Journal of High Energy Physics

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Published by IOP Publishing for SISSA

Received: *May 2, 2009*
Accepted: *May 27, 2009*
Published: *June 4, 2009*

On the $z = 4$ Hořava-Lifshitz Gravity

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Abstract: We consider $z = 4$ Hořava-Lifshitz gravity in both 3+1 and 4+1 dimensions. We find black hole solutions in the IR region for a kind of $z = 4$ Hořava-Lifshitz gravity which is inherited from the new massive gravity in three dimensions and an analog of the new massive gravity in four dimensions through the quantum inheritance principle. We analyze thermodynamic properties for the black hole solutions for $z = 4$ Hořava-Lifshitz gravity. We also write out the Friedmann equation in 3+1 dimensions for cosmological solutions.

Keywords: Models of Quantum Gravity, Black Holes

ArXiv ePrint: 0904.4104

doi:10.1088/1126-6708/2009/06/010

JHEP06(2009)010

Contents

1 Introduction

Recently, Hořava-Lifshitz gravity [1–3] has attracted much attention as a candidate quantum field theory of gravity with $z = 3$ in the UV, where z measures the degree of anisotropy between space and time. In 3+1 dimensions, the Hořava-Lifshitz theory has a $z = 3$ fixed point in the UV and flows to a $z = 1$ fixed point in the IR, which is just the classical Einstein-Hilbert gravity theory. This theory is renormalizable in the sense that the effective coupling constant is dimensionless in the UV.

Much work has been done in Hořava-Lifshitz gravity theory [4–16]. In [5–7, 10, 11, 15, 16], the possible effect of this quantum gravity from the sky was studied. In [9, 13, 14] the solutions for $z = 3$ Hořava-Lifshitz gravity in 3+1 dimensions have been considered and in [14] the thermodynamics of black hole solutions was discussed. In [12] the time problem in this quantum gravity theory was considered. The focus of all these work was on the Hořava-Lifshitz theory with $z = 3$ in the UV, while Hořava-Lifshitz theory with $z = 4$ in the UV can also be constructed from the curvature square terms in the three spatial dimensions [3], which is power-counting super-renormalizable in 3+1 dimensions.

There are several reasons that the $z = 4$ Hořava-Lifshitz theory is of much importance. First, in [4] it was shown that the spectral dimension calculated using the numerical Causal Dynamical Triangulations (CDT) approach [17] for quantum gravity in 3+1 dimensions prefers the $z = 4$ Hořava-Lifshitz gravity in the UV. Second, in [3] it was pointed out that in the case when the scalar mode of the metric is present as a physical field, in

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order for the theory to be power-counting renormalizable for general value of λ, which is a dimensionless coupling defined below, one way is to add super-renormalizable terms. Third, from the conjecture of the quantum inheritance principle, the construction of a four dimensional renormalizable gravity is intimately related to a three dimensional renormalizable relativistic theory. The potential of the $z = 3$ Hořava-Lifshitz theory in the UV can be constructed from the topological massive theory (TMG) [18–20] in three dimensions by the so called "detailed balance condition." There is also another renormalizable relativistic massive gravity in three dimensions, which is proposed recently by [21] and further investigated in [22–29]. This new massive gravity (NMG) can be used to construct a $z = 4$ Hořava-Lifshitz theory. Thus in the rest of this paper we will focus our attention on $z = 4$ Hořava-Lifshitz gravity .

In this paper, we would like to consider $z = 4$ Hořava gravity in both 3+1 and 4+1 dimensions. We will first write out the form of the action for the $z = 4$ Hořava-Lifshitz gravity in 3+1 dimensions constructed from general R^2 terms in three dimensions. Then we will search for the topological black hole solutions of the theory inherited from the new massive gravity and discuss thermodynamic properties of the black hole solutions. We also get the Friedmann equation in 3+1 dimensions for cosmological solutions. In 4+1 dimensions, $z = 4$ Hořava-Lifshitz gravity is renormalizable by power counting and the general form of the action is similar to the case of 3+1 dimensions. We also search for black hole solutions of the theory constructed from an analog of the new massive gravity in four dimensions and analyze the thermodynamic properties of these black hole solutions.

Our note is organized as follows. In section 2 we will formulate the $z = 4$ Hořava gravity in 3+1 dimensions and discuss its black hole solutions as well as cosmological solutions. In section 3 we will formulate the $z = 4$ Hořava-Lifshitz gravity in 4+1 dimensions and study its solutions. Section 4 is devoted to conclusions and discussions.

2 $z = 4$ Hořava-Lifshitz gravity in 3+1 dimensions

In this section, we will give the form of the action for $z = 4$ Hořava-Lifshitz gravity in 3+1 dimensions and search for solutions for the Hořava-Lifshitz gravity inherited from the new massive gravity.

2.1 Brief review of Hořava-Lifshitz gravity

First we will review the construction of Hořava-Lifshitz gravity in $D + 1$ dimensions [2, 3]. Similar to the ADM decomposition of the metric in general relativity, we can write the $D + 1$ dimensional metric as

$$ds^2 = -N^2c^2dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \tag{2.1}$$

where $i = 1, \cdots, D$ and c is the speed of light. The scaling dimension is modified at the fixed point with Lifshitz index z as $t \to b^z t, x^i \to bx^i$, under which g_{ij} and N are invariant while $N^i \to b^{1-z}N^i$. In the following we measure the scaling properties in the units of inverse spatial length, so we have $[t] = -z,\ [x^i] = -1,\ [c] = [N^i] = z-1$ at the fixed point. When $z = 1$, the scaling properties get back to our familiar relativistic case, and this is the

IR limit. In the UV region, the Hořava-Lifshitz theory switches to other z to make this theory renormalizable.

The kinetic term is given by

$$S_K = \frac{2}{\kappa^2}\int dtd^Dx\sqrt{g}N(K_{ij}K^{ij} - \lambda K^2), \tag{2.2}$$

where $K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ and the scaling dimension of the coupling constant κ at the fixed point is $[\kappa] = \frac{z-D}{2}$. Here λ is a dimensionless parameter and it is equal to 1 in the IR to restore general relativity. In the UV, to make sure that our theory is power-counting renormalizable we should have $z_{\rm UV} \geq D$. When $z_{\rm UV} = D$, the theory is renormalizable and when $z_{\rm UV} > D$, the theory is super-renormalizable.

The potential term is obtained by the "detailed-balance principle" of the form

$$S_V = \frac{\kappa^2}{8}\int dtd^Dx\sqrt{g}NE^{ij}\mathcal{G}_{ijk\ell}E^{k\ell} \tag{2.3}$$

with E^{ij} coming from a D dimensional relativistic action in the form

$$E^{ij} = \frac{1}{\sqrt{g}}\frac{\delta W_D[g_{ij}]}{\delta g_{ij}}, \tag{2.4}$$

and

$$\mathcal{G}_{ijk\ell} = \frac{1}{2}(g_{ik}g_{j\ell} + g_{i\ell}g_{jk}) - \tilde{\lambda}g_{ik}g_{j\ell}, \quad \tilde{\lambda} = \frac{\lambda}{D\lambda - 1}. \tag{2.5}$$

Because we consider the spatial isotropic theory, W_D must be the Euclidean action of a relativistic theory. The expression (2.4) connects a D-dimensional system described by the action W_D to a $D+1$-dimensional system described by the action $S_K - S_V$. This is the so called "detailed balance condition." It's well-known that it plays an important role in the quantum field theory for scalar fields and the role in the quantum field theory for gauge fields has also been studied in [1].

2.2 Action for $z = 4$ Hořava-Lifshitz gravity in 3+1 dimensions

To get the form of the action of the $z = 4$ Hořava-Lifshitz gravity in 3+1 dimensions from the detailed balance condition we follow [3] to begin from the action for general higher derivative corrections in three dimensional gravity theory

$$W[g_{ij}] = W_1 + W_2 + W_3, \tag{2.6}$$

$$W_1 = \mu\int d^3x\sqrt{g}(R - 2\Lambda_W), \tag{2.7}$$

$$W_2 = \frac{1}{w^2}\int d^3x\sqrt{g}\varepsilon^{ijk}\Gamma^m_{il}\left[\partial_j\Gamma^l_{km} + \frac{2}{3}\Gamma^l_{jn}\Gamma^n_{km}\right], \tag{2.8}$$

$$W_3 = \frac{1}{M}\int d^3x\sqrt{g}(R_{ij}R^{ij} + \beta R^2). \tag{2.9}$$

Here μ, ω^2, M and β are coupling constants. We take $\varepsilon_{ijk} = \sqrt{g}\epsilon_{ijk}$ with $\epsilon_{123} = 1$. When we only have W_1 and W_2, W is the action for TMG and the resulting Hořava-Lifshitz gravity just has $z=3$ in the UV. To get $z=4$ we need to keep the W_1 and W_3 terms and in this case when we choose $\beta=-3/8$, W represents the action of the Euclidean version of the NMG theory. Because the NMG itself is renormalizable in the Minkowski spacetime [29], we believe that the 3+1 dimensional $z=4$ theory constructed at this special β should be much simpler than at other β, which is indeed the case when we search for solutions in the next subsection. When we have all the W_1, W_2 and W_3 terms at $\beta=-3/8$, the action W reduces to the Euclidean version of the generalized massive gravity theory (GMG) which is the generalized version of three-dimensional massive gravity theory [21, 27].

Now we follow the quantum inheritance principle and the "detailed balance principle" to formulate the action of the 3+1 dimensional theory. We will keep β unfixed in this procedure. We can get E^{ij} from (2.4) as

$$\begin{aligned}
E^{ij} &= \frac{1}{\sqrt{g}}\frac{\delta W}{\delta g_{ij}} = \frac{2}{\omega^2}C^{ij} - \mu(G^{ij}+\Lambda_W g^{ij}) - \frac{1}{M}L^{ij}, \\
C^{ij} &= \varepsilon^{ik\ell}\nabla_k\left(R^j{}_\ell - \frac{1}{4}R\delta^j{}_\ell\right), \\
G^{ij} &= R^{ij} - \frac{1}{2}g^{ij}R, \\
L^{ij} &= (1+2\beta)(g^{ij}\nabla^2 - \nabla^i\nabla^j)R + \nabla^2 G^{ij} \\
&\qquad + 2\beta R\left(R^{ij} - \frac{1}{4}g^{ij}R\right) + 2\left(R^{imjn} - \frac{1}{4}g^{ij}R^{mn}\right)R_{mn}.
\end{aligned} \tag{2.10}$$

By combining the kinetic terms and the potential terms, we obtain the action

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1, \\
\mathcal{L}_0 &= \sqrt{g}N\left\{\frac{2}{\kappa^2}(K^{ij}K_{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda_W R - 3\Lambda_W^2)}{8(1-3\lambda)}\right\}, \\
\mathcal{L}_1 &= -\sqrt{g}N\frac{\kappa^2}{8}\left\{\frac{4}{\omega^4}C^{ij}C_{ij} - \frac{4\mu}{\omega^2}C^{ij}R_{ij} - \frac{4}{\omega^2 M}C^{ij}L_{ij} + \mu^2 G_{ij}G^{ij} + \frac{2\mu}{M}G^{ij}L_{ij}\right. \\
&\qquad \left. + \frac{2\mu}{M}\Lambda_W L + \frac{1}{M^2}L^{ij}L_{ij} - \tilde{\lambda}\left(\frac{L^2}{M^2} - \frac{\mu L}{M}(R - 6\Lambda_W) + \frac{\mu^2}{4}R^2\right)\right\},
\end{aligned} \tag{2.11}$$

where

$$L \equiv g^{ij}L_{ij} = \left(\frac{3}{2}+4\beta\right)\nabla^2 R + \frac{\beta}{2}R^2 + \frac{1}{2}R_{ij}R^{ij}. \tag{2.12}$$

When $\beta=-3/8$ and $w\to\infty$, the action (2.11) just represents the $z=4$ Hořava-Lifshitz gravity inherited from the new massive gravity, and this is the case we are most interested in because the renormalizability of the new massive gravity in Minkowski spacetime has been confirmed in [29].

In the UV, the terms with the highest scaling dimension in the potential terms will dominate, i.e. the terms inherited from W_3, and the theory exhibits a $z=4$ Lifshitz type fixed point in the UV region. In the IR, the terms with the lowest scaling dimension in the potential term will dominate, i.e. $\mathcal{L}_0$ will dominate. In order to obtain general relativity in the IR region, the effective coupling should be related to the speed of light c, the Newton coupling G and the effective cosmological constant as

$$c = \frac{\mu\kappa^2}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}}, \tag{2.13}$$

$$G_N = \frac{\kappa^2 c}{32\pi} \tag{2.14}$$

and

$$\Lambda = \frac{3}{2}\Lambda_W. \tag{2.15}$$

From the expression of the speed of light we see that in the IR region the cosmological constant can only be negative.

2.3 Black hole solutions

To search for black hole solutions in this theory, we follow [9, 14] to assume the form of the metric as

$$ds^2 = -\tilde{N}^2(r)f(r)c^2dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \tag{2.16}$$

where $d\Omega_k^2$ denotes the line element for a two dimensional Einstein space with constant scalar curvature $2k$ and volume Ω_k. Without loss of generality, we take $k = 0, \ \pm 1$ respectively. Then we can substitute this into the action (2.11) to get the equation of motion for $\tilde{N}(r)$ and $f(r)$ at any given energy scale. It is very complicated to find explicit solutions for a general β and we will focus on our most interesting case: $\beta = -3/8$, which is the Hořava-Lifshitz theory related to NMG in three dimensions. Coincidentally we find that at $\beta = -3/8$, the action has the least number of contributions which makes the calculation simplified. In this case when the Cotton tensor terms are also included, the three dimensional theory is the Euclidean generalized massive gravity theory (GMG), which has been recently studied in [21–29]. In addition, this GMG theory is renormalizable. As we know that the norm of the ground state of a $D+1$ dimensional Lifshitz theory is related to the partition function of the D dimensional relativistic theory [2], it is natural to consider this theory seriously.

We choose $\lambda = 1$ to seek for solutions in the IR region. Then we substitute the metric ansatz (2.16) into the action (2.11) and find that

$$\begin{aligned} I = \frac{\kappa^2\Omega_k}{16\sqrt{-\Lambda_W^3}}\int dt dx \tilde{N}\Bigg(\tilde{\mu}^2\left[x^3 - 2x(f-k) + \frac{(f-k)^2}{x}\right] \\ -2\tilde{\beta}\tilde{\mu}\left[\frac{(f-k)^3}{x^3} - \frac{(f-k)^2}{x}\right] + \tilde{\beta}^2\frac{(f-k)^4}{x^5}\Bigg)', \end{aligned} \tag{2.17}$$

where we have defined $\tilde{\mu} = -\mu\Lambda_W, \tilde{\beta} = \frac{\Lambda_W^2}{4M}$ and $x = \sqrt{-\Lambda_W}r$, and the prime denotes the derivative with respect to x. From the action, we can obtain the equations of motion as

$$\begin{aligned} 0 &= \tilde{N}'\left(\tilde{\beta}\frac{(f-k)^2}{x^2} - \tilde{\mu}(f-k-x^2)\right)\left(\frac{2\tilde{\beta}(f-k)}{x^2} - \tilde{\mu}\right), \\ c_0 &= \frac{1}{x}\left(\tilde{\beta}\frac{(f-k)^2}{x^2} - \tilde{\mu}(f-k-x^2)\right)^2, \end{aligned} \tag{2.18}$$

where $c_0 \geq 0$ is an integration constant. When $c_0 > 0$, from the equations of motion we can see that $N(r)$ should be a constant, which can always be set to 1 by a redefinition of the coordinate t. The function $f(r)$ can be obtained as

$$f = k + \frac{\tilde{\mu}}{2\tilde{\beta}} x^2 \left(1 \pm \sqrt{1 - \frac{4\tilde{\beta}}{x^2 \tilde{\mu}^2} (\tilde{\mu} x^2 \pm \sqrt{c_0 x})} \right). \tag{2.19}$$

To reduce the theory to the $z = 3$ one when $\tilde{\beta}$ goes to zero, the two $\pm$ in the solution (2.19) can only be chosen to be $-$. Thus the solution becomes

$$f = k + \frac{\tilde{\mu}}{2\tilde{\beta}} x^2 \left(1 - \sqrt{1 - \frac{4\tilde{\beta}}{x^2 \tilde{\mu}^2} (\tilde{\mu} x^2 - \sqrt{c_0 x})} \right). \tag{2.20}$$

In that case, we have to impose $4\tilde{\beta}/\tilde{\mu} \leq 1$ in order to have a well-defined vacuum solution. This solution is asymptotically AdS_4 and when $\tilde{\beta}$ goes to zero, this just comes back to the solution given in [14] up to a redefinition of the integration constant. When c_0 is zero, $\tilde{N}(r)$ can be an arbitrary function of r and $f(r)$ is just (2.19) with $c_0 = 0$. We suspect that other branches in (2.19) are unstable perturbatively. To confirm this, however, a careful analysis is needed. There is a potential singularity at the point where the square root vanishes in (2.20), in addition to the one at $x = 0$.

2.4 Thermodynamic properties of the black holes

The black hole horizon x_+ is given by the largest root of the equation $f(r) = 0$. The mass of the solution can be obtained by the Hamiltonian approach following [30, 31]

$$m = \frac{\kappa^2 \Omega_k}{16(-\Lambda_W)^{3/2}} c_0, \tag{2.21}$$

while c_0 can be expressed in terms of the horizon radius

$$c_0 = \frac{1}{x_+} \left(\frac{\tilde{\beta} k^2}{x_+^2} + \tilde{\mu}(k + x_+^2) \right)^2. \tag{2.22}$$

The Hawking temperature associated with the black hole is found to be

$$T = \frac{\sqrt{-\Lambda_W}}{8\pi x_+} \frac{3x_+^2 - k - \frac{5\tilde{\beta}k^2}{\tilde{\mu} x_+^2}}{1 + \frac{2\tilde{\beta}k}{\tilde{\mu} x_+^2}}. \tag{2.23}$$

When $\tilde{\beta} \to 0$, it goes back to the temperature of black hole in the $z = 3$ Hořava-Lifshitz gravity [14]. Following [14], one can obtain the black hole entropy by integrating the first law of black hole thermodynamics, which gives

$$S = \frac{\pi \kappa^2 \mu^2 \Omega_k}{4} \left(x_+^2 + 2k \ln x_+ - \frac{3\tilde{\beta}k^2}{\tilde{\mu} x_+^2} - \frac{\tilde{\beta}^2 k^3}{\tilde{\mu}^2 x_+^4} + \frac{4\tilde{\beta}k}{\tilde{\mu}} \ln x_+ \right) + S_0. \tag{2.24}$$

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Here S_0 is an integration constant, which cannot be determined by the thermodynamic method. Note that when $\tilde{\beta} \to 0$, the entropy reduces to the one given in [14]. The leading term is the area term, and the second is a logarithmic correction. Clearly one can see from the black hole entropy that $z = 4$ terms in the gravity action lead to the entropy correction terms $1/A$ and $1/A^2$, besides a new logarithmic term, here $A \sim x_+^2$ is the horizon area of the black hole. Note that for a Ricci flat black hole with $k = 0$, only the area term remains in the entropy expression.

2.5 Cosmological solutions

In this subsection, we are going to write out the Friedmann equation for cosmological solutions in the $z = 4$ Hořava-Lifshitz gravity theory in $3 + 1$ dimensions. We first assume that the Friedmann-Robertson-Walker (FRW) metric is of the form

$$ds^2 = -N^2(t)c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{2.25}$$

where $k = 1, 0, -1$ corresponding to a closed, flat and open universe respectively.

We substitute the ansatz (2.25) into the action (2.11), and obtain

$$I_g = \int dt d^3x\sqrt{g}\left\{-\frac{6(-1+3\lambda)\dot{a}^2}{\kappa^2 N a^2} + \frac{3\kappa^2 N}{8(-1+3\lambda)a^8}\left(2(1+3\beta)\frac{k^2}{M} - k\mu a^2 + \Lambda\mu a^4\right)^2\right\}, \tag{2.26}$$

where the dot denotes the derivative to t. Supposing that the matter contribution is equivalent to an ideal fluid and satisfies the "separated-detailed balance condition" [6], we have ρ and p of the matter to be

$$\rho = -\frac{1}{\sqrt{g}}\frac{\delta I_\phi}{\delta N}, \quad p = \frac{a}{3N\sqrt{g}}\frac{\delta I_\phi}{\delta a}. \tag{2.27}$$

The equation of motion for $N(t)$ gives

$$\frac{6}{\kappa^2}(-1+3\lambda)H^2 + \frac{3\kappa^2}{8(-1+3\lambda)a^8}\left(2(1+3\beta)\frac{k^2}{M} - k\mu a^2 + \Lambda_W\mu a^4\right)^2 = \rho, \tag{2.28}$$

and the equation of motion for $a(t)$ gives

$$\begin{aligned}\frac{4}{\kappa^2}(-1+3\lambda)\left(\dot{H} + \frac{3}{2}H^2\right) + \frac{\kappa^2}{4(-1+3\lambda)a^8}\left(2(1+3\beta)\frac{k^2}{M} - k\mu a^2 + \Lambda_W\mu a^4\right)\times \\ \times\left(-5(1+3\beta)\frac{k^2}{M} + \frac{1}{2}k\mu a^2 + \frac{3}{2}\Lambda_W\mu a^4\right) = -p,\end{aligned} \tag{2.29}$$

where

$$H \equiv \frac{\dot{a}}{Na}, \quad \dot{H} \equiv \frac{1}{N}\partial_t\left(\frac{\dot{a}}{Na}\right). \tag{2.30}$$

Form (2.28) and (2.29), we can obtain

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.31}$$

Note that for $k = 0$, there is no contribution from the higher-order derivative terms of the action. For $k = \pm 1$, the contribution of higher derivative terms dominates for small a and can be ignored at large a. This behavior is very analogous to the $z = 3$ case [6, 7, 9, 11] and indeed, when $M \to \infty$, we come back to the $z = 3$ case [7, 9]. We choose the gauge $N(t) = 1$ and for vacuum solutions with $\rho = p = 0$, in order to obtain de-Sitter solutions without matter, we should perform an analytic continuation of the parameters [9],

$$\mu \to i\mu, \quad \omega^2 \to -i\omega^2, \quad M \to -iM, \tag{2.32}$$

then $c = \frac{\mu\kappa^2}{4}\sqrt{\frac{\Lambda_W}{3\lambda-1}}$ and the Friedmann equations are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^4}{16(-1+3\lambda)^2}\left(2(1+3\beta)\frac{k^2}{Ma^4} - \frac{k\mu}{a^2} + \Lambda_W\mu\right)^2. \tag{2.33}$$

This equation can be integrated to give the explicit solution of $a(t)$ and because the solution is long and complicated, we don't write it out here. When $M \to \infty$, it comes back to the $z = 3$ case and it can be easily shown that for $k = 0$, it has de Sitter solutions.

3 $z = 4$ Hořava-Lifshitz gravity in 4+1 dimensions

In this section, we will study the $z = 4$ Hořava-Lifshitz gravity in 4+1 dimensions. The $z = 4$ theory in 4+1 dimensions is power-counting renormalizable. If the Hořava-Lifshitz gravity can indeed be a quantum gravity, it is interesting to investigate the AdS_5/CFT_4 correspondence in the framework of this theory. Thus it is interesting to study the $z = 4$ Hořava-Lifshitz gravity in 4+1 dimensions.

3.1 Action

The general four dimensional relativistic Lagrangian with higher derivative terms is of the form

$$W = \mu\int d^4x\sqrt{g}(R - 2\Lambda_W) + \frac{1}{M}\int d^4x\sqrt{g}(R_{ij}R^{ij} + \beta R^2). \tag{3.1}$$

Here the second term is the most general curvature square contribution because the Gauss-Bonnet term is a topological invariant in four dimensions.

Using (3.1), the E^{ij} can be obtained

$$\begin{aligned}
E^{ij} &= -\mu(G^{ij} + \Lambda_W g^{ij}) - \frac{1}{M}L^{ij}, \\
G^{ij} &= R^{ij} - \frac{1}{2}g^{ij}R, \\
L^{ij} &= (1+2\beta)(g^{ij}\nabla^2 - \nabla^i\nabla^j)R + \nabla^2 G^{ij} \\
&\quad + 2\beta R\left(R^{ij} - \frac{1}{4}g^{ij}R\right) + 2\left(R^{imjn} - \frac{1}{4}g^{ij}R^{mn}\right)R_{mn}.
\end{aligned} \tag{3.2}$$

Then the Lagrangian of the $z = 4$ Hořava-Lifshitz gravity in 4+1 dimensions are of the following form

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1, \\
\mathcal{L}_0 &= \sqrt{g}N\left\{\frac{2}{\kappa^2}(K^{ij}K_{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda_W R - 2\Lambda_W^2)}{4(1-4\lambda)}\right\}, \\
\mathcal{L}_1 &= -\sqrt{g}N\frac{\kappa^2}{8}\left\{\mu^2 G_{ij}G^{ij} + \frac{2\mu}{M}G^{ij}L_{ij} + \frac{2\mu}{M}\Lambda_W L + \frac{1}{M^2}L^{ij}L_{ij}\right. \\
&\qquad\left. -\tilde{\lambda}\left(\frac{L^2}{M^2} - \frac{2\mu L}{M}(R - 4\Lambda_W) + \mu^2 R^2\right)\right\}, \qquad (3.3)
\end{aligned}$$

where

$$L = 2(1+3\beta)\nabla^2 R. \qquad (3.4)$$

In the UV, this theory will exhibit a $z = 4$ Lifshitz-type fixed point and in the IR, the $\mathcal{L}_0$ term dominates and the theory flows to the $z = 1$ one. In order to get back to general relativity in the IR region, the effective couplings should be related to the speed of light c, the Newton coupling G and the effective cosmological constant Λ at IR in the way

$$c = \frac{\mu\kappa^2}{\sqrt{8}}\sqrt{\frac{\Lambda_W}{1-4\lambda}}, \qquad (3.5)$$

$$G_N = \frac{\kappa^2 c}{32\pi} \qquad (3.6)$$

and

$$\Lambda = \Lambda_W, \qquad (3.7)$$

respectively. In the IR region, to get back to general relativity λ should be chosen to be 1 and Λ_W can only be negative to have a physical speed of light.

3.2 Solutions

In this subsection we will seek for black hole solutions in the IR region. We make an ansatz for the metric form as

$$ds^2 = -\tilde{N}^2(r)f(r)c^2dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \qquad (3.8)$$

where $d\Omega_k^2$ is the three-dimensional Einstein manifold with constant scalar curvature $6k$, which we may choose to be $k = 0, \ \pm 1$, without loss of generality.

It is quite difficult to find the exact solution for a general β in this case, and motivated by the 3+1 dimensional case, we consider a special value of $\beta = -1/3$, which comes from an unsuccessful attempt to generalize the new massive gravity to the case in four dimensions [22]. We solve the solution following the method of [9, 14]. We substitute the metric ansatz into the action and find that

$$\begin{aligned}
I = \frac{\kappa^2\mu^2\Omega_k}{24}\int dtdx\tilde{N}(x)\frac{1}{x}\Big\{&12(1-\lambda)(f-k)^2 + 6x(f-k)(-2x + (1+2\lambda)f') \qquad (3.9)\\
&+x^2(4x^2 - 6xf' + 3(1-\lambda)f'^2)\Big\},
\end{aligned}$$

where $x = \sqrt{-\Lambda_W} r$. Then in the IR region, we further consider the case of $\lambda = 1$ and obtain

$$I = \frac{\kappa^2 \mu^2 \Omega_k}{24} \int dt dx \tilde{N} \Big(\big(3(f-k) - x^2 \big)^2 \Big)' . \tag{3.10}$$

The equations of motion are

$$\begin{aligned} 0 &= \tilde{N}' (3(f-k) - x^2), \\ c_0 &= \big(3(f-k) - x^2 \big)^2, \end{aligned} \tag{3.11}$$

where $c_0 \geq 0$ is an integration constant. When $c_0 > 0$, $\tilde{N}(r)$ should be a constant and we can always choose it to be 1 through a redefinition of the coordinate t. The solution of $f(r)$ can be easily obtained as

$$f = k + \frac{x^2}{3} \pm \sqrt{c_0}, \tag{3.12}$$

where $c_0 > 0$. When $c_0 = 0$, $\tilde{N}(r)$ can be an arbitrary function of r and $f(r)$ is just the solution (3.12) with $c_0 = 0$. In the calculation of the thermodynamic properties in the next subsection, we only focus on the solutions with $\tilde{N}(r) = 1$ and $c_0 \geq 0$, which are of interest. In addition let us notice that the solution (3.12) is the same as the topological black hole solution in the five-dimensional Chern-Simons gravity [30], or in the five-dimensional Gauss-Bonnet gravity with a special Gauss-Bonnet coefficient [32].

3.3 Thermodynamic properties of the black holes

Note that when $c_0 = 0$ and $\tilde{N} = 1$, the solution (3.12) is a five-dimensional AdS space. When $c_0 > 0$, the black hole horizon exists at $x_+^2 = 3(\mp\sqrt{c_0} - k)$ if $\mp\sqrt{c_0} - k \geq 0$. Therefore when $k = 1$ or $k = 0$, the black hole horizon exists only in the minus branch in (3.12), while it requires $\sqrt{c_0} < 1$ in the plus branch if $k = -1$. Of course, in the case of $k = -1$, the black hole horizon always exists for the minus branch. According to the Hamiltonian approach [30, 31], the mass of the solution is

$$m = \frac{\kappa^2 \mu^2 \Omega_k}{24} c_0. \tag{3.13}$$

Comparing the solution (3.12) with the one in the Gauss-Bonnet gravity [32], we expect that the solution with the plus branch is unstable perturbatively. Therefore in what follows, we focus on the minus branch only. In that case, the black hole horizon is at $x_+^2 = 3(\sqrt{c_0} - k)$. One can see immediately that when $k = 1$, there is a mass gap $c_0 = 1$; when $c_0 < 1$, black hole horizon does not exist. While $k = -1$, there is a minimal horizon radius $x_{+\min} = \sqrt{3}$.

The Hawking temperature of the black hole is easy to obtain. Either directly calculating surface gravity at black hole horizon, or requiring the absence of conical singularity at black hole horizon in the Euclidean sector of the black hole solution gives

$$T = \frac{\sqrt{-\Lambda_W}}{6\pi} x_+. \tag{3.14}$$

Clearly the temperature is a monotonically increasing function of horizon radius x_+. Therefore the black hole is always thermodynamically stable. Using the first law of black hole thermodynamics, $dm = TdS$, we obtain the black hole entropy

$$S = \frac{\pi\kappa^2\mu^2\Omega_k}{27\sqrt{-\Lambda_W}}\left(x_+^3 + 9kx_+\right) + S_0. \tag{3.15}$$

The leading term is proportional to the horizon area of the black hole, while the linear term vanishes for a Ricci flat black hole with $k = 0$.

4 Conclusion and discussion

In this note, we formulated the action of the $z = 4$ Hořava-Lifshitz gravity in both 3+1 and 4+1 dimensions following [3], and found static black hole solutions in the IR region. We also analyzed thermodynamic properties of the black hole solutions. We only find explicit black hole solutions for $\lambda = 1$ and $\beta = -3/8$ in 3+1 dimensions and black hole solutions for $\lambda = 1$ and $\beta = -1/3$ in 4+1 dimensions. These special value of β correspond to $z = 4$ Hořava-Lifshitz gravity theories inherited from NMG or GMG, which are of interest. It is interesting to find exact analytic solutions for general λ and β. We also write out the Friedmann equation for cosmological solutions in the $z = 4$ Hořava-Lifshitz gravity in 3+1 dimensions for general λ and β, which shows that for $k = 0$ there is no contribution from $z = 4$ terms.

The quantum inheritance principle is very important in the construction of the Hořava-Lifshitz gravity and still needs to be further understood in the framework of quantum gravity. A relevant problem is the relationship between the new massive gravity and the Hořava-Lifshitz gravity, which also needs to be further studied. If the Hořava-Lifshitz gravity is indeed a quantum gravity, the understanding of AdS/CFT correspondence in the frame work of this theory would be very important and it would be interesting to further analyze the asymptotically AdS solutions in five dimensions we obtained in this paper. In addition, it is very interesting to study the implications of this gravity theory in cosmology.

Acknowledgments

We would like to thank Hong Lü for useful correspondence. This work was supported in part by the Chinese Academy of Sciences with Grant No. KJCX3-SYW-N2 and the NSFC with Grant No. 10821504 and No. 10525060.

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