and use the fact that BPS and almost BPS solutions are related by a $\psi \leftrightarrow \phi$ exchange, in flat space. In this way one arrives at the following solutions

$$\mu_3 = Q_6 Q_I d_I \frac{\cos\theta}{2R\,V\Sigma^2}\,, \qquad \vec{\omega}_3 = Q_6 Q_I d_I \frac{r\sin^2\theta}{2R\,\Sigma^2}d\phi\,. \tag{3.18}$$

For the fourth equation one can easily verify that the following expressions

$$\mu_4^{(1)} = \left(h^2 + \frac{Q_6^2}{R^2}\right)\frac{C_{IJK}}{6}\frac{d_I d_J d_K}{V\,\Sigma^3}\,, \qquad \vec{\omega_4}^{(1)} = 0\,, \tag{3.19}$$

and

$$\mu_4^{(2)} = \left(h^2 + \frac{Q_6^2}{R^2}\right)\frac{C_{IJK}}{6}\,d_I d_J d_K \frac{r\cos\theta}{RV\,\Sigma^3}\,, \quad \vec{\omega_4}^{(2)} = \left(h^2 + \frac{Q_6^2}{R^2}\right)\frac{C_{IJK}}{6}\,d_I d_J d_K \frac{r^2\sin^2\theta}{R\,\Sigma^3}d\phi. \tag{3.20}$$

both solve the equation. Hence we will take

$$\mu_4 = \mu_4^{(2)} + \alpha(\mu_4^{(2)} - \mu_4^{(1)})\,, \qquad \vec{\omega}_4 = (1+\alpha)\vec{\omega}_4^{(2)}\,, \tag{3.21}$$

and, for the moment, we will keep the parameter, α, arbitrary.

The fifth equation is the only one whose solution cannot be found by simply recycling pieces of the black ring solutions in $\mathbb{R}^4$ or $\mathbb{R}^3 \times S^1$, because the right hand side vanishes in both limits ($Q_6 \to 0$ or $h \to 0$). However, it is possible to think about the right hand side as coming from a fake solution in $\mathbb{R}^4$ whose warp factor is

$$Z_{\text{fake}} \sim \frac{r^2 + R^2}{\Sigma^2}\,. \tag{3.22}$$

One can then express $Z_{\rm fake}$ in the x-y coordinate system used to find the black ring in $\mathbb{R}^4$ [10], solve the corresponding equations[7] for k_1 and k_2, and express the $\mathbb{R}^4$ solution as a solution of the almost BPS equations to read off $\mu_5 V$ and $\vec\omega_5$. This gives

$$\mu_5 = Q_6 h \frac{C_{IJK}}{6}\, d_I d_J d_K \frac{3r^2+R^2}{2R^2 V\, r\, \Sigma^3}\,, \tag{3.23}$$

$$\vec\omega_5 = Q_6 h \frac{C_{IJK}}{6}\, d_I d_J d_K \frac{r(3R^2+r^2) - R(3r^2+R^2)\cos\theta}{2R^3\,\Sigma^3} d\phi\,, \tag{3.24}$$

which one can also verify directly to be a solution of (3.17).

Finally one has the freedom to add a solution of the homogeneous equation, that is, a one-form in TN space with self-dual field strength. Such a one-form has the general form

$$k = \frac{M}{V}(d\psi + \vec A) + \vec\omega\,, \qquad *_3 d\vec\omega = -dM\,, \tag{3.25}$$

with M any harmonic form on $\mathbb{R}^3$. We take M of the form

$$M = m_0 + \frac{m}{\Sigma} + \frac{\tilde m}{r}\,. \tag{3.26}$$

We will see that, unlike the BPS solution, a pole in M at $r=0$ is necessary to produce a regular solution. Hence the final possible contributions to μ and $\vec\omega$ are

$$\mu_6 = \frac{m_0}{V} + \frac{m}{V\,\Sigma} + \frac{\tilde m}{V\,r}\,, \qquad \vec\omega_6 = -m\frac{r\cos\theta - R}{\Sigma} d\phi - \tilde m \cos\theta d\phi\,. \tag{3.27}$$

We should also note that one should think of the term proportional to α in μ_4 and $\vec\omega_4$ as coming from an extra harmonic term in M. Thus, the harmonic function M that determines the black ring solution is really

$$M = m_0 + \frac{m}{\Sigma} + \frac{\tilde m}{r} + \alpha\frac{C_{IJK}}{6R}\, d_I d_J d_K\left(h^2 + \frac{Q_6^2}{R^2}\right)\frac{\cos\theta_\Sigma}{\Sigma^2}\,, \tag{3.28}$$

where θ_Σ was defined in (3.2). In section 3.3 we will show that the coefficient of the dipole term, $\frac{\cos\theta_\Sigma}{\Sigma^2}$, is fixed by requiring regularity at the black ring horizon. We will see in section 5 that an analogous dipole term is also present in the black hole solution: in the black hole case, this term is not fixed by regularity at the horizon, and in fact is required for allowing the black hole to rotate.

Adding all the terms together, we arrive at the final answer

$$\begin{aligned}
\mu &= \frac{m_0}{V} + \frac{m}{V\,\Sigma} + \frac{\tilde m}{V\,r} + \frac{l_I d_I}{2\Sigma} + \frac{h Q_I d_I}{2V\,\Sigma^2} + Q_6 Q_I d_I \frac{\cos\theta}{2R\,V\Sigma^2} \\
&\quad + \frac{C_{IJK}}{6}\, d_I d_J d_K\left[\left(h^2 + \frac{Q_6^2}{R^2}\right)\left(\frac{r\cos\theta}{R\,V\,\Sigma^3} + \alpha\frac{r\cos\theta - R}{R\,V\,\Sigma^3}\right) + Q_6 h\frac{3r^2+R^2}{2R^2 V\, r\,\Sigma^3}\right]\,, \\
\vec\omega &= \Bigg[\kappa - m\frac{r\cos\theta - R}{\Sigma} - \tilde m\cos\theta + \frac{h l_I d_I}{2}\frac{r\cos\theta - R}{\Sigma} + \frac{Q_6 l_I d_I}{2}\frac{r - R\cos\theta}{R\Sigma} \\
&\quad + Q_6 Q_I d_I \frac{r\sin^2\theta}{2R\,\Sigma^2} + \left(h^2 + \frac{Q_6^2}{R^2}\right)\frac{C_{IJK}}{6}\, d_I d_J d_K (1+\alpha)\frac{r^2\sin^2\theta}{R\,\Sigma^3} \\
&\quad + Q_6 h \frac{C_{IJK}}{6}\, d_I d_J d_K \frac{r(3R^2+r^2) - R(3r^2+R^2)\cos\theta}{2R^3\,\Sigma^3}\Bigg] d\phi\,.
\end{aligned} \tag{3.29}$$

We have included a constant term $\kappa d\phi$ in $\vec\omega$ and this will be needed to cancel Dirac-Misner strings.

[7] Equations (46) and (47) in [1].

3.2 Regularity

The angular coordinates ψ and ϕ both shrink to zero size at the center of Taub-NUT space, $r = 0$. Hence regularity of the one-form k requires that μ and $\vec{\omega}$ vanish at $r = 0$ and imposes the following constraints on the parameters of the solution:

$$\mu_{r=0} = 0 \quad \Rightarrow \quad \frac{\tilde{m}}{Q_6} + \frac{l_I d_I}{2R} + \frac{C_{IJK}}{6}\frac{h\, d_I d_J d_K}{2R^3} = 0\,, \tag{3.30}$$

$$\vec{\omega}_{r=0} = 0 \quad \Rightarrow \quad \kappa + m - \frac{h l_I d_I}{2} - \left(\tilde{m} + \frac{Q_6 l_I d_I}{2R} + \frac{C_{IJK}}{6}\frac{Q_6 h\, d_I d_J d_K}{2R^3}\right)\cos\theta = 0\,. \tag{3.31}$$

Moreover the coordinate ϕ degenerates on the z axis (i.e. for $\theta = 0$ or π): one should thus require that $\vec{\omega}$ vanishes on this axis. The constraint one obtains for $\theta = \pi$ is

$$\vec{\omega}_{\theta=\pi} = 0 \;\Rightarrow\; \kappa + m - \frac{h l_I d_I}{2} + \left(\tilde{m} + \frac{Q_6 l_I d_I}{2R} + \frac{C_{IJK}}{6}\frac{Q_6 h\, d_I d_J d_K}{2R^3}\right) = 0\,, \tag{3.32}$$

and is thus already implied by the two previous constraints (3.30) and (3.31). Vanishing of $\vec{\omega}$ at $\theta = 0$ imposes the further condition

$$\vec{\omega}_{\theta=0} = 0 \quad \Rightarrow \quad \kappa - \tilde{m} + \mathrm{sign}(r-R)\left(-m + \frac{h l_I d_I}{2} + \frac{Q_6 l_I d_I}{2R} + \frac{C_{IJK}}{6}\frac{Q_6 h\, d_I d_J d_K}{2R^3}\right) = 0\,. \tag{3.33}$$

All the regularity conditions are solved by taking

$$\begin{aligned} m &= \left(h + \frac{Q_6}{R}\right)\frac{l_I d_I}{2} + \frac{C_{IJK}}{6}\frac{Q_6 h\, d_I d_J d_K}{2R^3}\,, \\ \tilde{m} &= \kappa = -Q_6\left(\frac{l_I d_I}{2R} + \frac{C_{IJK}}{6}\frac{h\, d_I d_J d_K}{2R^3}\right)\,. \end{aligned} \tag{3.34}$$

The parameter $\tilde{m}$ determines the value of μ at the center of Taub-NUT, and the second equation determines the value of this parameter that gives regular geometries (much like for BPS solutions). As we will see later, the parameter m gives the D0 charge of the ring, and hence the first equation determines the distance between the two centers, R, as a function of the charges. This equation is the generalization of the bubble equations [6, 28–30] to non-BPS black holes, and reduces to these equations in the BPS limits ($h \to 0$ or $Q_6 \to 0$). For BPS solutions this equation is a simple, linear equation for R, but for the non-BPS solutions this equation is cubic in R, and its structure is much richer. Since the charges of the black ring are quantized, for given values of the moduli this equation quantizes the possible values of R.

Note that the foregoing conditions do not depend upon the parameter α that governs the "dipole" piece, proportional to $\frac{\cos\theta_\Sigma}{\Sigma^2}$, in μ. We will see in the next subsection that a careful analysis of regularity near the horizon fixes α to a non-zero value.

We should note that the authors of [7] conjectured some expressions for the harmonic functions that underlie the non-BPS black ring solution. The proposed solutions for K_I, L_I and M had poles at the black ring location (much like for BPS black rings) but our analysis here shows that such a solution will always be pathological. Regular solutions must have a source in M at the center of Taub-NUT, with coefficient $\tilde{m}$ given by (3.34). Similarly, there must also be very specific, non-zero "dipole" pieces, proportional to α, in μ and $\vec{\omega}$.

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3.3 Near-horizon geometry

We now examine the metric in the vicinity of the horizon, which is located at $\Sigma = 0$. We will work in the coordinates (Σ, θ_Σ) defined in (3.2). Neglecting the torus directions x_i, the horizon is spanned by the coordinates ψ, ϕ and θ_Σ, and its induced metric (in the eleven-dimensional Einstein frame) is

$$\begin{aligned} ds^2_H &= \frac{I_4}{(Z_1Z_2Z_3)^{2/3}V^2}(d\psi + \vec{A})^2 - 2\frac{\mu\,\omega_\phi}{(Z_1Z_2Z_2)^{2/3}}(d\psi+\vec{A})d\phi \\ &\quad +(Z_1Z_2Z_3)^{1/3}\left(V\Sigma^2\sin^2\theta_\Sigma - \frac{\omega_\phi^2}{Z_1Z_2Z_3}\right)d\phi^2 + (Z_1Z_2Z_3)^{1/3}V\Sigma^2 d\theta_\Sigma^2\,, \end{aligned} \tag{3.35}$$

where

$$I_4 = Z_1Z_2Z_3V - \mu^2V^2\,. \tag{3.36}$$

The volume element of this metric is

$$\sqrt{g_H} = \Sigma(I_4\Sigma^2\sin^2\theta_\Sigma - \omega_\phi^2)^{1/2}\,. \tag{3.37}$$

For generic values of the parameter α one has

$$I_4 \sim \Sigma^{-5}\,, \qquad \omega_\phi \sim \Sigma^{-1}\,, \tag{3.38}$$

and thus $\sqrt{g_H} \sim \Sigma^{-1/2}$. So for generic α the geometry does not have a regular horizon of finite area. However the term of order Σ^{-5} in I_4 can be canceled by taking

$$\alpha = -\frac{h^2R^2}{h^2R^2 + Q_6^2}\,. \tag{3.39}$$

One can think about α as the coefficient of a harmonic function that determines a momentum one-form whose field strength is self-dual, and hence lies in the kernel of the $(1-*)d$ operator in equation (2.9). Adding this self-dual piece with the right coefficient is crucial for the regularity of the solution.

For this value of α, the metric coefficients have the following near-horizon expansions:

$$I_4 = \frac{J_4}{\Sigma^4} + \left(\frac{C_{IJK}}{6}\hat{d}_I\hat{d}_J\hat{d}_K\right)^2\frac{Q_6^2}{R^4\,V_R^4\,\Sigma^4}\sin^2\theta_\Sigma + O\left(\frac{1}{\Sigma^3}\right) \tag{3.40}$$

$$Z_I = \frac{C_{IJK}}{2}\frac{\hat{d}_J\hat{d}_K}{V_R\,\Sigma^2} + O\left(\frac{1}{\Sigma}\right) \tag{3.41}$$

$$\mu = \frac{C_{IJK}}{6}\frac{\hat{d}_I\hat{d}_J\hat{d}_K}{V_R^2\,\Sigma^3} + O\left(\frac{1}{\Sigma^2}\right) \tag{3.42}$$

$$\omega_\phi = \frac{C_{IJK}}{6}\frac{Q_6^2\,\hat{d}_I\hat{d}_J\hat{d}_K}{R^2\,V_R^2\,\Sigma}\sin^2\theta_\Sigma + O(\Sigma^0)\,, \tag{3.43}$$

where J_4 is the usual quartic invariant:

$$J_4(Q_I, \hat{d}_I, \hat{m}) = \frac{1}{2}\sum_{I<J}\hat{d}_I\hat{d}_J\,Q_IQ_J - \frac{1}{4}\sum_I \hat{d}_I^2\,Q_I^2 - \frac{C_{IJK}}{3}\hat{m}\,\hat{d}_I\hat{d}_J\hat{d}_K\,. \tag{3.44}$$

We have also defined the "effective" dipole and angular momentum parameters of the ring, $\hat{d}_I$, $\hat{m}$, via:

$$\hat{d}_I = V_R \, d_I \,, \qquad \hat{m} = V_R^{-1} \, m \,, \qquad V_R = \left(h + \frac{Q_6}{R} \right) . \tag{3.45}$$

One can see from these expressions that the horizon volume element has a finite limit for $\Sigma \to 0$:

$$\sqrt{g_H} \to J_4^{1/2} \sin \theta_\Sigma \,, \tag{3.46}$$

and that the five-dimensional horizon area is given by

$$A_H = (4\pi Q_6)(4\pi) J_4^{1/2} \,. \tag{3.47}$$

To compare this area to that of the BPS black ring in Taub-NUT, it is easiest to choose moduli so that the five-dimensional Newton's constant is given by $G_5 = \frac{\pi}{4}$ and the three tori have equal volume. When $Q_6 = 1$ one can compare the singular parts of the harmonic functions to those of [5], and observe that the integer M2, M5 and KK momentum charges are:

$$n_I = -\frac{d_I V_R}{2} = -\frac{\hat{d}_I}{2} \,, \qquad N_I = \frac{Q_I}{4} \,, \qquad J_{KK} = -\frac{m}{8 V_R} = -\frac{\hat{m}}{8} \,. \tag{3.48}$$

The entropy of the ring is then

$$S_{BR} = 2\pi \sqrt{J_4(N_I, n_I, J_{KK})} \,, \tag{3.49}$$

which is exactly the same as for BPS black rings of identical integer charges [14].

Furthermore, one can use (3.35) and the limiting values (3.40)–(3.43) to obtain the metric induced on the horizon:

$$ds_H^2 = \ell^{-4/3} J_4 (d\psi + Q_6 d\phi)^2 + \ell^{2/3} \left[d\theta_\Sigma^2 + \sin^2 \theta_\Sigma \left(d\phi - \frac{Q_6}{R^2 V_R^2} (d\psi + Q_6 d\phi) \right)^2 \right] \,, \tag{3.50}$$

where

$$\ell = \frac{C_{IJK}}{6} \, \hat{d}_I \hat{d}_J \hat{d}_K \,. \tag{3.51}$$

The factor of $\frac{Q_6}{R^2 V_R^2}$ in (3.50) appears naively to imply that the metric induced on the horizon has conical singularities at $\theta_\Sigma = 0$ and $\theta_\Sigma = \pi$. Nevertheless, by carefully investigating the periodicity of ψ and ϕ one can show that the angle that becomes degenerate[8] has periodicity 2π and hence no such singularities exist.

3.4 Asymptotic charges

To obtain the reduction to four dimensions of the eleven-dimensional metric (2.1) one must recast the Gibbons-Hawking U(1) fibration according to:

$$\begin{aligned} ds^2 = \; & \frac{I_4}{(Z_1 Z_2 Z_3)^{2/3} V^2} \left[d\psi + \vec{A} - \frac{\mu V^2}{I_4} (dt + \vec{\omega}) \right]^2 + \frac{V (Z_1 Z_2 Z_3)^{1/3}}{I_4^{1/2}} ds_E^2 \\ & + \left(\frac{Z_2 Z_3}{Z_1^2} \right)^{1/3} (dx_1^2 + dx_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2} \right)^{1/3} (dx_3^2 + dx_4^2) + \left(\frac{Z_1 Z_2}{Z_3^2} \right)^{1/3} (dx_5^2 + dx_6^2) \,, \end{aligned} \tag{3.52}$$

[8]More explicitly, this angle is $\frac{\psi}{2Q_6} - \frac{\phi}{2}$.

where

$$ds_E^2 = -I_4^{-1/2}(dt+\vec{\omega})^2 + I_4^{1/2} ds_3^2 \tag{3.53}$$

is the four-dimensional Lorentzian metric. In order for this metric to have the canonical normalization at infinity one needs that $I_4 \to 1$ at large r. This is achieved if one takes

$$\frac{C_{IJK}}{6} h\, l_I l_J l_K - m_0^2 = 1\,. \tag{3.54}$$

One could also impose that the ψ coordinate be canonically normalized (i.e. that $g_{\psi\psi} \to 1$ asymptotically) and this requires that

$$\frac{C_{IJK}}{6} h^3\, l_I l_J l_K = 1\,. \tag{3.55}$$

One can also see that, if $m_0 \neq 0$, μ does not vanish at infinity, producing a non-vanishing $g_{t\psi}$. This means that one is in a rotating frame at infinity, which can be undone by a re-definition of the coordinate ψ, as

$$\tilde{\psi} = \psi + h m_0\, t\,. \tag{3.56}$$

In terms of $\tilde{\psi}$ the metric is explicitly asymptotically flat and it is straightforward to compute the associated asymptotic charges. The M2 charges are:

$$\hat{Q}_I = Q_I + \frac{Q_6}{R^2}\frac{C_{IJK}}{2} d_J d_K\,, \tag{3.57}$$

while the KK-monopole charge is simply given by Q_6 and the M5 charges by d_I. The mass is given by the BPS-like formula:

$$M = \frac{C_{IJK}}{6}\frac{l_I l_J l_K}{4} Q_6 + \frac{h}{4}\frac{C_{IJK}}{2}\hat{Q}_I l_J l_K - \frac{m_0\, h}{2} l_I d_I\,. \tag{3.58}$$

Note that here Q_6 and $\hat{Q}_I$ denote the absolute values of the charges.

The momentum along the KK direction $\tilde{\psi}$ is:

$$P = h^2\left(\frac{C_{IJK}}{6} h\, l_I l_J l_K + m_0^2\right) l_I d_I - m_0\, h^2\,\frac{C_{IJK}}{2}\hat{Q}_I l_J l_K - m_0^3\, Q_6\,, \tag{3.59}$$

and the angular momentum in the non-compact $\mathbb{R}^3$ is:

$$\begin{aligned} J &= R\left(m - h\frac{l_I d_I}{2}\right) + \frac{Q_6}{2R} d_I Q_I + \frac{Q_6^2}{R^3}\frac{C_{IJK}}{6} d_I d_J d_K \\ &= \frac{Q_6}{2} l_I d_I + \frac{Q_6}{2R} d_I Q_I + \frac{Q_6}{2R^2}\left(h + \frac{2Q_6}{R}\right)\frac{C_{IJK}}{6} d_I d_J d_K\,. \end{aligned} \tag{3.60}$$

If $m_0 = 0$ and the l_I and h are equal to 1, the mass formula takes a more familiar form, as a sum of absolute values of charges:

$$M = \frac{Q_6}{4} + \frac{1}{4}\sum_I \hat{Q}_I\,, \tag{3.61}$$

and the KK momentum along the GH fiber is just the sum of the dipole charges (much like for BPS black rings):

$$P = \sum_I d_I = \sum_I \frac{\hat{d}_I}{1 + Q_6/R} . \tag{3.62}$$

Moreover, the four-dimensional angular momentum becomes

$$J = \frac{Q_6 P}{2} + \frac{Q_6}{2R} d_I Q_I + \frac{Q_6}{2R^2}\left(1 + \frac{2Q_6}{R}\right) \frac{C_{IJK}}{6} d_I d_J d_K , \tag{3.63}$$

where now we can identify the first piece as coming from the Poynting vector caused by the KK electric and magnetic charges and the other pieces as coming from the interactions between the electric M2 charges and the magnetic M5 charges. When the black ring becomes a supertube ($d_1 = d_2 = Q_3 = 0$), the latter interactions are zero, and the KK Poynting term $\frac{Q_6 P}{2}$ is the only one that survives.

4 Almost BPS supertubes

4.1 The supertube solution

From a supergravity perspective, a supertube [31] can be thought of as a particular black ring with only two charges and one dipole charge. One can thus trivially obtain an "almost BPS" supertube from the non-BPS solution above taking the following harmonic functions

$$K_1 = K_2 = 0 , \qquad K_3 = \frac{d_3}{\Sigma} \qquad V = 1 + \frac{Q_6}{r} \tag{4.1}$$

$$L_1 = 1 + \frac{Q_1}{\Sigma} , \qquad L_2 = 1 + \frac{Q_2}{\Sigma} , \qquad L_3 = 1 , \tag{4.2}$$

$$M = m_0 + \frac{m}{\Sigma} + \frac{\tilde{m}}{r} . \tag{4.3}$$

The solution simplifies considerably, and one finds

$$\begin{aligned}
a_1 &= a_2 = 0 , \qquad a_3 = K_3(d\psi + \vec{A}) + \vec{a}_3 , \qquad *_3 d\vec{a}_3 = V dK_3 - K_3 dV \\
&\Rightarrow \ \vec{a}_3 = d_3 \frac{r\cos\theta - R}{\Sigma} d\phi + Q_6 d_3 \frac{r - R\cos\theta}{R\Sigma} d\phi , \\
Z_I &= L_I \\
\mu &= \frac{M}{V} + \frac{1}{2} K_3 , \qquad *_3 d\vec{\omega} = -dM + \frac{1}{2}(V dK_3 - K_3 dV) \\
&\Rightarrow \ \vec{\omega} = \left(-m + \frac{d_3}{2}\right) \frac{r\cos\theta - R}{\Sigma} d\phi - \tilde{m}\cos\theta \, d\phi + \frac{Q_6 d_3}{2} \frac{r - R\cos\theta}{R\,\Sigma} d\phi .
\end{aligned} \tag{4.4}$$

The supertube is smooth in a duality frame in which the electric (M2) charges correspond to D1 and D5 branes and the magnetic (M5) dipole moment corresponds to a KK-monopole wrapped around the Taub-NUT direction. In this frame, the ten-dimensional string metric is:

$$ds^2 = -\frac{1}{\sqrt{Z_1 Z_2} Z_3}(dt + k)^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\left(dy + a_3 - \frac{dt + k}{Z_3}\right)^2 + \sqrt{Z_1 Z_2} ds_4^2 + \sqrt{\frac{Z_2}{Z_1}} \sum_{a=1}^{4} dx_a^2 , \tag{4.5}$$

where y the common D1-D5 direction. Standard BPS supertubes are regular in this frame and so we now consider the regularity of the metric of the "almost BPS" supertubes. The coefficient of $d\psi^2$ in the metric is:

$$\begin{aligned} g_{\psi\psi} &= \frac{1}{\sqrt{Z_1 Z_2}}(Z_3 a_{3,\psi}^2 - 2\mu a_{3,\psi} + Z_1 Z_2 V^{-1}) \\ &= \frac{1}{V\sqrt{L_1 L_2}}(L_1 L_2 - 2MK_3)\,, \end{aligned} \tag{4.6}$$

where in the second line we have used the expressions for Z_I, a_3 and μ given in (4.4). The requirement that $g_{\psi\psi}$ be finite for $\Sigma \to 0$ implies

$$m = \frac{Q_1 Q_2}{2d_3}\,. \tag{4.7}$$

In order for $\vec{\omega}$ not to have any Dirac-Misner string pathologies around the point $\Sigma = 0$ it is necessary that $\vec{\omega}$ vanish for $\theta = 0$ and r greater or smaller than R. These conditions imply:

$$m = \frac{V_R d_3}{2} \quad \text{with} \quad V_R = 1 + \frac{Q_6}{R}\,. \tag{4.8}$$

Combining these two relations for m one obtains an equation that determines the supertube location R:

$$V_R = \frac{Q_1 Q_2}{d_3^2}\,. \tag{4.9}$$

Finally one should look at regularity at the Taub-NUT center $r = 0$. As the coordinate ψ degenerates at $r = 0$, μ must vanish to prevent CTC's, which implies

$$\tilde{m} = -\frac{d_3 Q_6}{2R}\,. \tag{4.10}$$

4.2 Comparing BPS and "almost BPS" supertubes

Having found a smooth supertube metric that solves the "almost BPS" equations (2.7)–(2.9), we can compare it to that of a BPS supertube, and show that despite their rather different appearance, the two solutions are identical.

Denoting with a "hat" the quantities associated with the BPS solution, we recall that the BPS supertube solution is given by:

$$\begin{aligned} &\hat{a}_3 = \frac{\hat{K}_3}{V}(d\psi + \vec{A}) + \hat{\vec{a}}_3\,, && *_3 d\hat{\vec{a}}_3 = -d\hat{K}_3\,, && \hat{Z}_I = \hat{L}_I\,, \\ &\hat{k} = \left(\hat{M} + \frac{\hat{K}_3}{2V}\right)(d\psi + \vec{A}d\phi) + \hat{\vec{\omega}}\,, && *_3 d\hat{\vec{\omega}} = V d\hat{M} - \hat{M} dV - \frac{1}{2} d\hat{K}_3\,. && \end{aligned} \tag{4.11}$$

Since the supertube solution has $Z_3 = 1$, one can absorb the term $-dt/Z_3$ in equation (4.5) by the coordinate shift $y \to y + t$. Thus the dipole potential a_3 only enters in the metric via the combination $(dy + a_3 - k)^2$. Comparing the BPS expressions (4.11) to the "almost BPS" ones (4.4), one sees that, under the identifications

$$\hat{K}_3 = 2M\,, \qquad \hat{M} = \frac{K_3}{2}\,, \qquad \hat{L}_I = L_I \tag{4.12}$$

one has

$$\hat{a}_3 - \hat{k} = -(a_3 - k)\,, \qquad \hat{Z}_I = Z_I\,, \qquad \hat{k} = k\,. \tag{4.13}$$

Hence, the BPS and "almost BPS" supertube solutions can be related to each other by flipping the sign of y and interchanging harmonic functions.

5 General extremal non-BPS rotating black holes

In this section we present the other main result of this paper: a rotating five-charge extremal non-BPS black hole in four dimensions. This black hole can serve as the seed solution for the most generic under-rotating non-BPS extremal black hole in the *STU* model and in $\mathcal{N} = 8$ supergravity in four dimensions, and can be thought of as coming from the non-BPS extension of the five-dimensional BPS rotating (BMPV) black hole to an asymptotically Taub-NUT solution.[9]

We first construct and analyze this black hole, and then show that for special values of the charges it reduces to the under-rotating D0-D6 extremal black hole [19].

5.1 The solution

The harmonic functions associated with the KK-monopole and electric (M2) charges have the usual form

$$V = h + \frac{Q_6}{r}\,, \qquad L_I = 1 + \frac{Q_I}{r}\,, \tag{5.1}$$

where for simplicity we have set to one the constants l_I in the L_I harmonic functions. The solution with arbitrary moduli is presented in section 6.

The dipole charges vanish, and hence $K_I = 0$. The harmonic function, M, which encodes the angular momentum of the solution is taken to have the form:

$$M = m_0 + \frac{m}{r} + \alpha\frac{\cos\theta}{r^2}\,. \tag{5.2}$$

The term proportional to α is the harmonic potential is sourced by a dipole at the origin of Taub-NUT space and, as we will see, is needed to generate the angular momentum of the black hole.

With this choice of harmonic functions, the "almost BPS" equations (2.7), (2.9) are solved by[10]

$$\Theta_I = 0\,, \quad Z_I = L_I\,, \quad \mu = \frac{M}{V} = \frac{m_0}{V} + \frac{m}{Vr} + \alpha\frac{\cos\theta}{Vr^2}\,, \quad \vec{\omega} = -m\cos\theta d\phi + \alpha\frac{\sin^2\theta}{r}d\phi\,. \tag{5.3}$$

Absence of Dirac-Misner strings requires that $\vec{\omega}$ vanish both at $\theta = 0$ and $\theta = \pi$, and hence we must take

$$m = 0\,. \tag{5.4}$$

[9]For a recent discussion of the BPS extension of this black hole to Taub-NUT see [32].

[10]The vector potential $\vec{\omega}$ dual to the dipole field $\frac{\cos\theta}{r^2}$ follows from the identity

$$*_3\, d\left(\frac{\sin^2\theta}{r}d\phi\right) = -d\left(\frac{\cos\theta}{r^2}\right)\,.$$

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Nevertheless, α remains as a free parameter of the solution and it encodes the angular momentum. To see this more explicitly we compute the conserved charges. As shown in section 3.4, the four-dimensional Lorentzian metric is:

$$ds_E^2 = -I_4^{-1/2}(dt + \vec{\omega})^2 + I_4^{1/2} ds_3^2\,, \qquad I_4 = Z_1 Z_2 Z_3 V - \mu^2 V^2\,, \tag{5.5}$$

and the electric component of the KK gauge field coming from the reduction along the Taub-NUT fiber is

$$A_{KK} = -\frac{\mu V^2}{I_4}\,. \tag{5.6}$$

The normalization condition $I_4 \to 1$ for large r requires

$$h - m_0^2 = 1\,. \tag{5.7}$$

The KK momentum along ψ, found from the asymptotic expansion of A_{KK}, is

$$P = m_0(h^2(Q_1 + Q_2 + Q_3) + m_0^2 Q_6)\,, \tag{5.8}$$

and the $\mathbb{R}^3$ angular momentum, encoded in $\vec{\omega}$, is

$$J = \alpha\,. \tag{5.9}$$

One can also show that this solution has a regular horizon of finite area. In the near-horizon ($r \to 0$) limit, one has

$$I_4 \to \frac{Q_1 Q_2 Q_3 Q_6 - \alpha^2 \cos^2\theta}{r^4}\,, \qquad \omega_\phi \to \alpha \frac{\sin^2\theta}{r}\,, \tag{5.10}$$

and thus the volume element of the metric induced on the horizon is

$$\sqrt{g_H} = r(I_4 r^2 \sin^2\theta - \omega_\phi^2)^{1/2} \approx \sin\theta (Q_1 Q_2 Q_3 Q_6 - \alpha^2)^{1/2}\,. \tag{5.11}$$

The horizon area is

$$A_H = (4\pi Q_6)(4\pi)\sqrt{Q_1 Q_2 Q_3 Q_6 - \alpha^2}\,, \tag{5.12}$$

which coincides with the area of the BMPV black hole supersymmetrically embedded in Taub-NUT.

5.2 The extremal rotating D0-D6 black hole

We now discuss the relationship between the solution presented above to the one of Rasheed and Larsen [19]. First of all, the solution of Rasheed and Larsen can be compared to ours only in the "slowly rotating" or "ergo-free" extremal limit: $a \to 0$, $m \to 0$, keeping $a/m = J$ fixed. In this limit the metric of [19] can be recast in a form similar to the one of (5.5):

$$ds^2 = -\frac{r^2}{\sqrt{H_1 H_2}}(dt + \mathbf{B})^2 + \frac{\sqrt{H_1 H_2}}{r^2} ds_3^2, \tag{5.13}$$

where

$$\mathbf{B} = \frac{(pq)^{3/2}}{2(p+q)}\, J\, \frac{\sin^2\theta}{r}\, d\phi, \tag{5.14}$$

$$H_1 = r^2 + rp + \frac{p^2 q}{2(p+q)} - \frac{p^2 q}{2(p+q)} J\cos\theta, \tag{5.15}$$

$$H_2 = r^2 + rq + \frac{q^2 p}{2(p+q)} + \frac{q^2 p}{2(p+q)} J\cos\theta. \tag{5.16}$$

This solution has a single scalar field running

$$z = i\sqrt{\frac{H_2}{H_1}} \tag{5.17}$$

and a vanishing axion. The physical D0 and D6 charges Q and P are related to p and q by

$$Q^2 = \frac{q^3}{4(p+q)}, \qquad P^2 = \frac{p^3}{4(p+q)}. \tag{5.18}$$

This solution is related, by a U-duality transformation, to the solution presented above. We will establish this by applying an appropriate transformation to the scalar field (5.17) and showing that the resulting fields and charges fall in a special subset of those presented above. Since we are starting from a special configuration with only two charges turned on and no axion, we do not expect to be able to generate the most general solution, but we will obviously obtain some constraints on the allowed values for the moduli at infinity.

In order to simplify computations, we consider the $\mathcal{N}=2$ truncation of the M-theory description used earlier. Hence we will look at compactifications on $T^6/(Z_2\times Z_2)\times S^1$, where the last S^1 is parametrized by ψ and the orbifold action is the trivial one preserving the 2-forms $dx_1\wedge dx_2$, $dx_3\wedge dx_4$ and $dx_5\wedge dx_6$. The resulting $\mathcal{N}=2$ effective theory is described by an STU model, with scalar fields in the vector multiplets parametrizing:

$$\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right]^3 \simeq \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)}. \tag{5.19}$$

The three complex moduli for our solution are given by

$$t_I = \frac{4M}{VZ_I} + 4i\,\mathrm{e}^{-\phi}B_I, \tag{5.20}$$

where

$$B_I = \frac{(\frac{1}{2}C_{IJK}Z_JZ_K)^{1/3}}{Z_I^{2/3}}, \tag{5.21}$$

and the dilaton is

$$\mathrm{e}^{-2\phi} = \frac{I_4}{(Z_1Z_2Z_3)^{2/3}V^2}. \tag{5.22}$$

The duality action on the three scalar fields then acts as follows:

$$t_I \to \frac{a_I t_I + b_I}{c_I t_I + d_I} \qquad \text{(no sum)} \tag{5.23}$$

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where

$$M_I = \begin{pmatrix} a_I & b_I \\ c_I & d_I \end{pmatrix}, \tag{5.24}$$

are SL$(2,\mathbb{R})$ matrices.

Without rotation one can immediately check that our solution reduces to

$$t_I = \frac{4}{VZ_I}(m_0 + i\,\mathrm{e}^{-2U}), \tag{5.25}$$

with $\mathrm{e}^{-2U} = \sqrt{I_4}$, which is the one presented in Equation (4.34) of [15]. This is easily dualized to the generating solution by [16] by taking

$$M_I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.26}$$

which yields

$$t_I = \frac{1}{2\,C_{IJK}Z_J Z_K}(m_0 - i\,\mathrm{e}^{-2U}). \tag{5.27}$$

At this point one can further dualize to D0-D6 charges by following the duality rotations described in [16]. The complete duality transformation mapping the D6-D2-D2-D2 system into the D0-D6 is then given by

$$M_I = -\frac{1}{\sqrt{2\lambda\rho_I}}\begin{pmatrix} -\rho_I & 1 \\ -\rho_I\lambda & -\lambda \end{pmatrix}, \tag{5.28}$$

where

$$\lambda = \left(\frac{P}{Q}\right)^{1/3}, \qquad \rho_I = \sqrt{\frac{p^0 q_I}{\frac{1}{2}C_{IJK}q_J q_K}}, \tag{5.29}$$

with $16p^0 = Q_6$, $q_I = Q_I$ and $(PQ)^2 = 4p^0 q_1 q_2 q_3$.

Following the inverse route, we can start from (5.13)–(5.17) and apply the inverse transformation:

$$M_I = -\frac{1}{\sqrt{2\lambda\rho_I}}\begin{pmatrix} -\lambda & -1 \\ \rho_I\lambda & -\rho_I \end{pmatrix}. \tag{5.30}$$

The four-dimensional dilaton can be identified to the diagonal scalar $t_1 = t_2 = t_3 = z$. After applying the duality transformation we obtain

$$t_I = -\frac{1}{\rho_I}\frac{\lambda z + 1}{\lambda z - 1} \tag{5.31}$$

which we expect to match the moduli of our metric (5.20), which become[11]

$$t_I = \frac{4}{VZ_I}\left(\mu V + i\sqrt{I_4}\right). \tag{5.32}$$

[11] As in [16], we use conventions in which $|t_I| = \frac{1}{\rho_I}$ at infinity.

Using the explicit expression for z given in (5.17) we can see that one needs to identify

$$VZ_I = \frac{2\rho_I}{\lambda}\frac{H_1 + \lambda^2 H_2}{r^2} \tag{5.33}$$

and

$$V\mu = \frac{1}{2\lambda}\frac{H_1 - \lambda^2 H_2}{r^2}. \tag{5.34}$$

This can be achieved for

$$\lambda = \sqrt{\frac{p}{q}}, \qquad \rho_I = \frac{p+q}{2(pq)^{3/2}} Q_6 q_I, \tag{5.35}$$

which is equivalent to (5.29) and

$$h = \frac{p+q}{pq} Q_6, \qquad l_I = \frac{p+q}{pq} q_I, \qquad m_0 = \frac{q-p}{2\sqrt{pq}}, \qquad \alpha = -\frac{(pq)^{3/2}}{2(p+q)} J, \tag{5.36}$$

where the l_I are the constants in the harmonic functions L_I, which, for simplicity, we have set to one in equation (5.1), but which we will explicitly include in the next section (see equation (6.1)) when discussing the general black-hole-black-ring solution.

Hence for special values of the charges and of the moduli, our solution can be dualized to the under-rotating extremal limit of the D0-D6 Rasheed-Larsen black hole. However, our solution has generic charges and moduli and hence it is more general; its duality orbit includes all the under-rotating extremal black hole solutions of the STU model or of $\mathcal{N} = 8$ supergravity in four dimensions.

6 Non-BPS black ring in a black-hole background

Making use of the linear structure underlying the equations (2.7)–(2.9), it is possible to superimpose the solutions constructed in the previous sections to generate the metric describing a non-BPS black ring with a rotating black hole at the origin of Taub-NUT space. Starting from the black ring solution of section 3, adding the rotating black hole corresponds to adding a $1/r$ term to the harmonic functions L_I, which therefore becomes

$$L_I = l_I + \frac{Q_I}{\Sigma} + \frac{\tilde{Q}_I}{r}\,, \tag{6.1}$$

and a "dipole" source centered at $r = 0$ to the harmonic function M:

$$M = m_0 + \frac{m}{\Sigma} + \frac{\tilde{m}}{r} + \tilde{\alpha}\frac{\cos\theta_\Sigma}{\Sigma^2} + \beta\frac{\cos\theta}{r^2}\,. \tag{6.2}$$

The dipole potentials a_I are left untouched, and are still given by the expressions in (3.4). The warp factors Z_I are obtained by replacing the old functions L_I with the new ones given in (6.1):

$$Z_I = l_I + \frac{Q_I}{\Sigma} + \frac{\tilde{Q}_I}{r} + \frac{C_{IJK}}{2}\frac{d_J d_K}{\Sigma^2}\left(h + \frac{Q_6 r}{R^2}\right)\,. \tag{6.3}$$

The new $1/r$ term in Z_I adds the contribution

$$\left(h+\frac{\tilde{Q}_6}{r}\right)\frac{\tilde{Q}_I}{r}d\left(\frac{d_I}{\Sigma}\right) \tag{6.4}$$

to the r.h.s. of the equation for k (3.12). Hence k receives two new contributions. The first one is given by the solution of

$$d(V\,\mu_7)+*_3 d\vec{\omega}_7 = h\frac{\tilde{Q}_I}{r}d\left(\frac{d_I}{\Sigma}\right)\,. \tag{6.5}$$

This equation is easily solved by

$$\mu_7=\frac{h\,\tilde{Q}_I d_I}{2V\,r\Sigma}\,,\qquad \vec{\omega}_7=\frac{h\,\tilde{Q}_I d_I}{2}\frac{r-R\cos\theta}{R\Sigma}d\phi\,. \tag{6.6}$$

The other new term in k is found by solving

$$d(V\,\mu_8)+*_3 d\vec{\omega}_8=\frac{Q_6\,\tilde{Q}_I}{r^2}d\left(\frac{d_I}{\Sigma}\right)\,. \tag{6.7}$$

Again one can find the solution by using the corresponding solution for a flat base. The result is

$$\mu_8=\frac{Q_6\,\tilde{Q}_I d_I}{R\,V\,r\,\Sigma}\cos\theta\,,\qquad \vec{\omega}_8=\frac{Q_6\,\tilde{Q}_I d_I}{R\,\Sigma}\sin^2\theta\,d\phi\,. \tag{6.8}$$

Furthermore the term proportional to β in M generates an extra contribution given by

$$\mu_9=\beta\frac{\cos\theta}{Vr^2}\,,\qquad \vec{\omega}_9=\beta\frac{\sin^2\theta}{r}d\phi\,. \tag{6.9}$$

Adding the new terms to the previous black ring result, one finds the full solution for k:

$$\begin{aligned}
\mu\;=\;&\frac{m_0}{V}+\frac{m}{V\,\Sigma}+\frac{\tilde{m}}{V\,r}+\beta\frac{\cos\theta}{Vr^2}+\frac{l_I d_I}{2\Sigma}+\frac{hQ_I d_I}{2V\,\Sigma^2}+Q_6 Q_I d_I\frac{\cos\theta}{2R\,V\Sigma^2}+\frac{h\,\tilde{Q}_I d_I}{2V\,r\Sigma}+\frac{Q_6\,\tilde{Q}_I d_I}{R\,V\,r\,\Sigma}\\
&+\frac{C_{IJK}}{6}d_I d_J d_K\left[\left(h^2+\frac{Q_6^2}{R^2}\right)\left(\frac{r\cos\theta}{R\,V\,\Sigma^3}+\alpha\frac{r\cos\theta-R}{R\,V\,\Sigma^3}\right)+Q_6 h\frac{3r^2+R^2}{2R^2V\,r\,\Sigma^3}\right]\,,\\
\vec{\omega}\;=\;&\Bigg\{\kappa-m\frac{r\cos\theta-R}{\Sigma}-\tilde{m}\cos\theta+\beta\frac{\sin^2\theta}{r}d\phi+\frac{hl_I d_I}{2}\frac{r\cos\theta-R}{\Sigma}+\frac{Q_6 l_I d_I}{2}\frac{r-R\cos\theta}{R\Sigma}\\
&+Q_6 Q_I d_I\frac{r\sin^2\theta}{2R\,\Sigma^2}+\frac{h\,\tilde{Q}_I d_I}{2}\frac{r-R\cos\theta}{R\Sigma}+\frac{Q_6\,\tilde{Q}_I d_I}{R\,\Sigma}\sin^2\theta\\
&+\frac{C_{IJK}}{6}d_I d_J d_K\left[\left(h^2+\frac{Q_6^2}{R^2}\right)(1+\alpha)\frac{r^2\sin^2\theta}{R\,\Sigma^3}+Q_6 h\frac{r(3R^2+r^2)-R(3r^2+R^2)\cos\theta}{2R^3\,\Sigma^3}\right]\Bigg\}\,d\phi\,.
\end{aligned} \tag{6.10}$$

The absence of Dirac-Misner strings requires that $\vec{\omega}$ vanishes on the z axis. This imposes the following constraints, which are the generalization of (3.34)

$$\begin{aligned}
m\;&=\;\left(h+\frac{Q_6}{R}\right)\frac{l_I d_I}{2}+\frac{C_{IJK}}{6}\frac{Q_6 h d_I d_J d_K}{2R^3}+\frac{h}{2R}\tilde{Q}_I d_I\\
\tilde{m}\;&=\;\kappa=-Q_6\left(\frac{l_I d_I}{2R}+\frac{C_{IJK}}{6}\frac{h d_I d_J d_K}{2R^3}\right)-\frac{h}{2R}\tilde{Q}_I d_I\,.
\end{aligned} \tag{6.11}$$

The first equation can again be thought of as the generalization of the bubble equations [6, 28–30] to the most generic two-center non-BPS extremal solution.

The topology of the black ring horizon at $\Sigma = 0$ is not affected by the black hole. As above, if α is chosen as in (3.39), this solution has horizon of finite area at $\Sigma = 0$ with an $S^2 \times S^1$ geometry. The area of this horizon is:

$$A_H = 16\pi^2 Q_6 \, \tilde{J}_4^{1/2} \,, \tag{6.12}$$

where

$$\tilde{J}_4^{1/2} = \frac{1}{2}\sum_{I<J} \hat{d}_I \hat{d}_J \, Q_I Q_J - \frac{1}{4}\sum_I \hat{d}_I^2 \, Q_I^2 - \frac{C_{IJK}}{6}\hat{d}_I \hat{d}_J \hat{d}_K \left(2\hat{m} + \frac{Q_6}{R^2 \, V_R^2}\tilde{Q}_I \hat{d}_I \right) . \tag{6.13}$$

As for BPS black rings in black-hole backgrounds [33], the integer D0 charge of the ring is no longer proportional to $\hat{m}$ but rather to the combination that appears in equation (6.13):

$$\hat{m} + \frac{Q_6}{2R^2 \, V_R^2}\tilde{Q}_I \hat{d}_I \,. \tag{6.14}$$

The black hole at the center of the Taub-NUT space has five-dimensional horizon area equal to:

$$A_{BH} = (4\pi Q_6)(4\pi)\sqrt{Q_6 \frac{C_{IJK}}{6}\tilde{Q}_I \tilde{Q}_J \tilde{Q}_K - \beta^2} \,. \tag{6.15}$$

This black hole carries electric D6 and D2 charges (Q_6 and $\tilde{Q}_I$), and angular momentum β.

7 Conclusions and future directions

We have explicitly constructed three-charge three-dipole charge extremal non-BPS black rings in Taub-NUT, both in the absence and in the presence of a three-charge black hole. These rings are locally identical to the supersymmetric black rings, but break supersymmetry because the D6 brane that can be thought of as sourcing the Taub-NUT space has a reversed orientation compared to the BPS embedding. Our solutions become identical to the BPS rings both in the limit when Taub-NUT becomes $\mathbb{R}^4$ and in the limit when it becomes $\mathbb{R}^3 \times S^1$, where the orientation of the D6 brane becomes irrelevant.

We have also constructed the solution for the non-BPS embedding of a two-charge supertube, and have shown that this solution is the same as that of a BPS supertube, despite the rather different form of the ingredients that enter in its construction. This agrees with the intuition that supersymmetry is broken by the incompatible supersymmetry constraints imposed by multiple branes. When only two D2 charges are present flipping the charge of the D6 brane creates no such incompatibility: There are still consistent supersymmetries with all three branes.

We have also found an extremal rotating non-BPS five-charge (D6-D2-D2-D2-D0) black hole in four dimensions. This solution is the seed for the most general extremal (under)rotating non-BPS black hole solution of the STU model or of $\mathcal{N} = 8$ supergravity in four dimensions. For particular values of the charges and moduli we have shown that this

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solution can be dualized to the Kaluza Klein rotating black hole solution of Rasheed and Larsen [19] or its U-duals [20].

Using our solution-generating method we have also constructed a solution that contains both a rotating black hole and a black ring. This solution descends in four dimensions to a two-black-hole non-BPS bound state, where one of the black holes has five charges (D6-D2-D2-D2-D0) and the other has seven charges (D4-D4-D4-D2-D2-D2-D0).[12] The bubble equations that determine the distance between the two black holes are cubic in this distance, and hence are more complicated than those governing BPS multi-center solutions [6, 28–30].

This two-center solution appears to be the most general one can construct within the framework of [7]. Furthermore, the charges of its centers can be dualized to those of more generic extremal non-BPS two-black-hole configurations. It would be interesting to further investigate how generic these charges are, and whether our solution lies in the duality orbit of the most generic two-center extremal solution.

As we discussed in the Introduction, our work bridges the gap between two rather disconnected bodies of research — the construction of extremal supergravity solutions using fake superpotentials [15, 16, 21, 22, 24] and the embedding of five-dimensional black holes and black rings in Taub-NUT [25–27]. It would be interesting to see whether the fake superpotential approach can be extended to describe multi-center solutions, and whether the two-center solution we obtain is the most general one can find within this framework. It would be equally interesting to see if the construction in [27] can be extended to electrically and magnetically-charged black rings in Taub-NUT, and whether the extremal limit of these rings can be compared to the extremal non-BPS rings we construct.

Another very important extension of this work would be to encompass families of multi-center solutions. The method we have outlined here allows one to recycle a considerable part of the known BPS multi-center solutions in $\mathbb{R}^4$ and in $\mathbb{R}^3 \times S^1$. Indeed, just by writing these solutions as non-BPS solutions one can read off the warp factors, as well as all the angular momentum terms that are not cubic in the dipole charges. Nevertheless, the terms cubic in the dipole charges (that satisfy an equation similar to (3.17)) appear to be somewhat harder to obtain.

Last, but not least, it would be interesting to use the ansatz of [7] to extend the construction of BPS smooth multi-center bubbling solutions [28–30] to non-BPS smooth extremal solutions, which would correspond to microstates of extremal non-BPS black holes. It would be particularly interesting if non-BPS solutions exhibited scaling behavior.

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[12] Note that that the charges at the two centers are mutually nonlocal, and hence this solution is more general than the one constructed in [34].

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