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Abstract: We construct solutions for thin black rings in Anti-deSitter and deSitter spacetimes using approximate methods. Black rings in AdS exist with arbitrarily large radius and satisfy a bound $|J| \leq LM$, which they saturate as their radius becomes infinitely large. For angular momentum near the maximum, they have larger area than rotating AdS black holes. Thin black rings also exist in deSitter space, with rotation velocities varying between zero and a maximum, and with a radius that is always strictly below the Hubble radius. Our general analysis allows us to include black Saturns as well, which we discuss briefly. We present a simple physical argument why supersymmetric AdS black rings must not be expected: they do not possess the necessary pressure to balance the AdS potential. We discuss the possible existence or absence of 'large AdS black rings' and their implications for a dual hydrodynamic description. An analysis of the physical properties of rotating AdS black holes is also included.

Keywords: AdS-CFT Correspondence, Black Holes.

Contents

1. Introduction

The discovery of black rings [1, 2] and other novel black holes in asymptotically flat higher-dimensional gravity [3] naturally prompts the question of whether they admit counterparts in the presence of a cosmological constant. In particular, their existence (or absence) in Anti-deSitter space is of special interest for the possible implications in the context of the AdS/CFT duality. However, in spite of attempts since early on, an exact solution describing an (Anti-)deSitter black ring remains elusive.

Nevertheless, there appears no obvious physical reason why these solutions should not be possible. Black rings represent states of equilibrium between the tension and the centrifugal force of a rotating circular black string. Putting a ring in Anti-deSitter space should have the effect of increasing the gravitational centripetal pull on it, but, at least within some parameter ranges, this can be plausibly balanced by spinning the black ring faster. On the other hand, if we put the ring in deSitter space, the cosmological expansion should act against the tension, and so the required rotation should be smaller and possibly reach zero. Thus, the failure to find any exact (Anti-)deSitter black rings seems to be no more than a technical snag. The situation in this respect is similar to that of asymptotically flat black rings in $d \geq 6$, for which exact analytic solutions seem unlikely, but nevertheless thin black rings have been constructed via approximate methods in every dimension $d \geq 5$ [4].

It seems natural, then, to apply the methods of [4] to the approximate construction of cosmological black rings. This is the subject of this paper, and our results prove the above conjectures correct. Namely, thin black rings exist in $\mathrm{AdS}_{d\geq 5}$ with arbitrarily large radius, which spin comparatively faster than their asymptotically flat counterparts; and black rings are also possible in $\mathrm{dS}_{d\geq 5}$ with rotation velocities that range between zero and a maximum. An exact black ring solution in $\mathrm{(A)dS}_5$ may still be found, but we doubt that this is possible in $d \geq 6$.

By construction, our black rings, with horizon topology $S^1 \times S^{d-3}$, are *thin*, in the sense that their S^{d-3}-radius is much smaller than both the S^1-radius *and* the cosmological radius. These black rings have negative specific heat, and in the AdS/CFT context belong in the same class as 'small AdS black holes':[1] metastable excitations of the thermal dual CFT that do not admit a dual hydrodynamic description. However, one may still wonder whether 'large AdS black rings' exist beyond the reach of our approximate methods, and whether they could correspond to hydrodynamic phases of the thermal CFT that have been overlooked so far. We shall discuss the available evidence to explore this possiblity.

May black rings also exist in other backgrounds? More precisely, given a stationary spacetime with a spatial U(1) isometry, is it possible to place a (thin) black ring along the direction of an orbit of the isometry? We have made some headway in answering in the affirmative this question for static backgrounds. The perturbation analysis of [4] can be easily extended to a wide class of static backgrounds, for which we prove that a boosted neutral black string that satisfies a suitable, easy to verify, equilibrium condition, remains regular on and outside its horizon when bent into a circular shape.

The approach of [4] can also be readily extended to black rings charged under gauge fields. If we apply it to the supersymmetric black rings of five-dimensional supergravity [5], we find a simple, physical explanation for why supersymmetric AdS_5 black rings have not turned up despite many attempts to find them:[2] supersymmetric black strings have zero

[1]Small in the sense that one of the horizon dimensions is smaller than the AdS radius. All our solutions have regular horizons of finite area in the second-derivative theory.

[2]For published attempts, see *e.g.,* [6 – 8].

pressure, so if we bend them and place them in AdS, they can not counterbalance the centripetal pull of the AdS potential. This argument fits nicely with the analysis of [7, 8], which concluded from a study of near-horizon solutions that supersymmetric black rings can not exist without conical singularities on the plane of the ring.

The paper is organized as follows. Section 2 describes how to obtain the equilibrium condition for black rings in AdS, and using this, proceeds to analyze their physical properties. The technical details of the perturbation theory for curving a black string into a black ring are deferred to appendices A and B, which actually deal with a much more general situation than required for (A)dS spacetimes. Section 3 compares the properties of black rings and rotating AdS black holes with a single angular momentum. In section 4 we discuss a plausible extension of the phases of thin black rings that we have constructed, and then we examine whether large AdS black rings may exist that could be described within the fluid-dynamical approach to rotating black holes in AdS [9, 10]. Section 5 demonstrates how a simple application of our methods allows to conclude that black rings, including static ones, are possible in deSitter space, and how a class of black Saturns can be easily obtained too. Section 6 argues that thin supersymmetric black rings should not be expected in AdS_5. Section 7 concludes with remarks on the outlook from the paper. Besides the appendices mentioned above, appendix C contains a brief discussion of some general properties of rotating AdS black holes with several angular momenta.

Notation and terminology. Throughout this work we find convenient to introduce dimensionless magnitudes, denoted by sans-serif fonts, corresponding to quantities measured in units of the cosmological radius L or 'cosmological mass' scale L^{d-3}/G. For instance, for the S^1-radius, mass, angular momentum and horizon area of the ring we define

$$\mathsf{R} = \frac{R}{L}\,, \qquad \mathsf{M} = \frac{GM}{L^{d-3}}\,, \qquad \mathsf{J} = \frac{GJ}{L^{d-2}}\,, \qquad \mathsf{A_H} = \frac{A_H}{L^{d-2}}\,. \tag{1.1}$$

Equivalently, we might have set $L = 1 = G$, but the meaning of some formulas is clearer if we retain L.

To avoid misnomers, we shall refer to the solutions that generalize the Kerr and Myers-Perry black holes to include a cosmological constant [11−13] as *rotating (A)dS black holes.* Even if (A)dS black rings and pinched black holes are also (A)dS black holes, we hope that the distinction is clear from the context.

2. Thin black rings in AdS

2.1 The approximation

Our analysis of thin black rings in Anti-deSitter space extends the basic idea developed in [4]. The rings, with horizon topology $S^1 \times S^{d-3}$, are characterized by the two radii R and r_0 of the S^1 and S^{d-3} respectively. They are built by bending a straight, thin, boosted black string into a circular shape. At large distances from the horizon of the black ring, the latter is approximated by a distributional source of energy-momentum, while near the ring the solution is obtained by perturbing the straight black string into an arc of large radius R.

In the present case, besides the circle radius R we have another scale from the Anti-deSitter curvature radius L, which we also take to be much larger than r_0. It is clear that if $R \ll L$ the solutions are only slight modifications of the asymptotically flat black rings of [4]. However, we do not want to assume any hierarchy among R and L, in particular we want to describe black rings for which $R > L$, which may behave differently than those with $R \ll L$. In appendix A we show that our approximations are in fact valid whenever[3]

$$r_0 \ll \min(R, L), \tag{2.1}$$

which allows us to treat rings longer than the cosmological scale, $R > L$, as long as the S^{d-3} is small on that scale, $r_0 < L$.

To be specific, consider global AdS_d spacetime,

$$ds^2 = -V(\rho)\, d\tau^2 + \frac{d\rho^2}{V(\rho)} + \rho^2 \left(d\Theta^2 + \sin^2\Theta\, d\Omega_{d-4}^2 + \cos^2\Theta\, d\psi^2\right), \tag{2.2}$$

where

$$V(\rho) = 1 + \frac{\rho^2}{L^2}. \tag{2.3}$$

We shall place the ring at $\rho = R$ on the $\Theta = 0$ plane. Observe that R measures the proper length of the circle. It is natural to define proper time and length coordinates, t and z, along the worldsheet of the ring as

$$t = \sqrt{V(R)}\,\tau = \sqrt{1 + \mathsf{R}^2}\,\tau, \qquad z = R\,\psi. \tag{2.4}$$

The coordinate z must be regarded as periodic, $z \sim z + 2\pi R$. Observe that the AdS background has the effect of introducing a redshift between the proper time coordinate t near the ring and the canonically-normalized asymptotic time τ.

At large distance in the directions transverse to the ring, the gravitational field it creates is the same as that of an equivalent circular distribution of energy-momentum, $T_{\tau\tau}$, $T_{\tau\psi}$, $T_{\psi\psi}$, centered at $\rho = R$ on the $\Theta = 0$ plane in (2.2). We specify the source locally using the orthonormal coordinates (t, z) along the string worldsheet, in which it takes the same form as for a straight boosted string in flat space,

$$\begin{aligned}
T_{tt} &= \frac{r_0^{d-4}}{16\pi G}\left((d-4)\cosh^2\alpha + 1\right)\ \delta^{(d-2)}(r)\,,\\
T_{tz} &= \frac{r_0^{d-4}}{16\pi G}\ (d-4)\cosh\alpha\sinh\alpha\ \delta^{(d-2)}(r)\,,\\
T_{zz} &= \frac{r_0^{d-4}}{16\pi G}\left((d-4)\sinh^2\alpha - 1\right)\ \delta^{(d-2)}(r)\,,
\end{aligned} \tag{2.5}$$

where α is the boost parameter along the string and $r = 0$ is the location of the ring[4] (see appendix A for the construction of coordinates adapted to the ring).

[3] Actually we derive a more precise form, $r_0 \ll R/\sqrt{1 + \mathsf{R}^2}$, equivalent to (2.1) within factors of order one: this is the condition that r_0 be much smaller than the extrinsic curvature radius of the S^1.

[4] We normalize $\int_{B^{d-2}} \delta^{(d-2)}(r) = \Omega_{d-3}$ for a ball B^{d-2} that intersects $r = 0$ once.

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2.2 Equilibrium condition

Not every energy-momentum source can be consistently coupled to gravity: it must satisfy the conservation equations $\nabla_\mu T^\mu{}_\nu = 0$. For a thin brane, such as our black ring, this is equivalent to imposing the 'equations of motion' $K_{\mu\nu}{}^\rho T^{\mu\nu} = 0$, derived by Carter in [14], where $K_{\mu\nu}{}^\rho$ is the extrinsic curvature tensor of the brane embedding.

For a circular source at $\rho = R$, $\Theta = 0$ in a metric of the generic form (2.2), these equations imply

$$\frac{RV'(R)}{2V(R)} T^\tau{}_\tau + T^\psi{}_\psi = 0\,, \tag{2.6}$$

so in the case of Anti-deSitter (2.3),[5]

$$\frac{\mathsf{R}^2}{1+\mathsf{R}^2} T^\tau{}_\tau + T^\psi{}_\psi = 0\,. \tag{2.7}$$

Observe that, whereas asymptotically flat black rings ($\mathsf{R} = 0$) are sourced by a pressureless distribution of matter, the attraction caused by the AdS potential demands a non-vanishing pressure of the circular source to achieve equilibrium.

Since $T^\tau{}_\tau = T^t{}_t = -T_{tt}$ and $T^\psi{}_\psi = T^z{}_z = T_{zz}$, using (2.5) we find that the equilibrium condition (2.7) amounts to

$$\sinh^2\alpha = \frac{1 + (d-2)\,\mathsf{R}^2}{d-4}\,. \tag{2.8}$$

This condition fixes the value of the boost that provides the correct centrifugal force to counterbalance the ring tension and the AdS gravitational potential. As $\mathsf{R} \to 0$ we recover the finite, flat-space value obtained in [4]. The cosmological constant has the effect that the equilibrium boost depends on the ring radius: the longer the ring, the stronger the cosmological pressure on it, so the boost increases with R and approaches the speed of light as $\mathsf{R} \to \infty$.

Eq. (2.8) is a necessary condition for the existence of a regular solution for a thin black ring. But in addition, we must also construct the metric perturbation of the boosted black string that curves it into a circular shape, and check that the perturbed horizon remains regular. This calculation is described in appendix B.

2.3 Physical properties

We compute the mass and angular momentum of the distributional ring as

$$M = \int_{\Sigma_\tau} T_{\mu\nu} n^\mu \xi^\nu\,, \tag{2.9}$$

$$J = \int_{\Sigma_\tau} T_{\mu\nu} n^\mu \chi^\nu\,, \tag{2.10}$$

where Σ_τ is a spatial section at constant time τ, the vector n is the unit future-directed normal to Σ_τ and $\xi = \partial_\tau$ and $\chi = \partial_\psi$ are the asymptotically canonically normalized Killing vectors conjugate to the mass and angular momentum. Since

$$T_{\mu\nu} n^\mu \xi^\nu = \sqrt{1+\mathsf{R}^2}\, T_{tt}\,, \qquad T_{\mu\nu} n^\mu \chi^\nu = R\, T_{tz} \tag{2.11}$$

[5] The general case for $V(\rho)$ will be studied in section 5.

we find, substituting the equilibrium value (2.8) into (2.5),

$$M = \frac{r_0^{d-4}\,L}{8G}\Omega_{d-3}(d-2)\mathsf{R}\left(1+\mathsf{R}^2\right)^{3/2}\,, \tag{2.12}$$

and

$$J = \frac{r_0^{d-4}\,L^2}{8G}\Omega_{d-3}\mathsf{R}^2\left[\left(1+(d-2)\mathsf{R}^2\right)\left(d-3+(d-2)\mathsf{R}^2\right)\right]^{1/2}\,. \tag{2.13}$$

(without loss of generality, we shall assume throughout the paper that J is positive). Thus we obtain that the equilibrium of rings in AdS requires that the radius be fixed in terms of the angular momentum per unit mass by

$$\frac{J}{ML} = \frac{1}{d-2}\frac{\mathsf{R}}{\left(1+\mathsf{R}^2\right)^{3/2}}\left[\left(1+(d-2)\mathsf{R}^2\right)\left(d-3+(d-2)\mathsf{R}^2\right)\right]^{1/2}\,. \tag{2.14}$$

Other physical properties of the black ring are easily obtained. The horizon area is the area of a boosted black string of length $\Delta z = 2\pi R$,

$$A_H = 2\pi R\,\Omega_{d-3}r_0^{d-3}\cosh\alpha = 2\pi L r_0^{d-3}\Omega_{d-3}\mathsf{R}\sqrt{\frac{d-3+(d-2)\mathsf{R}^2}{d-4}}\,. \tag{2.15}$$

The horizon of the boosted black string is generated by the Killing vector

$$\begin{aligned}\hat{\zeta} &= \partial_t + \tanh\alpha\,\partial_z = \frac{1}{\sqrt{1+\mathsf{R}^2}}\left(\partial_\tau + \frac{\sqrt{1+\mathsf{R}^2}}{R}\tanh\alpha\,\partial_\psi\right)\\ &= \frac{1}{\sqrt{1+\mathsf{R}^2}}\zeta\end{aligned} \tag{2.16}$$

where $\zeta = \partial_\tau + \Omega_H\partial_\psi$ is the null generator of the horizon in terms of the Killing generators of time translations and rotations at infinity. Thus we find the angular velocity of the horizon

$$\Omega_H = \frac{\sqrt{1+\mathsf{R}^2}}{R}\tanh\alpha = \frac{1}{L}\sqrt{\frac{(1+\mathsf{R}^2)(1+(d-2)\mathsf{R}^2)}{\mathsf{R}^2(d-3+(d-2)\mathsf{R}^2)}}\,. \tag{2.17}$$

Observe that

$$\Omega_H > \frac{1}{L}\,, \tag{2.18}$$

and $\Omega_H L \to 1$ as $\mathsf{R}\to\infty$. For the rotating AdS black hole solutions an angular velocity $\Omega_H L > 1$ was argued in [15] to be associated with the existence of ergoregions relative to asymptotic observers, which in AdS are believed to yield a superradiant instability [16, 17]. Certainly we expect our thin black rings to possess these ergoregions[6] and thus they should suffer from this instability. However, the time scale of the instability may be very long for thin rings.

On the other hand, our black rings should suffer from Gregory-Laflamme-type instabilities [18] along the direction of the ring. In fact the boost enhances the instability of the

[6] The non-superradiant ergoregion of the boosted black string becomes superradiant when bent into a ring-shape since the asymptotic behavior of modes changes.

black string by reducing the wavelength of the threshold mode, $\lambda_{\rm GL} \sim r_0/\cosh\alpha$ [19], so for a given radius R the black ring in AdS fits more easily the unstable wavelengths than in the asymptotically flat case.

The surface gravity κ of the Killing horizon generated by ζ is related through the redshift factor to the surface gravity $\hat\kappa$ associated to $\hat\zeta$,

$$\kappa = \sqrt{1+\mathsf{R}^2}\,\hat\kappa = \sqrt{1+\mathsf{R}^2}\frac{d-4}{2r_0\cosh\alpha} = \frac{(d-4)^{3/2}\sqrt{1+\mathsf{R}^2}}{2r_0\sqrt{d-3+(d-2)\mathsf{R}^2}}\,. \tag{2.19}$$

Our results for M, J, A_H, Ω_H and κ are in principle valid up to corrections of order $r_0/\min(R,L)$. For $d\geq 6$ they are very likely correct up to the next order: in the same way as argued in [4], the dipole corrections to the metric induced at order $r_0/\min(R,L)$ do not generate any corrections to the physical magnitudes. The only possibility for corrections at this order is through gauge modes becoming physical through boundary conditions. In [4], these effects were present only in $d=5$, and this is likely the case as well in AdS.

It is straightforward to check that the first law

$$dM = \frac{\kappa}{8\pi G}dA_H + \Omega_H dJ\,, \tag{2.20}$$

is satisfied. However, the naive Smarr relation is not satisfied, since

$$\frac{d-3}{d-2}M \neq \frac{\kappa}{8\pi G}A_H + \Omega_H J\,. \tag{2.21}$$

This is due to the presence of the cosmological length scale, which invalidates the standard scaling argument that connects the first law and the Smarr relation.

A "BPS bound" on the angular momentum of regular solutions in AdS was proven in [20], which, for solutions with a single (positive) angular momentum, takes the form

$$J \leq ML\,. \tag{2.22}$$

It is easy to check that this bound is satisfied by our black rings. It is also saturated in the limit of very long rings, $\mathsf{R}\to\infty$, where the difference between ML and J at fixed mass decreases to zero as $\mathsf{R}\to\infty$,

$$ML - J = \frac{ML}{\mathsf{R}^2} + \mathcal{O}(\mathsf{R}^{-4})\,. \tag{2.23}$$

For a given L, the solutions are characterized by the dimensionless mass M and angular momentum J defined in (1.1). The BPS bound is equivalently expressed as $\mathsf{J}\leq\mathsf{M}$. For fixed mass M, the horizon area decreases to zero as the bound is approached like

$$\mathsf{A_H} \propto \mathsf{M}^{\frac{d-3}{d-4}}\left(1-\frac{\mathsf{J}}{\mathsf{M}}\right)^{\frac{d-2}{d-4}}\left(1+\mathcal{O}(\mathsf{M}-\mathsf{J})\right) \tag{2.24}$$

(we omit the numerical proportionality factor, which we will not require).

3. Black rings vs. rotating AdS black holes

In order to make a comparison to rotating AdS black holes, we first analyze the properties of the latter. The solutions were found in [12, 13] and their physical parameters correctly computed in [21]. We consider the situation with only one spin turned on,[7] for which the solutions are parametrized by the mass and rotation parameters m and a, and the physical magnitudes are

$$M = \frac{\Omega_{d-2}}{4\pi G}\frac{m}{\Xi^2}\left(1+\frac{(d-4)\Xi}{2}\right), \tag{3.1}$$

$$J = \frac{\Omega_{d-2}}{4\pi G}\frac{ma}{\Xi^2}, \tag{3.2}$$

$$A_H = \Omega_{d-2} r_+^{d-4}\frac{r_+^2+a^2}{\Xi}, \tag{3.3}$$

$$\Omega_H = \frac{a}{L^2}\frac{r_+^2+L^2}{r_+^2+a^2}, \tag{3.4}$$

$$\kappa = r_+\left(1+\frac{r_+^2}{L^2}\right)\left(\frac{d-3}{2r_+^2}+\frac{1}{r_+^2+a^2}\right)-\frac{1}{r_+}, \tag{3.5}$$

with

$$\Xi \equiv 1-\frac{a^2}{L^2} \tag{3.6}$$

and r_+ being the largest positive real root of

$$r_+^{d-5}(r_+^2+L^2)(r_+^2+a^2) = 2mL^2\,. \tag{3.7}$$

The rotation parameter is restricted to $a \le L$. It is then easy to see that these solutions satisfy the bound $J \le ML$, with $J/ML \to 1$ as $a \to L$.

3.1 Range of angular momentum

There are significant differences between $d = 5$ and $d \ge 6$ in how the spin is bounded above at any given mass, and in the nature of the solutions at maximum angular momentum. These are the AdS counterparts of properties of the asymptotically flat case. In many respects, the limit $a \to L$ and the bound $\mathsf{J} = \mathsf{M}$ in AdS correspond to the limit $a \to \infty$ and the ultra-spinning regime of infinite spin for fixed mass of Myers-Perry (MP) black holes [22, 23].

In $d = 5$ eq. (3.7) can be solved explicitly. Real values of r_+ that yield black hole solutions exist provided that $a^2 < 2m$. This implies that we cannot take $a \to L$ and at the same time keep the mass M fixed at a finite value, so the BPS bound $\mathsf{J} = \mathsf{M}$ is never saturated. In the limit $a^2 \to 2m$ we have $r_+ \to 0$ and the surface gravity $\kappa \to 0$: this is an extremal naked singularity of zero area, with maximum angular momentum for given mass,

$$\mathsf{J}^2_{\max}(\mathsf{M}) = \mathsf{M}^2 + \frac{9\pi}{16}\mathsf{M} + \frac{27\pi^2}{512} - \sqrt{\pi}\left(\mathsf{M}+\frac{9\pi}{64}\right)^{3/2} < \mathsf{M}^2 \qquad (d=5)\,. \tag{3.8}$$

[7] Some properties of solutions with several spins are discussed in appendix C.

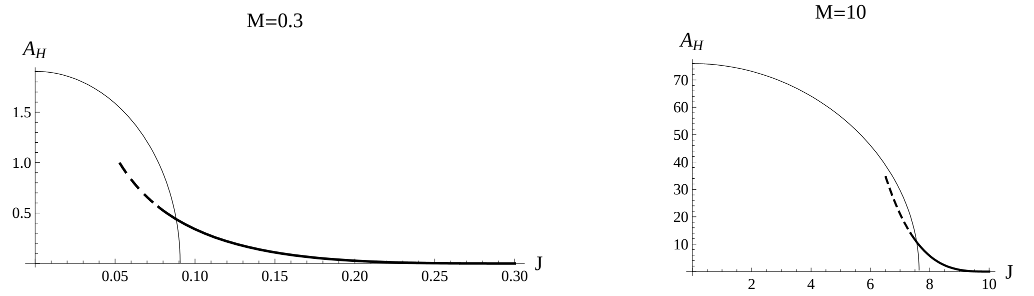

Figure 1: Plots in five dimensions of $\mathsf{A_H}(\mathsf{J})$ for fixed M, at small M (left) and large M (right). Thin lines correspond to rotating AdS black holes, thick lines to black rings. The thick solid line continues to a thick dashed line when the thin-ring approximation $r_0 \ll R/\sqrt{1+\mathsf{R}^2}$ breaks down (here and in the next figures the solid-dashed divide is arbitrarily taken at $r_0 = \frac{R}{5\sqrt{1+\mathsf{R}^2}}$ and the dashed line is extended up to $r_0 = \frac{R}{2\sqrt{1+\mathsf{R}^2}}$). The spin of black rings reaches up to the BPS bound $\mathsf{J} = \mathsf{M}$, but the spin of rotating AdS black holes is bounded by $\mathsf{J} \leq \mathsf{J}_{\max}(\mathsf{M}) < \mathsf{M}$, eq. (3.8). For small M, and J not too close to M, the curves are very similar to the asymptotically flat case.

At small M this reduces to the known asymptotically flat extremality bound, $\mathsf{J}_{\max}^2 \to 32\mathsf{M}^3/27\pi$, while at very large M the extremal angular momentum approaches the BPS bound, $\mathsf{J}_{\max} \to \mathsf{M}(1 - \sqrt{\frac{\pi}{4\mathsf{M}}})$.

In $d \geq 6$, instead, the same argument of the asymptotically flat case [22] shows that for any value of $a \in [0, L)$ and for finite mass there is always a real positive solution to (3.7). In the limit $a \to L$, which requires taking $m \to 0$ in order to keep the mass finite, we find

$$\mathsf{J}_{\max}(\mathsf{M}) = \mathsf{M} \qquad (d \geq 6)\,, \tag{3.9}$$

so the BPS bound is saturated. In this limit one finds $r_+ \to 0$. However, the solutions that saturate the bound are not extremal, in the sense that the surface gravity does not vanish, but instead diverges like $\kappa \to (d-5)/2r_+$. The area vanishes in the limit, decreasing to zero like

$$\mathsf{A_H} \propto \mathsf{M}^{\frac{d-4}{d-5}} \left(1 - \frac{\mathsf{J}}{\mathsf{M}}\right)^{\frac{d-3}{d-5}} (1 + \mathcal{O}(\mathsf{M} - \mathsf{J}))\,. \tag{3.10}$$

Comparing to (2.24), we see that near the maximum J, black rings always have larger area than rotating AdS black holes with the same mass.

We plot the curves $\mathsf{A_H}(\mathsf{J})$ for fixed M, at small and large M, in five dimensions, figure 1, and in seven dimensions, figure 2, as a representative of $d \geq 6$.

3.2 Limits to black membranes

There are other features that justify referring to the limit $a \to L$ in $d \geq 6$ as an analogue of the 'ultraspinning limit' of asymptotically flat MP black holes. Here we follow [23]. The sectional areas of the black hole in the directions transverse and parallel to the rotation plane allow to identify a transverse and parallel size of the horizon,

$$\ell_\perp \sim r_+\,, \qquad \ell_\parallel \sim \left(\frac{r_+^2 + a^2}{\Xi}\right)^{1/2} \qquad (a \to L) \tag{3.11}$$

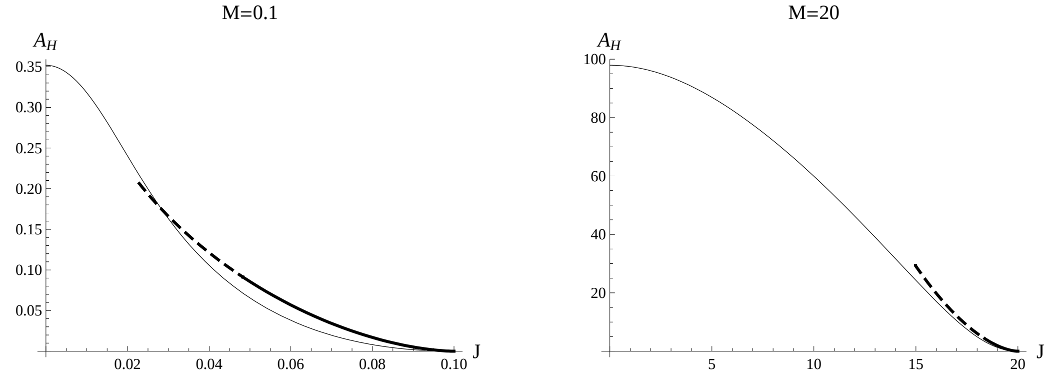

Figure 2: Plots in seven dimensions of $\mathsf{A_H}(\mathsf{J})$ for fixed M, at small M (left) and large M (right). Thin lines correspond to rotating AdS black holes, thick lines to black rings (dashed line when the thin-ring approximation breaks down). Both rotating AdS black holes and black rings extend up to the BPS bound $\mathsf{J} = \mathsf{M}$, but black rings have larger area near the maximum J. The asymptotic curves near this point are (2.24) and (3.10).

(it is slightly convenient to use Ξ as the small parameter). In this limit r_+ decreases like $\sim L\,\mathsf{M}^{\frac{1}{d-5}}\,\Xi^{\frac{2}{d-5}}$ so at fixed mass the ratio $\ell_\parallel/\ell_\perp$ diverges like $\sim \Xi^{-\frac{d-1}{2(d-5)}}$: the horizon pancakes out along the plane of rotation. The geometry in fact approaches that of a black membrane as the upper bound on J is approached. To see this, take the solution in Boyer-Lindquist coordinates $(t, r, \theta, \phi, \Omega_{d-4})$ of [13], and define a new mass parameter and new coordinates

$$\hat{\mu} = \frac{2m}{L^2\Xi^2}\,, \qquad \hat{t} = \Xi^{-\frac{2}{d-5}}t\,, \qquad \hat{r} = \Xi^{-\frac{2}{d-5}}r\,, \qquad \sigma = \Xi^{-\frac{d-7}{2(d-5)}}L\sin\theta \tag{3.12}$$

that remain finite as $a \to L$, *i.e.,* as $\Xi \to 0$. In the limit the metric becomes

$$ds^2 \to \Xi^{\frac{4}{d-5}}\left[-\left(1-\frac{\hat{\mu}}{\hat{r}^{d-5}}\right)d\hat{t}^2 + \frac{d\hat{r}^2}{1-\hat{\mu}/\hat{r}^{d-5}} + d\sigma^2 + \sigma^2 d\phi^2 + \hat{r}^2 d\Omega_{d-4}^2\right]. \tag{3.13}$$

This is the metric of a black membrane, only rescaled by a conformal factor that reflects the fact that if we keep the mass of the black hole fixed, then its size goes to zero in the limit $a \to L$. The metric (3.13) should give a good approximation to the horizon geometry up to $\sigma \sim L\,\Xi^{-\frac{d-1}{2(d-5)}}$.

Like in [23], it is natural to conjecture that a Gregory-Laflamme-type of instability appears for black holes that approach this limit. For a fixed value of M, the onset of the black membrane-like behavior of the black holes can be located at the value of J for which the surface gravity κ reaches a minimum, which is also an inflection point for $\mathsf{A_H}$. One can find the critical value of r_+/L for which the minimum of κ occurs at a given mass. It turns out that this critical value r_{+c} is always in a range $r_{+c} \leq \sqrt{\frac{d-5}{d-1}}L$, approaching the latter value for large mass. The corresponding critical spin J_c is at $(\mathsf{M} - \mathsf{J}_c)/\mathsf{M} \sim \mathsf{M}^{-1/2}$, very near the maximum spin. Since the unstable regime takes place at values $r_+ < r_{+c}$, we see that the black hole only becomes unstable when its size $\ell_\perp$ is smaller than the AdS radius.

I.e., large AdS black holes, with $\ell_{\|}, \ell_{\perp} > L$, should not suffer from Gregory-Laflamme-type instabilities.

It is worth discussing a limit of the rotating AdS solutions that does *not* have an analogue in the absence of the cosmological constant, so it necessarily belongs in the regime of very large M. In the limit (3.12) we kept the mass fixed, with the effect that the black hole size decreased to zero as $a \to L$. Instead, if we insist on keeping the horizon size finite, then as $a \to L$ we only scale the polar angle $\theta \to 0$ while keeping finite a new coordinate σ

$$\sin\theta = \sqrt{\Xi}\,\sinh(\sigma/2)\,. \tag{3.14}$$

The limiting metric is

$$ds^2 = -f(r)\left(dt - L\sinh^2(\sigma/2)d\phi\right)^2 + \frac{dr^2}{f(r)} + \frac{L^2}{4}\left(1+\frac{r^2}{L^2}\right)\left(d\sigma^2 + \sinh^2\sigma d\phi^2\right) + r^2 d\Omega^2_{d-4}\,, \tag{3.15}$$

where

$$f(r) = 1 - \frac{2m}{r^{d-5}(r^2+L^2)} + \frac{r^2}{L^2}\,. \tag{3.16}$$

This solution (a particular case of solutions in [24, 25]) has a horizon at the largest real root $r = r_+$ of $f(r)$. The horizon geometry is $\mathbb{H}^2 \times S^{d-4}$, where the hyperboloid $\mathbb{H}^2$ has coordinates (σ, ϕ). The geometry may then be regarded as a *rotating black hyperboloid membrane*.[8] The boundary geometry is not the Einstein universe $\mathbb{R}_t \times S^{d-2}$ that appears at finite mass, but rather a rotating spacetime with spatial sections $\mathbb{H}^2 \times S^{d-4}$. Observe that the solution makes sense in $d = 5$ as well, where it will have a horizon provided that $m > L^2/2$. In this case the solution, with horizon $\mathbb{H}^2 \times S^1$, can also be regarded as a rotating hyperbolic black string. Static hyperbolic black strings were obtained in [27].

The mass and angular momentum of the black hole diverge in this limit: since we are keeping m finite, M and J in (3.1), (3.2) diverge as $a \to L$. However, their ratio remains finite, $J/ML \to 1$. The hyperboloid membrane must then be interpreted as the limit of very large M of the black holes with spins very near the maximum $\mathsf{J}_{\max}$. For small r_+/L (which only happens in $d \geq 6$ for small m) the $\mathbb{H}^2$ radius is $\sim L$, much larger than the S^{d-4} size $\sim r_+$. In fact, when r_+ is very small the geometry near the axis $\sigma = 0$ reduces to the black membrane of (3.13). Presumably we should expect a Gregory-Laflamme instability in this case. When r_+ is large the characteristic radii of the hyperboloid and the sphere are both $\sim r_+$ and a GL instability seems unlikely.[9]

We can summarize the results of this section as follows: the properties of rotating AdS black holes at any given mass are qualitatively similar to those of their asymptotically flat counterparts, only with a phase diagram where the infinite range of angular momenta for MP black holes is 'compressed' to the range $\mathsf{J} \leq \mathsf{M}$ for rotating AdS black hole solutions.

[8] Distinct from hyperbolic AdS black holes [26], whose horizon is $\mathbb{H}^{d-2}$. It is also different than the conventional limit of very large mass in which one recovers a static black hole with planar horizon $\mathbb{R}^{d-2}$.

[9] Ref. [28] studied the GL instability of static hyperbolic black strings, and found them to be stable. In this case the perturbations consisted of ripples along the S^1, not the $\mathbb{H}^2$. But in any case, our argument suggests that in $d = 5$ the latter perturbations should also be stable.

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The onset of the membrane behavior, and accompanying instabilities, happens at values of the spin that, for small mass, are similar to the asymptotically flat ones, but for large mass this region is pushed close to the upper end of the phase space, $(\mathsf{M}-\mathsf{J})/\mathsf{M} \sim \mathsf{M}^{-1/2}$. In this regime, the differences to the asymptotically flat case that persist as the mass is scaled up are captured by the limiting geometry (3.15).

4. Phase connections

Given the conclusion of the previous section, it seems reasonable to conjecture that the curves for black rings in the phase diagrams at fixed mass in figures 1 and 2 are completed in a manner analogous to that proposed in [4], only compressed to a range of $\mathsf{J} \in [0, \mathsf{M}]$. However, while the pattern of connections proposed in [4] can presumably be imported without much distortion to AdS for small M, we cannot expect it to provide more than a qualitative picture of the mergers between phases when M is large and the cosmological length scale plays an important role.

4.1 The girths of a black ring

Black rings in AdS are characterized by the relative sizes of three scales, r_0, R, and L, *i.e.*, the radius of the ring's S^2, S^1, and of AdS, respectively. So far we have referred to rings that satisfy (2.1) as thin, but for the purpose of the following discussion it is convenient to introduce a more refined terminology:

- *Thin* rings have $r_0 \ll R$; *fat* rings have $r_0 \sim R$.
- *Small* rings have $r_0 < L$; *large* rings have $r_0 > L$.
- *Short* rings have $R < L$; *long* rings have $R > L$.

Thus our approximation (2.1) applies to thin small rings, either short or long. The definition of r_0 and R for fat rings may not be unambiguous, but we use the characterization $r_0 \sim R$ to mean that the self-gravitational attraction of the ring is very strong and large curvature may be expected on the horizon, in particular near the inner circle of the ring. With a suitable definition, r_0 will never be larger than R, so not all combinations of girths are possible. The same characterization could be used for rotating AdS black holes if in place of r_0 and R we use the sizes of the horizon in the directions transverse and parallel to the rotation plane.

We are interested in the different possibilities for black rings at a given value of the mass M. To begin with, we investigate the range of M that the approximation (2.1) allows to cover. If we consider rings at small R, for which (2.12) reduces to

$$\mathsf{M} \sim \left(\frac{r_0}{L}\right)^{d-4} \mathsf{R} \tag{4.1}$$

then (2.1) implies that our approximation is only valid for very small masses

$$\mathsf{M} \ll \mathsf{R}^{d-3} \qquad (\mathsf{R} \ll 1)\,. \tag{4.2}$$

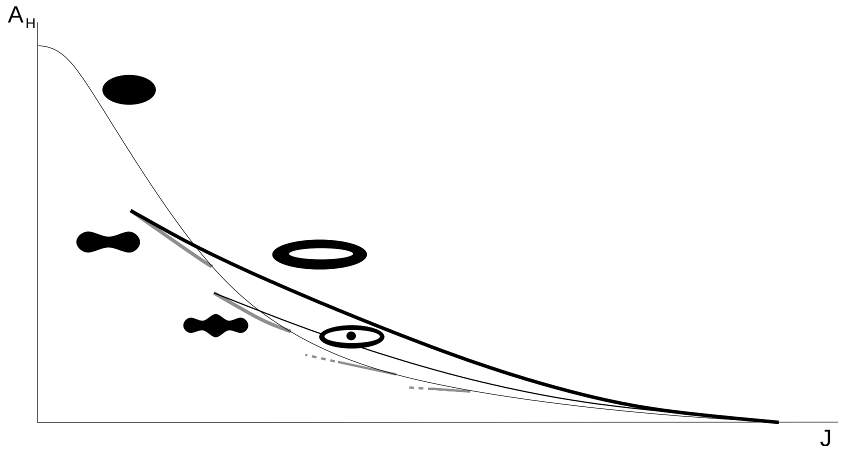

Figure 3: Proposal for the completion of phase curves at small M in $d \geq 6$ (the situation in $d = 5$ is a simple adaptation of the same idea). The patterns proposed in [4] are compressed here to the range $\mathsf{J} \leq \mathsf{M}$. We stress that the details of the connections (*e.g.,* first order vs. second order transitions) remain unknown and are arbitrarily drawn.

This is actually the same condition as in the asymptotically flat case, $GM \ll R^{d-3}$, and is hardly surprising: at small R the ring is much smaller and shorter than L and hence must be light on the cosmological scale. However, at large R we find from (2.12) and (2.1) that we must require

$$\mathsf{M} \sim \left(\frac{r_0}{L}\right)^{d-4} \mathsf{R}^4 \ll \mathsf{R}^4 \qquad (\mathsf{R} \gg 1)\,. \tag{4.3}$$

Thus within our approximations we can get to the regime of large M for extremely long rings.

4.2 The issue of large AdS black rings and dual fluids

Let us now explore heuristically how these branches of solutions may extend when the approximation (2.1) breaks down. It seems reasonable to assume that the curves of black rings, which we have seen extend up to $\mathsf{J} = \mathsf{M}$, can only end by merging to a black hole of spherical topology (possibly an extremal singular one in $d = 5$, and a pinched black hole in $d \geq 6$), and that this requires the ring to become fat, $r_0 \sim R$: the merger happens when the central hole of the ring closes off.

For very small M the rings are all small and short, $r_0 < R \ll L$. This situation is very similar to the asymptotically flat case, in which our approximation breaks down when the ring becomes fat, $r_0 \sim R \ll L$. Fat rings then connect to topologically spherical black holes or, possibly in $d = 5$, to a singular extremal solution. The scale L introduces only small corrections. Figure 3 illustrates qualitatively this situation in $d \geq 6$.

The regime of very large M is more intriguing. In this case, we have seen that our approximate methods reproduce some of these solutions as very long, small black rings, $r_0 \ll L \ll R$, at least when $R \gg \mathsf{M}^{1/4} L$. These cover the range of spins very close to the upper bound $\mathsf{J} \simeq \mathsf{M}$. As we move away from this point, reducing J while keeping M fixed, we are increasing r_0 and decreasing R. At some point, if M is very large, we must

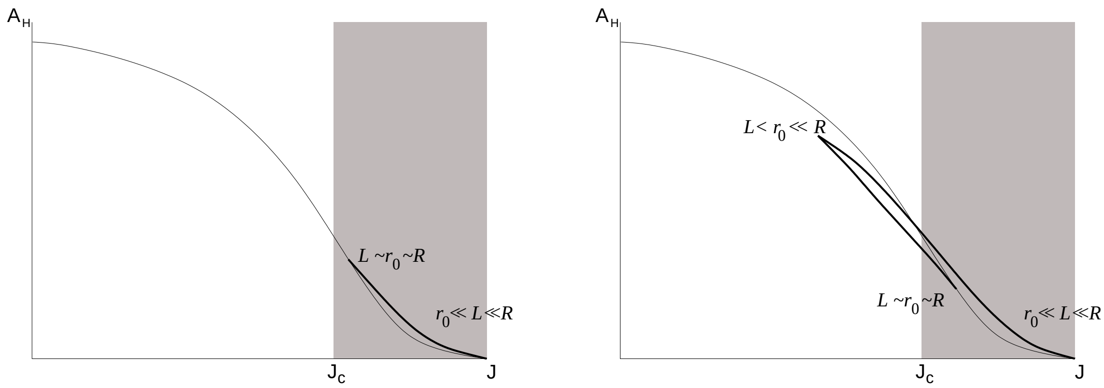

Figure 4: Two alternatives for the completion of phase curves at $\mathsf{M} \gg 1$. The gray-shaded area, which extends from $\mathsf{J} \sim \mathsf{J}_c$ where $(\mathsf{M} - \mathsf{J}_c)/\mathsf{M} \sim \mathsf{M}^{-1/2}$, to the upper limit $\mathsf{J} = \mathsf{M}$, corresponds to solutions that do *not* qualify as large AdS black holes or rings. In the left picture, no large AdS black rings exist. In the right picture, thin large AdS black rings, with $L < r_0 \ll R$ exist. This second alternative requires an explanation of why these black rings do not appear in dual hydrodynamic studies.

encounter a regime where $L \sim r_0 \ll R$ (increasing r_0 and decreasing R cannot lead us to $R < L$ since these small short rings would not have large M). It is unclear, however, whether r_0 can grow much larger than L while remaining much smaller than R, leading to a regime of thin large rings.

This would appear problematic for two reasons. The first one concerns the point at which black rings merge with black holes. In the previous section we have observed that in $d \geq 6$, rotating AdS black holes at large M develop a membrane-like behavior at a radius $r_+ \sim L$. It is natural to expect that the threshold of a Gregory-Laflamme instability sets in at around this point, giving rise to a branch of pinched black holes. If these then connect to (fat) black rings with parameters in a similar range, then the correspondence $r_+ \sim r_0$ would imply that these black rings would not be large. Unless the curve of AdS black rings extends far from this merger point, thin large black rings may not be realized, as r_0 would remain of the same order as L. Admittedly, though, this argument is merely suggestive.

The second concern is that, if black rings exist with parameters $L \ll r_0 \ll R$ then naively they might admit a dual fluid-dynamical description: their curvature would remain smaller than $1/L$ and their temperature larger than $1/L$. These conditions are required for the applicability of the fluid-dynamical approximation (fat large black rings, instead, probably cannot be described as dual fluids, as they involve large curvatures). However, a thorough analysis of the relativistic Navier-Stokes equation for a rotating fluid on the boundary geometry $\mathbb{R} \times S^{d-2}$ has not revealed other solutions than the duals of rotating AdS black holes [9] (see also [10]).

The apparent absence of thin large AdS black rings may not be too striking. When $r_0 \sim L$ the ring's S^{d-3} may be highly distorted: the ring is very massive and due to the centripetal cosmological pull, it may 'spread down towards the center of AdS' and thus become fat. Since thin large rotating AdS black holes do exist which are highly pancaked —just take $r_+ > L$ in (3.11), which makes $\mathsf{M} \gg 1$— one may naively envisage 'drilling a hole' through them and spinning them up faster to get a black ring. However, this may not be possible since these black holes (which limit to the geometry (3.15)) are already very close to the maximum spin.

A more quantitative argument that points to difficulties for a regime of black rings with $L \ll r_0$ follows by first observing that, quite naturally, r_0 should not be larger than the distance from the ring to the center of AdS (otherwise the ring would close its hole). To estimate this upper limit on r_0 for a given $R \gg L$, we can compute the proper radial distance to $\rho = R$ in empty AdS space (2.2), namely

$$r_0^{\max} \sim \int_0^R \sqrt{g_{\rho\rho}}\, d\rho = \int_0^R \frac{d\rho}{\sqrt{1+\rho^2/L^2}} \simeq L \log(R/L)\,. \tag{4.4}$$

Although this estimate neglects possible strong curvature effects of the ring, it suggests that the geometry of AdS severely limits r_0/L to grow only logarithmically with R, instead of the linear growth in flat space. Thus, there may not exist a regime proper of thin large AdS rings. Figure 4 illustrates the alternatives.

In conclusion, although the arguments are clearly not compellingly strong, the absence of thin large black rings seems plausible, even for very large M. If nevertheless they existed, then they should presumably extend to values of J in a range $(\mathsf{M}-\mathsf{J})/\mathsf{M} \gg \mathsf{M}^{-1/2}$ outside the regime of small long rings, possibly resembling large rotating AdS black holes with a central hole. It would remain to explain why they do not show up in the fluid-dynamical analysis of [9, 10]. A likely possibility is that for a black ring not all of the horizon is close to the boundary (the inner rim of the central hole could lie deeper inside AdS), and hence the fluid description could not capture all of it. In the dual parlance, the fluid would become too thin near the poles of the S^{d-2} to be captured within the approximations of [9, 10]. Or maybe there is a subtlety missing in the analysis. But at the moment, the simplest possibility is that the regime of thin large AdS rings is never realized.

5. Black rings galore

5.1 Black rings in deSitter

The extension of our analysis to black rings in deSitter is straightforward: just continue analytically the results for Anti-deSitter, $L \to iL$. Thus, considering a metric of the form (2.2) with the lapse function

$$V(\rho) = 1 - \frac{\rho^2}{L^2}\,, \tag{5.1}$$

equilibrium fixes the boost parameter to

$$\sinh^2\alpha = \frac{1-(d-2)\,\mathsf{R}^2}{d-4}\,. \tag{5.2}$$

This is a decreasing function of R that vanishes for a maximum radius $\mathsf{R} = \mathsf{R}_{st}$,

$$\mathsf{R}_{st} = \frac{1}{\sqrt{d-2}}\,. \tag{5.3}$$

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For this value of the radius, we find a static thin black ring in deSitter spacetime: the cosmological expansion exactly balances the string tension. For $\mathsf{R} > \mathsf{R}_{st}$ the equilibrium condition cannot be satisfied: these rings cannot avoid getting inflated away. Hence there is an upper limit to the size of thin black rings in deSitter. Observe that this upper limit is $\mathsf{R}_{st} < 1$. Even if it is clear that we could not have super-Hubble rings with $\mathsf{R} > 1$, our thin rings do not even approach this limit. However, it may still be that fat black rings of larger size exist in deSitter, and they might even reach the size of the cosmological horizon.

The existence of black rings in *four-dimensional* deSitter space is more difficult to rule out than in asymptotically flat or Anti-deSitter spacetimes, where they are forbidden by topological censorship theorems [29]. It is clear that with our methods we can not construct them, since the requisite vacuum four-dimensional black strings do not exist. What this implies is that, if black rings with horizon topology $S^1 \times S^1$ existed at all in four-dimensional deSitter, they could not admit a limit in which any of the ring's S^1 became much smaller than the cosmological radius, *i.e.,* they should always be *fat*. They could neither be type-D solutions [30], so we find them very unlikely to exist.

5.2 Black Saturns

Using our general formalism, we can easily study black rings living in any static spherically symmetric spacetime. Appendices A and B show that neutral black strings can be placed at an equatorial circle in these spacetimes to yield thin neutral black rings that are regular on and outside the horizon.

A large class of physically interesting backgrounds is given by the metric (2.2), where $V(\rho)$ is an arbitrary function of the radial coordinate. For example, with a proper choice of the function $V(\rho)$ this allows to describe the bending of a black string in the background of static black holes and thus construct thin black Saturns, in the sense that the black ring surrounding the black hole is thin. Exact solutions for black Saturns in five-dimensional vacuum gravity were obtained in [31]. Our approach here allows to extend them and could be used to refine the arguments in [32].

The rescaling encoding the redshift of the black string, and the proper length along the string, are given by

$$t = \sqrt{V(R)}\,\tau\,, \qquad z = R\psi\,. \tag{5.4}$$

Using eqs. (A.19) and (B.7) we find that a circular source with energy-momentum (2.5) centered at $\rho = R$ on the $\Theta = 0$ plane satisfies the equilibrium condition (2.6) as long as the boost parameter α of the string verifies

$$\sinh^2\alpha = \frac{1}{d-4}\,\frac{2V(R) + (d-3)RV'(R)}{2V(R) - RV'(R)}\,. \tag{5.5}$$

Taking into account the redshift factors coming from the rescalings (5.4), we can compute the physical quantities characterizing the black ring in this general background. Its mass and angular momentum read

$$M = \frac{r_0^{d-4}}{8G}\Omega_{d-3}R\,\frac{2(d-2)V^{3/2}}{2V-RV'}\,, \tag{5.6}$$

$$J = \frac{r_0^{d-4}}{8G}\Omega_{d-3}R^2\,\frac{\left[(2(d-3)V+RV')\,(2V+(d-3)RV')\right]^{1/2}}{2V-RV'}\,, \tag{5.7}$$

where $V \equiv V(R)$ and $V' \equiv V'(R)$. These M and J should be understood as quantities conjugate to the time τ and angular coordinate ψ —for each particular spacetime one has to determine how these are related to the canonical choice of time, if the spacetime admits one. For asymptotically flat or asymptotically AdS solutions, the results above are the canonical ones.

The area is given by

$$A_H = 2\pi r_0^{d-3}\Omega_{d-3}R\sqrt{\frac{2(d-3)V+RV'}{(d-4)(2V-RV')}}\,, \tag{5.8}$$

while the angular velocity, defined using the horizon generator $\partial_\tau + \Omega_H\partial_\psi$, is

$$\Omega_H = \frac{\sqrt{V}}{R}\sqrt{\frac{2V+(d-3)RV'}{2(d-3)V+RV'}} \tag{5.9}$$

and the surface gravity, for the same Killing vector, is

$$\kappa = \frac{(d-4)^{3/2}}{2r_0}\sqrt{\frac{(2V-RV')V}{2(d-3)V+RV'}}\,. \tag{5.10}$$

For instance, choosing

$$V(\rho) = 1 - \frac{2m}{\rho^{d-3}} \pm \frac{\rho^2}{L^2}, \tag{5.11}$$

the above formulas yield the physical properties of thin (Anti-)deSitter black Saturns with a static central black hole. Note that in this approximation, the total mass of the system is the sum of the mass of the central Schwarzschild-(A)dS black hole and the mass M of the thin black ring surrounding it, while its total angular momentum is given by J.

Observe that there are two kinds of critical values of the ring radius: the values $R = R_l$, where

$$\frac{V'(R_l)}{V(R_l)} = \frac{2}{R_l} \tag{5.12}$$

for which the boost (5.5) reaches speed of light (this is in fact the radius of null circular orbits) so $r_0 \to 0$ is required if the mass, spin, and area remain finite; and the values $R = R_{st}$ where the boost vanishes

$$\frac{V'(R_{st})}{V(R_{st})} = -\frac{2}{(d-3)R_{st}} \tag{5.13}$$

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and so the black ring is static. The set of these critical radii determine the ranges of existence of black rings in these backgrounds. We have seen one example in deSitter, eq. (5.3). For a black Saturn in (A)dS, described by (5.11), the speed-of-light radius

$$R_l^{d-3} = (d-1)m = \frac{d-1}{2}\rho_{bh}^{d-3}\left(1 \pm \frac{\rho_{bh}^2}{L^2}\right) \tag{5.14}$$

sets a lower bound on the ring radius. It is easy to see that for either positive or negative cosmological constant this radius is always outside the black hole horizon at ρ_{bh}. Below this R_l the attraction of the black hole is too strong for the ring to resist its pull.

We believe it is possible to extend this analysis to the case where the central black hole is rotating, but this lies outside the scope of this paper.

6. No supersymmetric black rings in Anti-deSitter

Supersymmetric black rings exist in five-dimensional ungauged supergravities, so it is natural to look for their counterparts in gauged supergravities. Imagine that such solutions exist with horizon topology $S^1 \times S^2$, and that the radius of the S^2 can be made arbitrarily smaller than the S^1-radius and the cosmological length, *i.e.,* that thin small supersymmetric black rings exist. As a limit we should recover a straight black string solution of the ungauged theory. Moreover, if the circular ring solution is supersymmetric, the limiting straight string should be supersymmetric as well. In general the straight solution will have no fewer (super)symmetries than the circular one, since some of the symmetries of the straight string geometry may be broken when bent into a circle.[10]

Thus AdS supersymmetric black rings, if they exist and admit a thin small string limit, should be amenable to our approximate methods. Let us consider, for simplicity, minimal five-dimensional gauged supergravity— the case with arbitrary vector multiplets presents no qualitative differences. Our arguments above imply that a thin supersymmetric black ring, when expanded in powers of r_0/R, r_0/L, should be described to zeroth order by the supersymmetric black string of ungauged supergravity [33]

$$\begin{aligned} ds^2 &= -f^2(dt + A_z dz)^2 + f^{-1}(dr^2 + r^2 d\Omega_3^2)\,, \\ f^{-1} &= 1 + \frac{Q}{r} + \frac{q^2}{4r^2}\,, \qquad\qquad A_z = -\left(\frac{3q}{2r} + \frac{3qQ}{4r^2} + \frac{q^3}{8r^3}\right) \end{aligned} \tag{6.1}$$

(here $r_0 = q/2$), which indeed arises as a limit of supersymmetric black rings in the ungauged theory [5]. To obtain a circular ring by curving this string in AdS, we proceed as in section 2 and identify the distributional source that reproduces the linearized field (6.1) at distances $r \gg r_0$. This is easy to obtain, since terms $\propto q^2, qQ$ in the metric are non-linear and can be dropped. We find

$$T_{tt} = \frac{3Q}{16\pi G}\,\delta^{(3)}(r)\,, \qquad T_{tz} = \frac{3q}{32\pi G}\,\delta^{(3)}(r)\,, \qquad T_{zz} = 0\,. \tag{6.2}$$

[10] In fact, if we curve a straight supersymmetric string, in general it will not remain supersymmetric.

Crucially, observe that the tension vanishes. We believe this to be a generic consequence ('cancellation of forces') of supersymmetry.

The next step is solving the linearized field equations when this source is placed along a circle of radius R in AdS. But we have seen in eq. (2.7) that a consistent coupling of this *circular* source to gravity in AdS requires non-vanishing pressure, which the supersymmetric string cannot provide.[11] Thus there cannot be supersymmetric black rings in AdS that admit a limit to (6.1).

More precisely, if we insist on bending the string (6.1) into a circle, the tension required in eq. (2.7) will appear in the form of conical singularities on the plane of the ring. These conical singularities arose in the analysis of [7, 8] as the obstruction preventing the existence of near-horizon geometries of supersymmetric AdS black rings. Our argument above confirms these results and gives a simple physical origin to the conical singularities.

7. Final remarks

Our analysis of black rings in the presence of a cosmological constant demonstrates the power of the approach initiated in [4] for the investigation of black holes with non-spherical topologies in higher-dimensional theories. A simple, straightforward technique provides non-trivial information in cases where the construction of exact solutions has been unsuccessful. Of all the solutions we have constructed approximately, the one that would seem more likely to admit a closed exact analytic solution is the static black ring in deSitter in five dimensions, generalizing the fairly simple metric for a vacuum static black ring [34]. The problem in this case seems to lie more in the form of the metric far from the horizon than close to it.

Although we have not done it here, one could very easily include additional spins for the black ring, corresponding to rotation of the S^{d-3}. This is simply done by considering a rotating boosted black string, and bending it to form a ring. The equilibrium condition (2.7) is not affected by this rotation in transverse directions, so it is straightforward to compute all physical magnitudes.

We believe that the same approach should be useful to study black rings in many other backgrounds. For instance, it may be interesting to apply it to black rings in Gödel spacetimes and study how the ring behaves as its radius becomes larger than the scale at which closed timelike curves appear.

Some of our conclusions may also be applicable to other extended objects (p-branes). Our argument for the absence of supersymmetric black rings in AdS is based on the simple observation that the AdS potential requires pressure along the p-brane in order to achieve mechanical balance when the p-brane is curved into some contractible cycle. But supersymmetry is presumably incompatible with this pressure, and so, according to this argument, in AdS we should not expect supersymmetric branes whose worldvolumes have

[11] Note that the stress-energy tensor of the electromagnetic field will only enter at quadratic order in q and Q. Also, besides conservation of energy-momentum one must impose conservation of charge, but this is trivially satisfied for a uniform circular profile.

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contractible cycles — that is, unless some other force is present to achieve the balance without breaking all supersymmetries.

We have discussed our black rings as living in AdS space, but in the context of string theory it is more natural to consider them as solutions in a larger space, say, $\mathrm{AdS}_5 \times S^5$, with the radius of S^5 being given by L. In this case, our solutions describe black rings which are completely smeared over the S^5, *i.e.,* their horizons are $S^1 \times S^2 \times S^5$. Our approximations require that the S^2 be much smaller than the S^5, in which case a Gregory-Laflamme instability towards localization in S^5 is expected. This is in addition to the GL-instability along the S^1. If the final fate of both instabilities is the fragmentation of the black ring, then we would end up with ten-dimensional black holes, with S^8 horizons, flying apart (and possibly merging eventually into a single black hole with S^8 horizon). For black rings beyond our approximations, depending on the possible regimes of existence (discussed in section 4), they may localize in S^5 without fragmenting along S^1, or remain extended along S^5 while fragmenting along S^1, giving $S^3 \times S^5$ horizons. It is also possible that large stable black rings exist, in particular if they also possess charges or dipoles.

An interesting approach to the physics of new black holes in AdS is based on their dual fluid-dynamical description. We have already discussed how the analysis in [9, 10] has not found any evidence for large AdS black rings nor for any phases of large pinched black holes (which are in any case unlikely, since pinched black holes should branch off from small AdS black holes with $r_+ < L$). On the other hand, plasma rings and plasma balls, including pinched plasma balls, have turned up in the study of hydrodynamic solutions in a theory with a confining vacuum [35, 36]. The gravitational duals of such solutions, which are not any of the black holes and black rings of this paper, are in many respects more similar to asymptotically flat black holes. Perhaps an extension of the hydrodynamic methods will allow to describe some of the solutions we have constructed, and, maybe, clarify some of the puzzles that we have left unsolved.

Acknowledgments

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Note: in appendices A and B we denote the total number of spacetime dimensions as

$$d = n + 4$$

in order to facilitate comparison with the results of [4].

A. Adapted coordinates and near-zone 'potential'

A.1 Anti-deSitter

Close to the circle at $\rho = R$, $\Theta = 0$ in (2.2) we can choose a set of coordinates $(t, r, \theta, \Omega_i, z)$ in AdS which are adapted to ring-like objects, in the sense that the radial coordinate r

measures transverse distance away from the circle at $r = 0$ and surfaces of constant r have ring-like topology $S^1 \times S^{n+1}$. The geometry close to the ring can then be studied by expanding the metric in powers of r/R.

To obtain these coordinates, first observe that the constant τ hypersurfaces of (2.2) are $(d-1)$-dimensional hyperbolic spaces, as is obvious introducing the new radial coordinate $\hat{\rho}$,

$$\rho = \frac{\hat{\rho}}{1 - \hat{\rho}^2/(4L^2)}\,, \tag{A.1}$$

which brings AdS_d into homogeneous (spatially conformally flat) coordinates

$$ds^2 = -\left(\frac{1+\hat{\rho}^2/(4L^2)}{1-\hat{\rho}^2/(4L^2)}\right)^2 d\tau^2 + \frac{1}{(1-\hat{\rho}^2/(4L^2))^2}\left(d\hat{\rho}^2 + \hat{\rho}^2 d\Theta^2 + \hat{\rho}^2\sin^2\Theta\, d\Omega_n^2 + \hat{\rho}^2\cos^2\Theta d\psi^2\right). \tag{A.2}$$

Note that $\hat{\rho}$ ranges from zero at the origin to $\hat{\rho} = 2L$ at spatial infinity of AdS. The ring will be located at $\Theta = 0$, $\hat{\rho} = \hat{R}$, such that

$$R = \frac{\hat{R}}{1 - \hat{R}^2/(4L^2)}\,. \tag{A.3}$$

Define coordinates $(\hat{r}, \theta)$ as

$$\hat{r}\sin\theta = \hat{\rho}\sin\Theta\,, \qquad \hat{r}\cos\theta = \hat{\rho}\cos\Theta - \hat{R}\,. \tag{A.4}$$

The ring source lies at $\hat{r} = 0$, and spatial surfaces at $\hat{r} =$ const. have topology $S^1 \times S^{n+1}$. We may now expand the metric near $\hat{r} = 0$ in powers of $\hat{r}/\hat{R}$, but before doing that it is convenient to make a few additional changes. Notice first that to zero-th order in $\hat{r}/\hat{R}$ we have

$$ds^2 = -\left(1 + \mathsf{R}^2\right) d\tau^2 + \frac{R^2}{\hat{R}^2}\left[d\hat{r}^2 + \hat{r}^2\left(d\theta^2 + \sin^2\theta\, d\Omega_n^2\right)\right] + R^2 d\psi^2 + \mathcal{O}(\hat{r}/R) \tag{A.5}$$

Thus, orthonormal coordinates (t, z) in which t is proper time and z proper length along the string worldheet directions, can be defined as in (2.4). We also rescale the radial coordinate to measure proper radius r,

$$r = \frac{R}{\hat{R}}\,\hat{r}\,. \tag{A.6}$$

In terms of these, the metric including the first corrections in r/R, is

$$\begin{aligned} ds^2 = &-\left(1 + \frac{\mathsf{R}^2}{\sqrt{1+\mathsf{R}^2}}\frac{2r\cos\theta}{R} + \mathcal{O}(r^2/R^2)\right) dt^2 \\ &+\left(1 + \left(\sqrt{1+\mathsf{R}^2} - 1\right)\frac{2r\cos\theta}{R} + \mathcal{O}(r^2/R^2)\right)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\Omega_n^2\right) \\ &+\left(1 + \sqrt{1+\mathsf{R}^2}\,\frac{2r\cos\theta}{R} + \mathcal{O}(r^2/R^2)\right) dz^2\,. \end{aligned} \tag{A.7}$$

This metric is of the generic form

$$\begin{aligned}
ds^2 = &-\left(1 + C_t \frac{2r\cos\theta}{R} + \mathcal{O}(r^2/R^2)\right) dt^2 \\
&+ \left(1 + C_r \frac{2r\cos\theta}{R} + \mathcal{O}(r^2/R^2)\right) \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\Omega_n^2\right) \\
&+ \left(1 + C_z \frac{2r\cos\theta}{R} + \mathcal{O}(r^2/R^2)\right) dz^2 \,.
\end{aligned} \tag{A.8}$$

with constant $C_{t,r,z}$, which describes a generic class of backgrounds in which the straight line at $r = 0$ is deformed into a large circle, whose extrinsic curvature radius in the plane $\theta = 0$ at constant t is R/C_z. The deformations in the metric only involve a dipole perturbation ($\propto \cos\theta$) of the S^{n+1}. A gauge transformation of the form

$$r \to r + a\frac{r^2\cos\theta}{R} + \mathcal{O}(R^{-2})\,, \qquad \theta \to \theta + a\frac{r\sin\theta}{R} + \mathcal{O}(R^{-2})\,, \tag{A.9}$$

does not modify g_{tt} nor g_{tz} nor does it introduce crossed terms $g_{r\theta}$, but it shifts

$$C_r \to C_r + 2a\,. \tag{A.10}$$

We can use this gauge freedom to obtain a more convenient radial coordinate. If we demand that $\Box r^{-n} = 0$, to order $\mathcal{O}(R^{-2})$, so r labels scalar equipotential surfaces, then we must adjust a to have

$$C_r = -\frac{C_t + C_z}{n}\,. \tag{A.11}$$

We make this choice to finally obtain

$$\begin{aligned}
ds^2 = &-\left(1 + \frac{\mathsf{R}^2}{\sqrt{1+\mathsf{R}^2}}\frac{2r}{R}\cos\theta\right) dt^2 \\
&+ \left(1 - \frac{1+2\mathsf{R}^2}{n\sqrt{1+\mathsf{R}^2}}\frac{2r}{R}\cos\theta\right) \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\Omega_n^2\right) \\
&+ \left(1 + \sqrt{1+\mathsf{R}^2}\,\frac{2r}{R}\cos\theta\right) dz^2 + \mathcal{O}(r^2/R^2)\,.
\end{aligned} \tag{A.12}$$

If we keep only terms of order $\mathcal{O}(r^0/R^0)$ we obviously recover flat space where $r = 0$ is a straight line along z. The corrections included in (A.12) capture the bending of this line into an arc of large radius R. But the Riemann tensor of this metric vanishes up to terms $\mathcal{O}(r^2/R^2)$, so to this order the metric (A.12) (and (A.8)) is in fact still *flat* space. Even if it was obtained starting from AdS space, the cosmological constant $\Lambda \propto -L^{-2}$ does not contribute terms of lower order than $(r/L)^2 = \mathsf{R}^2(r/R)^2$, and so it can only appear in the metric through the combination $\mathsf{R}^2 = R^2/L^2$.

The terms of order r/R in eq. (A.12) can be regarded as a 'gravitational potential': a black string placed in this field will be bent into an arc of large radius R. If the thickness of the black string is r_0, the approximation requires that the correction terms in (A.12) are small for $r \sim r_0$, *i.e.*,

$$r_0 \ll \frac{R}{\sqrt{1+\mathsf{R}^2}}\,. \tag{A.13}$$

Up to factors of order one, this implies the validity range (2.1).

A.2 General static spherically symmetric backgrounds

Here we show that the near-zone static metric (A.8) includes all static spherically symmetric spacetimes. These always admit a spatially conformally flat static metric,

$$ds^2 = -f(\hat\rho)d\tau^2 + g(\hat\rho)\left(d\hat\rho^2 + \hat\rho^2 d\Theta^2 + \hat\rho^2\sin^2\Theta\, d\Omega_n^2 + \hat\rho^2\cos^2\Theta\, d\psi^2\right) . \tag{A.14}$$

To obtain the geometry of the region near the ring located at $\Theta = 0$, $\hat\rho = \hat R$, we perform the coordinate transformation (A.4), and define the coordinates t and z that measure the proper time and proper length along the string world-sheet directions, and the proper radius r by

$$t = \sqrt{f(\hat R)}\,\tau, \qquad z = \hat R\sqrt{g(\hat R)}\,\psi, \qquad r = \sqrt{g(\hat R)}\,\hat r. \tag{A.15}$$

This casts the metric into adapted coordinates for the circular string, which takes, up to order r/R, the form (A.8) with coefficients

$$C_t = \left.\frac{(\ln f)'}{2\sqrt{g}}\right|_{\hat\rho=\hat R} R\,, \qquad C_r = \left.\frac{(\ln g)'}{2\sqrt{g}}\right|_{\hat\rho=\hat R} R\,, \qquad C_z = \left.\frac{2+\hat R(\ln g)'}{2\hat R\sqrt{g}}\right|_{\hat\rho=\hat R} R\,. \tag{A.16}$$

Here, R is a constant with the dimensions of length that sets the scale of the perturbations. We can conveniently take it as the proper circumference radius of the circle

$$R = \frac{\Delta z}{2\pi} = \hat R\sqrt{g(\hat R)}\,. \tag{A.17}$$

The particular class of metrics (2.2) is related to the generic metric (A.14) through a redefinition of the radial coordinate $\rho = \rho(\hat\rho)$ such that

$$\frac{d\rho}{d\hat\rho} = \frac{\rho}{\hat\rho}\sqrt{V(\rho)}\,. \tag{A.18}$$

Using this relation, and $R = \rho(\hat R)$, it is easy to show that the coefficients $C_{t,r,z}$ simplify to

$$C_t = \frac{RV'(R)}{2\sqrt{V(R)}}\,, \qquad C_r = \sqrt{V(R)} - 1\,, \qquad C_z = \sqrt{V(R)}\,, \tag{A.19}$$

with $z = R\psi$ and the rescaling encoding the redshift of the black string given by $t = \sqrt{V(R)}\,\tau$. The constant C_r can be gauge-transformed to the value (A.11) in the manner described above.

B. Bending the black string

In this appendix we extend the analysis of [4] to describe the perturbations of a boosted black string induced by placing it in a 'background potential' of the generic form (A.8) (with gauge choice (A.11)). It seems likely that any static background that possesses a hypersurface-orthogonal spacelike Killing vector should reduce, near an orbit of the isometry, to this form.[12]

[12] For stationary backgrounds, one can choose corotating coordinates that eliminate the term g_{tz} to zero-th order in r/R, but in general not to higher orders. Thus one must supplement (A.8) with a term $g_{tz} \propto r\cos\theta/R$.

Our main aim is to show that the horizon of the black string remains regular after we bend it to a circular shape. The analysis is simply an extension of that of [4], so our description will be very succinct.

Typically, in a matched asymptotic expansion with two separate scales $r_0 \ll R$ we would begin by solving first the equations in the far-zone $r \gg r_0$ and then use this solution in the intermediate-zone $r_0 \ll r \ll R$ in order to provide boundary conditions for the solution in the near-zone $r \ll R$.

In the case of AdS we have found it quite difficult to solve the far-zone equations. However, we have managed to fully determine the solution in the intermediate-zone, up to gauge transformations. This solution, which we obtain in the next subsection, is enough to provide the boundary conditions for the perturbations of the black string. Analyzing the problem in this manner has the advantage that we can deal with the larger class of backgrounds (A.8). Recall that the particular case of AdS corresponds to

$$C_t = \frac{\mathsf{R}^2}{\sqrt{1+\mathsf{R}^2}}\,, \qquad C_z = \sqrt{1+\mathsf{R}^2}\,, \tag{B.1}$$

while the results of [4] are recovered with $C_t = 0$, $C_z = 1$.

B.1 Intermediate-zone analysis

The intermediate zone is defined by

$$r_0 \ll r \ll \min(R, L)\,, \tag{B.2}$$

or, in more generality for a background of the form (A.8),

$$r_0 \ll r \ll R \min(C_t^{-1}, C_z^{-1})\,. \tag{B.3}$$

In this case the ring will be a small perturbation around (A.8) and we can solve the equations in the linearized approximation. Like in [4], we model the ring by a generic string-like distributional stress-energy tensor of the form

$$\begin{aligned} T_{tt} &= \frac{n(n+2)}{n+1}\,\mu\,\frac{r_0^n}{16\pi G}\,\delta^{(n+2)}(r)\,,\\ T_{tz} &= n\,p\,\frac{r_0^n}{16\pi G}\,\delta^{(n+2)}(r)\,,\\ T_{zz} &= \frac{n(n+2)}{n+1}\,\tau\,\frac{r_0^n}{16\pi G}\,\delta^{(n+2)}(r)\,, \end{aligned} \tag{B.4}$$

where μ, p, τ, are dimensionless quantities characterizing the mass and momentum densities and the pressure along the string. For the particular case of a boosted black string (cf. eq. (2.5)),

$$\begin{aligned} \frac{n(n+2)}{n+1}\mu &= n\cosh^2\alpha + 1\,,\\ \frac{n(n+2)}{n+1}\tau &= n\sinh^2\alpha - 1\,,\\ p &= \cosh\alpha\sinh\alpha\,. \end{aligned} \tag{B.5}$$

Conservation of this stress-energy tensor, $\nabla_\mu T^{\mu\nu} = 0$, in the background (A.8) implies that

$$\tau = \frac{C_t}{C_z}\mu\,, \tag{B.6}$$

which is actually equivalent to (2.7). We shall impose this condition henceforth — if at this stage we did not, it would be later enforced on us as a regularity condition on the perturbed geometry like in [4]. When applied to (B.5), eq. (B.6) fixes the equilibrium value of the boost to

$$\sinh^2\alpha = \frac{C_z + (n+1)C_t}{n(C_z - C_t)}\,. \tag{B.7}$$

This reproduces (2.8) for the Anti-deSitter background (B.1), and (5.5) for the background (2.2) with generic function $V(\rho)$.

Observe that, according to the observation made after eq. (A.12), the equations that we must solve in this region are the vacuum Einstein equations $R_{\mu\nu} = 0$. Following the same steps as in [4], we find that the regular solution to the linearized Einstein's equations is

$$\begin{aligned}
g_{tt} &= -\left(1 + C_t\frac{2r\cos\theta}{R}\right)\left(1 - \left(\mu + \frac{\tau}{n+1}\right)\frac{r_0^n}{r^n}\right)\,,\\
g_{tz} &= -p\frac{r_0^n}{r^n}\left(1 + \frac{C_t + C_z}{2}\frac{2r\cos\theta}{R}\right)\,,\\
g_{zz} &= \left(1 + C_z\frac{2r\cos\theta}{R}\right)\left(1 - \left(\tau + \frac{\mu}{n+1}\right)\frac{r_0^n}{r^n}\right)\,,\\
g_{rr} &= \left(1 - \frac{C_t + C_z}{n}\frac{2r\cos\theta}{R}\right)\left(1 + \frac{\mu-\tau}{n+1}\frac{r_0^n}{r^n} - k(n-1)\frac{r_0^n}{r^n}\frac{2r\cos\theta}{R}\right)\,,\\
g_{ij} &= \hat{g}_{ij}\left(1 - \frac{C_t + C_z}{n}\frac{2r\cos\theta}{R}\right)\left(1 + \frac{\mu-\tau}{n+1}\frac{r_0^n}{r^n} + k\frac{r_0^n}{r^n}\frac{2r\cos\theta}{R}\right)\,,
\end{aligned} \tag{B.8}$$

where in the angular part we have the factor

$$\hat{g}_{ij}dx^i dx^j = r^2(d\theta^2 + \sin^2\theta d\Omega_n^2)\,. \tag{B.9}$$

There is an undetermined constant k from the solution to the homogeneous differential equations. This was set to zero in [4] using an argument that does not apply here, but in any case it is just a gauge perturbation: to the required perturbation order, we can set it to zero by a coordinate transformation of the form

$$r \to r - k\frac{n-1}{n-2}\frac{r_0^n}{r^{n-2}}\frac{\cos\theta}{R}\,, \qquad \theta \to \theta + \frac{k}{n-2}\frac{r_0^n}{r^{n-1}}\frac{\sin\theta}{R}\,. \tag{B.10}$$

In the following, for simplicity we set $k = 0$[13] and assume that the string is a boosted black string with energy-momentum parameters (B.5). For the purpose of analyzing the

[13]We would actually need the far-zone solution to check that this choice is consistent with the asymptotic boundary conditions, which may fix partially this gauge freedom. We could in any case leave k free and our conclusions would not be modified.

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near-horizon perturbations it is convenient to pass to yet another gauge, $r \to r - \frac{r_0^n}{2nr^{n-1}}$, in which the solution takes the form

$$
\begin{aligned}
g_{tt} &= -1 + c_\alpha^2 \frac{r_0^n}{r^n} - C_t \frac{2r\cos\theta}{R}\left[1 - \frac{r_0^n}{r^n} c_\alpha^2 \left(1 + \frac{1}{2nc_\alpha^2}\right)\right], \\
g_{tz} &= -c_\alpha s_\alpha \frac{r_0^n}{r^n}\left(1 + \frac{C_t + C_z}{2}\frac{2r\cos\theta}{R}\right), \\
g_{zz} &= 1 + s_\alpha^2 \frac{r_0^n}{r^n} + C_z \frac{2r\cos\theta}{R}\left[1 + \frac{r_0^n}{r^n} s_\alpha^2 \left(1 - \frac{1}{2ns_\alpha^2}\right)\right], \\
g_{rr} &= 1 + \frac{r_0^n}{r^n} - \frac{C_t + C_z}{n}\frac{2r\cos\theta}{R}\left[1 + \frac{r_0^n}{r^n}\left(1 - \frac{1}{2n}\right)\right], \\
g_{ij} &= \hat{g}_{ij}\left[1 - \frac{C_t + C_z}{n}\frac{2r\cos\theta}{R}\left(1 - \frac{1}{2n}\frac{r_0^n}{r^n}\right)\right],
\end{aligned}
\tag{B.11}
$$

where we abbreviate

$$
c_\alpha = \cosh\alpha\,, \qquad s_\alpha = \sinh\alpha\,. \tag{B.12}
$$

B.2 Perturbations near the horizon

We now study the perturbations of the boosted black string, considering that $r \ll R$ but without assuming $r \gg r_0$. This generalizes the analysis of [4] to allow for a general value of the boost α. The effect of the background potential (A.8) will only enter when we impose boundary conditions. Only then will we need to fix the boost to the value (B.7).

The same arguments as in [4] allow us to reduce the perturbations to the form

$$
g_{tt} = -1 + c_\alpha^2 \frac{r_0^n}{r^n} + \frac{\cos\theta}{R} a(r)\,, \tag{B.13}
$$

$$
g_{tz} = -c_\alpha s_\alpha \left[\frac{r_0^n}{r^n} + \frac{\cos\theta}{R} b(r)\right], \tag{B.14}
$$

$$
g_{zz} = 1 + s_\alpha^2 \frac{r_0^n}{r^n} + \frac{\cos\theta}{R} c(r)\,, \tag{B.15}
$$

$$
g_{rr} = \left(1 - \frac{r_0^n}{r^n}\right)^{-1}\left[1 + \frac{\cos\theta}{R} f(r)\right], \tag{B.16}
$$

$$
g_{ij} = \hat{g}_{ij}\left[1 + \frac{\cos\theta}{R} g(r)\right], \tag{B.17}
$$

where $\hat{g}_{ij}$ is the metric (B.9). This form is convenient since it fixes the location of the horizon at $r = r_0$. There remains a gauge freedom

$$
r \to r + \gamma(r)\,\frac{r_0}{R}\,\cos\theta\,, \tag{B.18}
$$

$$
\theta \to \theta + \frac{r_0}{R}\,\sin\theta \int^r dr' \frac{\gamma(r')}{r'^2\left(1 - \frac{r_0^n}{r'^n}\right)}\,, \tag{B.19}
$$

that we deal with by working with invariant variables

$$\mathsf{A}(r) = a(r) - \frac{c_\alpha^2}{s_\alpha^2}\, c(r)\,, \tag{B.20}$$

$$\mathsf{B}(r) = b(r) - \frac{1}{s_\alpha^2}\, c(r)\,, \tag{B.21}$$

$$\mathsf{F}(r) = f(r) + \frac{2}{n\, s_\alpha^2} r_0 \left(\frac{r^{n+1}}{r_0^{n+1}}\, c(r)\right)' - \frac{1}{\left(1-\frac{r_0^n}{r^n}\right)\, s_\alpha^2}\, c(r)\,, \tag{B.22}$$

$$\mathsf{G}'(r) = g'(r) + \frac{2}{n\, s_\alpha^2}\frac{r_0}{r}\left(\frac{r^{n+1}}{r_0^{n+1}}\, c(r)\right)' + \frac{2}{n s_\alpha^2\, r\left(1-\frac{r_0^n}{r^n}\right)}\, c(r)\,. \tag{B.23}$$

The Einstein equations can now be shown to imply that, for any boost α, the functions A and B satisfy the same fourth-order differential equation as derived in [4]. As a consequence we can immediately write their solution as

$$\begin{aligned} \mathsf{A}(r) &= A_1\, u_1(r) + A_2\, u_2(r)\,, \\ \mathsf{B}(r) &= B_1\, u_1(r) + B_2\, u_2(r)\,, \end{aligned} \tag{B.24}$$

with

$$\begin{aligned} u_1(r) &= {}_2F_1\left(-\frac{1}{n}, -\frac{n+1}{n}; 1; 1-\frac{r_0^n}{r^n}\right) r\,, \\ u_2(r) &= {}_2F_1\left(-\frac{1}{n}, \frac{n-1}{n}; 1; 1-\frac{r_0^n}{r^n}\right) \frac{r_0^n}{r^{n-1}}\,. \end{aligned} \tag{B.25}$$

We have discarded solutions that blow up at $r = r_0$.

Note that $\mathsf{A}(r)$ and $\mathsf{B}(r)$ are not independent but one can be obtained from the other through an equation that in general differs slightly from the one in [4]. However, we shall not need this equation since the boundary behavior we obtained in the previous subsection is enough to fix all the integration constants A_i and B_i. The remaining functions $\mathsf{F}(r)$ and $\mathsf{G}(r)$ are obtained from $\mathsf{A}(r)$ and $\mathsf{B}(r)$ as

$$\begin{aligned} \mathsf{F}(r) =& -\frac{2(1+c_\alpha^2 n)r^{2n} - (4+c_\alpha^2(3n-2))r^n r_0^n - 2s_\alpha^2 r_0^{2n}}{n\,(r^n-r_0^n)\,r_0^n}\,\mathsf{A} - \frac{2r\left((1+c_\alpha^2 n)r^n + s_\alpha^2 r_0^n\right)}{n(n+1)r_0^n}\,\mathsf{A}' \\ &+\frac{2s_\alpha^2 c_\alpha^2\left(nr^n\,(2r^n-3r_0^n) + 2r_0^n\,(r^n-r_0^n)\right)}{n\,(r^n-r_0^n)\,r_0^n}\,\mathsf{B} + \frac{4s_\alpha^2 c_\alpha^2 r\,(nr^n+r_0^n)}{n(n+1)r_0^n}\,\mathsf{B}'\,, \end{aligned} \tag{B.26}$$

$$\begin{aligned} \mathsf{G}'(r) =& -\frac{2((1+nc_\alpha^2)r^n - (1+(n-1)c_\alpha^2)r_0^n)r^{n-1}}{n(r^n-r_0^n)r_0^n}\,\mathsf{A} - \frac{2(1+nc_\alpha^2)r^n + (n+2)s_\alpha^2 r_0^n}{n(n+1)r_0^n}\,\mathsf{A}' \\ &+\frac{4s_\alpha^2 c_\alpha^2(nr^n-(n-1)r_0^n)r^{n-1}}{n(r^n-r_0^n)r_0^n}\,\mathsf{B} + \frac{2s_\alpha^2 c_\alpha^2(2nr^n+(n+2)r_0^n)}{n(n+1)r_0^n}\,\mathsf{B}'\,. \end{aligned} \tag{B.27}$$

Asymptotic boundary conditions. We fix the integration constants A_i, B_i by demanding that the solution asymptotes to (B.11) as $r\to\infty$, *i.e.*, that

$$\mathsf{A}(r) = 2r\left(\mathsf{a}_0 + \mathsf{a}_1\left(\frac{r_0}{r}\right)^n + O(r^{-n-2})\right)\,,\quad \mathsf{B}(r) = 2r\left(\mathsf{b}_0 + \mathsf{b}_1\left(\frac{r_0}{r}\right)^n + O(r^{-n-2})\right) \tag{B.28}$$

with

$$\mathsf{a}_0 = -\left(C_t + \frac{c_\alpha^2}{s_\alpha^2}C_z\right), \qquad \mathsf{a}_1 = c_\alpha^2(C_t - C_z) + \frac{1}{2n}\left(C_t + \frac{c_\alpha^2}{s_\alpha^2}C_z\right),$$
$$\mathsf{b}_0 = -\frac{C_z}{s_\alpha^2}, \qquad \mathsf{b}_1 = \frac{C_t - C_z}{2} + \frac{C_z}{2ns_\alpha^2}. \tag{B.29}$$

We assume that the boost takes on the equilibrium value (B.7). Expanding (B.24) at large r and comparing to (B.28) determines

$$A_1 = \frac{2}{\pi}\frac{n+1}{n^3}\Gamma\left(\frac{1}{n}\right)^2\Gamma\left(-\frac{n+2}{n}\right)\sin\left(\frac{2\pi}{n}\right)\mathsf{a}_0, \qquad A_2 = 2A_1\left(\frac{1}{n} + \frac{n+2}{n+1}\frac{\mathsf{a}_1}{\mathsf{a}_0}\right) \tag{B.30}$$

and

$$B_1 = A_1\frac{\mathsf{b}_0}{\mathsf{a}_0}, \qquad B_2 = 2A_1\left(\frac{1}{n}\frac{\mathsf{b}_0}{\mathsf{a}_0} + \frac{n+2}{n+1}\frac{\mathsf{b}_1}{\mathsf{a}_0}\right). \tag{B.31}$$

Regularity of the horizon. The solution is now fully determined, up to gauge transformations, and it only remains to check that the horizon stays regular. This requires that the angular velocity and surface gravity are constant quantities on the horizon and do not depend on the polar angle θ.

Constancy of the horizon angular velocity is easily seen to be equivalent to

$$\mathsf{A}(r_0) = (c_\alpha^2 + s_\alpha^2)\mathsf{B}(r_0) \tag{B.32}$$

which translates into $A_1 + A_2 = (c_\alpha^2 + s_\alpha^2)(B_1 + B_2)$. This *is* satisfied by (B.30), (B.31). Additionally, one must partially fix the gauge freedom by requiring that the gauge-variant function $c(r)$ at the horizon be $c(r_0) = -c_\alpha^2 s_\alpha^2(B_1 + B_2)r_0$.

Uniformity of the surface gravity requires in turn

$$a'(r_0) - 2s_\alpha^2\, b'(r_0) + \frac{s_\alpha^2}{c_\alpha^2}\, c'(r_0) + \frac{n}{c_\alpha^2 r_0} f(r_0) = 0. \tag{B.33}$$

Despite appearances, this condition is gauge-invariant for regular gauge transformations. Using the explicit form of the solution derived above, this equation is seen to be identically satisfied.

Thus we conclude that we can construct black rings that are regular on and outside the horizon by placing a boosted black string in any spacetime that, locally near the ring, takes the form (A.8).

C. Phase space of rotating AdS black holes

Here we discuss the phase space of the rotating AdS black hole solution of [13] with more than one angular momentum (the case of a single angular momentum is discussed in the main text). Their mass, angular momenta, area and surface gravity, as computed in [21], are

$$M=\frac{m\,\Omega_{d-2}}{4\pi\prod_j\Xi_j}\left(\sum_{i=1}^{N}\frac{1}{\Xi_i}-\frac{1-\epsilon}{2}\right),\qquad J_i=\frac{m\,\Omega_{d-2}a_i}{4\pi\Xi_i\prod_j\Xi_j}\,, \tag{C.1}$$

$$\mathcal{A}=\frac{\Omega_{d-2}}{r_+^{1-\epsilon}}\prod_i\frac{r_+^2+a_i^2}{\Xi_i}\,,\qquad \kappa=r_+\left(1+\frac{r_+^2}{L^2}\right)\left(\sum_i\frac{1}{r_+^2+a_i^2}+\frac{\epsilon}{2r_+^2}\right)-\frac{1}{r_+}\,, \tag{C.2}$$

where $N=\left[\frac{d-1}{2}\right]$ is the maximal number of independent angular momenta, a_i are the N angular velocity parameters, m is the mass parameter, $\epsilon=(d-1)\mathrm{mod}\,2$, $\Xi_i=1-a_i^2/L^2$ and r_+ is the horizon position, given by the largest root of the equation

$$r^{\epsilon-2}(1+r^2/L^2)\prod_i(r^2+a_i^2)-2\,m=0. \tag{C.3}$$

Note that this equation is the same as the equation for the horizon of the asymptotically flat MP black hole in $d+2$ dimensions, where the additional rotation is $a_{N+1}=L$ and mass parameter is $\mu=2L^2m$ [3]. Hence, the root structure and the horizons of the rotating AdS_d black hole can be inferred from the MP_{d+2} solution; in particular, for odd d, a horizon always exists provided that any two of the spin parameters vanish, while for even d, its existence is guaranteed if any one of the spins vanishes. Therefore, under this assumption, an ultraspinning limit can be achieved for all but two (one) of the $a_i\to L$ in odd (even) dimensions.

The phase space $(\mathsf{M},\mathsf{J}_\phi,\mathsf{J}_\psi)$ covered by the rotating AdS black holes is shown in figure 5 for the five and six dimensional cases, filling the interior of the pyramids (the five-dimensional diagram was already presented in [3]). The extremal solutions lie at the faces of the pyramid. Figure 6 presents cuts of these surfaces at a constant value of the mass. These are

We can show that these black holes comply with the "BPS bound"

$$ML\geq\sum_{i=1}^{N}|J_i| \tag{C.4}$$

proven in [20]. To this effect, write

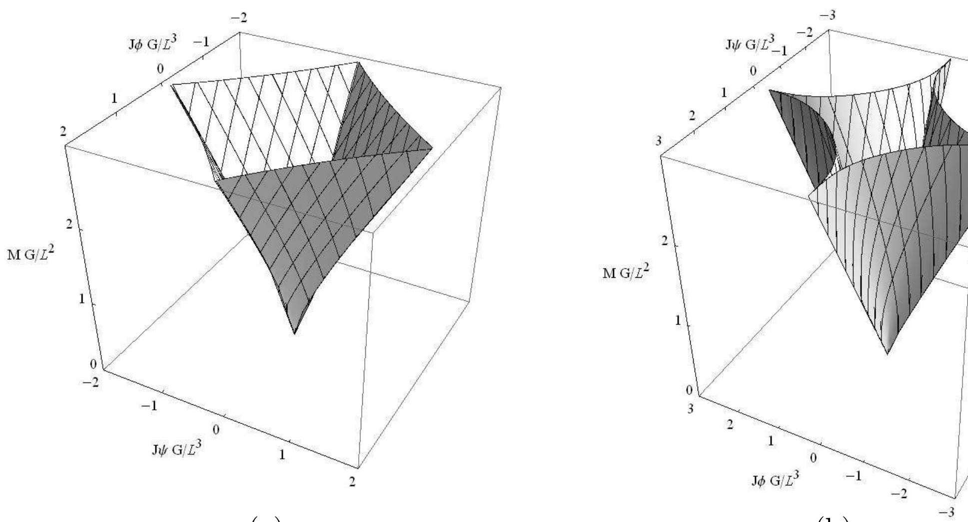
$$\frac{1}{ML}\sum_{i=1}^{N}|J_i|=\frac{\sum_i\frac{|a_i|}{L\Xi_i}}{\sum_j\frac{1}{\Xi_j}-\frac{1-\epsilon}{2}}\,. \tag{C.5}$$

If d is even, $\epsilon=1$, and since $|a_i|/L\leq 1$ we have

$$\sum_i\frac{|a_i|}{L\Xi_i}\leq\sum_j\frac{1}{\Xi_j}\,, \tag{C.6}$$

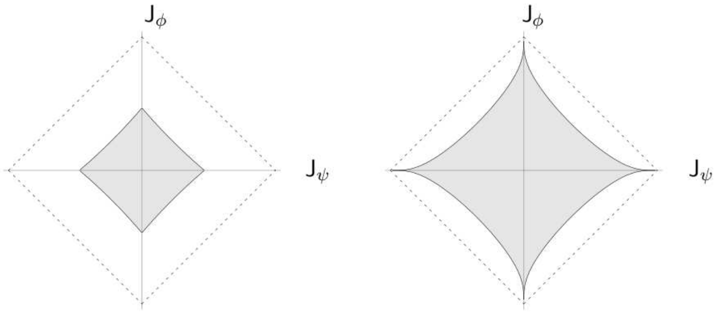
proving equation (C.4). If d is odd, $\epsilon=0$ and we can rewrite equation (C.5) as

$$\frac{1}{ML}\sum_{i=1}^{N}|J_i|=\frac{\sum_i\frac{|a_i|/L}{\Xi_i}}{\sum_j\frac{1-\Xi_j/2N}{\Xi_j}}\,. \tag{C.7}$$



(a) (b)

Figure 5: Boundaries of the phase space $(\mathsf{J}_\phi, \mathsf{J}_\psi, \mathsf{M})$ covered by doubly spinning rotating AdS black holes in five (a) and six (b) dimensions. The mass increases along the vertical axis. The surfaces correspond to extremal, zero-temperature black holes, except at the edges where they become naked singularities. The interior of the pyramids is filled by non-extremal black holes.



(a) (b)

Figure 6: Cuts at constant M of the surfaces of figure 5. Black hole solutions exist for angular momenta in the shaded regions. The dashed lines denote the BPS limit $\sum_{i=1}^{N} |\mathsf{J}_i| = \mathsf{M}$. In six dimensions this is saturated only at the corners. These diagrams can be compared to figure 2 of [3]. In more than six dimensions, we expect the diagrams to resemble the ones of the asymptotically flat case, see [3], only 'compressed' to lie within the 'BPS diamond'.

The polynomial

$$P_i(a_i) = 1 - \frac{\Xi_i}{2N} - \frac{1}{L}|a_i| = \frac{a_i^2}{2NL^2} - \frac{|a_i|}{L} + 1 - \frac{1}{2N} \tag{C.8}$$

admits always, as smaller positive root, $a_i = L$, and is therefore positive for $|a_i| < L$. Hence,

$$\frac{|a_i|/L}{\Xi_i} \leq \frac{1 - \Xi_j/2N}{\Xi_j}\,, \tag{C.9}$$

and (C.4) follows.

It is easy to show that this bound can only be saturated in the ultra-spinning regime, in which one or more spin parameters tend to L. However, it can never be saturated when all the angular momenta are non-zero. Indeed, suppose n spin parameters approach the ultraspinning limit. To keep the mass finite, we need to scale the parameters as

$$\Xi_{\alpha=1\dots n} = \xi_\alpha \nu \,, \qquad m = \mu \nu^{n+1}, \tag{C.10}$$

where $\nu \to 0$ in the ultraspinning limit, while keeping $\xi_1, \dots, \xi_n$ and μ constant. As we observed previously, this limit is allowed provided any one (two) of the a_i vanish in even (odd) dimensions. Then the root r_+ of equation (C.3) tends to zero, while the mass and angular momenta reach the values

$$M = \frac{\mu \Omega_{d-2}}{4\pi \Pi_\alpha \xi_\alpha \Pi_I \Xi_I} \sum_\alpha \frac{1}{\xi_\alpha}\,, \qquad J_\alpha = \frac{\mu \Omega_{d-2}}{4\pi \xi_\alpha \Pi_\beta \xi_\beta \Pi_I \Xi_I}\,, \qquad J_I = 0\,, \tag{C.11}$$

(with $\alpha, \beta = 1 \dots n$ running on the spin parameters that tend to L and $I = n+1, \dots, N$ denoting the others) and saturate the BPS bound (C.4). However, these black holes are not extremal, since the surface gravity diverges like $\kappa \to (2k + \epsilon - 2)/2r_+$, where k is the number of vanishing spin parameters. In this limit the area of the horizon decreases to zero like

$$\mathsf{A_H} \propto \mathsf{M}^{\frac{2k+\epsilon-1}{2k+\epsilon-2}} \left(1 - \frac{\mathsf{J}}{\mathsf{M}}\right)^{\frac{2k+n+\epsilon-1}{2k+\epsilon-2}} \left(1 + \mathcal{O}(\mathsf{M} - \mathsf{J})\right)\,, \tag{C.12}$$

and the limiting geometry describes a black membrane with horizon topology $\mathbb{R}^{2n} \times S^{d-2(n+1)}$. Since in this limit the black holes are pancaked out along the planes of rotation, it is reasonable to presume that they will develop a Gregory-Laflamme type of instability.

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